IJMMS 2004:9, 479–485
PII. S0161171204208274
http://ijmms.hindawi.com

INTERPOLATION METHODS TO ESTIMATE EIGENVALUE DISTRIBUTION OF SOME INTEGRAL OPERATORS

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Received 20 August 2002

We study the asymptotic distribution of eigenvalues of integral operators T_k defined by kernels k which belong to Triebel-Lizorkin function space $F^{\sigma}_{pu}(F^{\tau}_{qv})$ by using the factorization theorem and the Weyl numbers x_n. We use the relation between Triebel-Lizorkin space $F^{\sigma}_{pu}(\Omega)$ and Besov space $B^{\tau}_{pq}(\Omega)$ and the interpolation methods to get an estimation for the distribution of eigenvalues in Lizorkin spaces $F^{\sigma}_{pu}(F^{\tau}_{qv})$.

2000 Mathematics Subject Classification: 46B20, 47B10.

1. Lizorkin kernels. We will use the following notation: $l_{p,q}$, $S^{(x)}_{pq}$, B^s_{pq}, and F^s_{pq} to denote Lorentz sequence space, Schatten class, Besov function space, and Triebel-Lizorkin function space, respectively. By π_p, s_n, and x_n we denote p-summing norms, s-number function, and Weyl numbers, respectively, see [2, 4, 5].

Theorem 1.1 (see [1]). *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and let $q,u,v \in (0,\infty]$. Then $B^s_{pu} \subset F^s_{pq} \subset B^s_{pv}$ if and only if*

$$0 < u \le \min(p,q), \qquad \max(p,q) \le v \le \infty. \tag{1.1}$$

That is, if and only if $0 < u \le q \le v$.

Proposition 1.2 (see [4]). *Let $\Phi \in [B^{\sigma}_{pu}(0,1),X]$ and $r = \max(p,u)$. Then*

$$\Phi_{\text{op}} : a \longrightarrow (\Phi(\cdot),a) \tag{1.2}$$

(where Φ_{op} is an approximable operator from X' into $B^{\sigma}_{pu}(0,1)$) defines an absolutely r-summing operator from X' into $B^{\sigma}_{pu}(0,1)$. Moreover,

$$||\Phi_{\text{op}} \mid \pi_r|| \le ||\Phi \mid [B^{\sigma}_{pu},X]||. \tag{1.3}$$

We restate the previous proposition in the following form in the case of Triebel-Lizorkin space $F^{\sigma}_{pu}(\Omega)$.

Proposition 1.3. *Let X be a Banach space, $\Omega \subset \mathbb{R}^N$ a bounded domain, $\sigma > 0$, and $1 \le p < \infty$. Let $\Phi \in F^{\sigma}_{pu}(\Omega;X)$ and $r = \max(p,u)$. Then*

$$\Phi_{\text{op}} : x \longrightarrow (\Phi(\cdot),x) \tag{1.4}$$

defines an absolutely r-summing operator from X' into $F^{\sigma}_{pu}(X)$. Moreover,

$$\pi_r(\Phi_{\text{op}}) \le \|\Phi\|_{p,u,\sigma;\Omega,X}. \tag{1.5}$$

Proof. Given $x_1,\dots,x_n \in X'$, Jessen's inequality [4] yields

$$\left(\int_\Omega \left[\sum_{i=1}^n |(\Phi(\xi),x_i)|^r\right]^{p/r} d\xi\right)^{1/p} \le \left(\sum_{i=1}^n \left[\int_\Omega |(\Phi(\xi),x_i)|^p d\xi\right]^{r/p}\right)^{1/r}. \tag{1.6}$$

Therefore,

$$\left\|\left(\sum_{i=1}^n |(\Phi(\cdot),x_i)|^r\right)^{1/r}\right\|_{L_p} \le \left(\sum_{i=1}^n \|(\Phi(\cdot),x_i)\|_{L_p}^r\right)^{1/r} \le \|\Phi\|_{L_p}\|(x_i)\|_{\pi_r}. \tag{1.7}$$

Applying this result to $\Delta_\tau^m\Phi$, we obtain

$$\left\|\left(\sum_{i=1}^n |(\Delta_\tau^m\Phi(\cdot),x_i)|^r\right)^{1/r}\right\|_{L_p} \le \|\Delta_\tau^m\Phi\|_{L_p}\|(x_i)\|_{\pi_r}. \tag{1.8}$$

Hence,

$$\begin{aligned}
&\left[\int_\Omega \left(\sum_{i=1}^n \left[\tau^{-\sigma}\|(\Delta_\tau^m\Phi(\cdot),x_i)\|_{L_p}\right]^r\right)^{u/r}\frac{d\tau}{\tau}\right]^{1/u} \\
&\quad\le \left[\sum_{i=1}^n \left(\int_\Omega \left[\tau^{-\sigma}\|\Delta_\tau^m\Phi(\cdot),x_i\|_{L_p}\right]^u\right)^{r/u}\right]^{1/r} \\
&\quad\le \left\|\left(\left[\int_\Omega \tau^{-\sigma}|\Delta_\tau^m\Phi|\right]^u\frac{d\tau}{\tau}\right)^{1/u}\right\|_{L_p}\|(x_i)\|_{\pi_r}.
\end{aligned} \tag{1.9}$$

Finally, we conclude from the preceding inequalities that

$$\begin{aligned}
&\left\|\left(\sum_{i=1}^n |(\Phi(\cdot),x_i)|^r\right)^{1/r}\right\|_{F^{\sigma}_{pu}} \\
&\quad\le \left\|\left(\sum_{i=1}^n |(\Phi(\cdot),x_i)|^r\right)^{1/r}\right\|_{L_p} + \left[\int_\Omega\left(\sum_{i=1}^n [\tau^{-\sigma}\|(\Delta_\tau^m\Phi(\cdot),x_i)\|_{L_p}]^r\right)^{u/r}\frac{d\tau}{\tau}\right]^{1/u} \\
&\quad\le \left(\|\Phi\|_{L_p} + \left\|\left(\left[\int_\Omega \tau^{-\sigma}|\Delta_\tau^m\Phi|\right]^u\frac{d\tau}{\tau}\right)^{1/u}\right\|_{L_p}\right) = \|\Phi\|_{F^{\sigma}_{pu}}\|(x_i)\|_{\pi_r}.
\end{aligned} \tag{1.10}$$

This shows that Φ_{op} is absolutely r-summing. □

Corollary 1.4 (see [2]). *Let X and Y be Banach spaces, $2 \le p < \infty$, and $T \in \pi_{p,2}(X,Y)$. Then $T \in S^{x}_{p,\infty}(X,Y)$, and for any $n \in \mathbb{N}$,*

$$x_n(T) \le n^{-1/p}\pi_{p,2}(T). \tag{1.11}$$

We are interested in the following theorem.

Theorem 1.5 (see [3]). *Let $1 \le p \le \max(2,q) \le \infty$. Then*

$$x_n\left(I^m_{p,q} : l^m_p \longrightarrow l^m_q\right) \asymp \begin{cases} n^{1/q-1/p} & \text{for } 1 \le p \le q \le 2, \\ n^{1/2-1/p} & \text{for } 1 \le p \le 2 \le q < \infty, \\ 1 & \text{for } 2 \le p \le q < \infty. \end{cases} \tag{1.12}$$

Theorem 1.6 ((multiplication theorem) [4]). *If $1/p+1/q = 1/r$ and $1/u+1/v = 1/w$, then*

$$S^{(x)}_{pu} \circ S^{(x)}_{qv} \subseteq S^{(x)}_{rw}. \tag{1.13}$$

Theorem 1.7 ((eigenvalue theorem) [2]). *Let $0 < p < \infty$, $0 < q \le \infty$, and let X be a Banach space. Then any operator $T \in L(X)$ which has Weyl numbers $(x_n(T)) \in l_{p,q}$, $T \in S^{(x)}_{p,q}(X)$ is a Riesz operator, the eigenvalue sequence of which is in $l_{p,q}$, and the following inequality holds*

$$\left\|(\lambda_n(T))\right\|_{p,q} \le c\left\|(x_n(T))\right\|_{p,q}. \tag{1.14}$$

2. Eigenvalue theorem for Lizorkin kernels. The following theorem contains the main result of this note.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $1 \le p,q,u,v < \infty$, and $\sigma+\tau > N(1/p+1/q-1)$. Define r by $1/r = (\sigma+\tau)/N + 1/q^+$, where $q^+ = \max(q',2)$. Then the eigenvalues of any kernel $k \in F^{\sigma}_{pu}(\Omega; F^{\tau}_{qv}(\Omega))$ belong to the Lorentz sequence space $l_{r,p}$ with*

$$\left\|(\lambda_n(k))_{n\in\mathbb{N}}\right\|_{l_{r,p}} \le c\|k\|_{F^{\sigma}_{pu}(F^{\tau}_{qv})}. \tag{2.1}$$

The constant c depends only on the indices and Ω.

Proof. First, we assume that $p \le q'$.

We will show that there exists an imbedding map $\mathrm{id} : F^{\sigma}_{pu}(\Omega) \hookrightarrow F^{\tau}_{qv}(\Omega)'$ and then estimate its Weyl numbers $x_n(\mathrm{id})$. We factories an imbedding map $\mathrm{id} : F^{\sigma}_{pu}(\Omega) \hookrightarrow F^{\tau}_{qv}(\Omega)'$ with the help of maps A and B such that

$$\mathrm{id} = B \circ \mathrm{id}^l \circ A,$$

$$\begin{array}{ccc} F^{\sigma}_{pu}(\Omega) & \xrightarrow{\ \mathrm{id}\ } & F^{\tau}_{qv}(\Omega)' \\ {\scriptstyle A}\big\downarrow & & \big\uparrow{\scriptstyle B} \\ l^m_p(\Omega) & \xrightarrow{\ \mathrm{id}^l\ } & l^m_{q'}(\Omega). \end{array} \tag{2.2}$$

This means that

$$x_n(\mathrm{id}) \le \|A\| x_n(\mathrm{id}^l)\|B\| \tag{2.3}$$

if we are able to estimate $\|A\|$ and $\|B\|$ suitably; from [6], operators A and B are defined exactly as they are in [4], and assume that Ω contains the unit cube in $\mathbb{R}^N$ and divide the unit cube in the usual way into 2^{jN} congruent cubes with side length 2^{-j}.

From [1], we have

$$\|A\| \le c_1 2^{-j(\sigma - N/p)}, \qquad \|B\| \le c_2 2^{j(-\tau - N/q')}. \tag{2.4}$$

Substituting (2.4) in (2.3), we get

$$x_n(\mathrm{id}) \le c 2^{-j(\sigma+\tau)+jN(1/p-1/q')} x_n(\mathrm{id}^l). \tag{2.5}$$

By Theorem 1.5, we have

$$x_n\left(\mathrm{id} : F^{\sigma}_{pu}(\Omega) \hookrightarrow F^{\tau}_{qv}(\Omega)'\right) < n^{-\rho}, \tag{2.6}$$

where

$$\rho = \frac{\sigma+\tau}{N} + \begin{cases} 0, & \text{if } 1 \le p \le q' \le 2, \\ \dfrac{1}{2} - \dfrac{1}{q}, & \text{if } 1 \le p \le 2 \le q' < \infty, \\ 1 - \dfrac{1}{p} - \dfrac{1}{q}, & \text{if } 2 \le p \le q' < \infty, \end{cases} \tag{2.7}$$

and $n = 2^{Nj}$.

Hence,

$$\mathrm{id} \in S^{(x)}_{1/\rho,\infty}\left(F^{\sigma}_{pu}(\Omega) \hookrightarrow F^{\tau}_{qv}(\Omega)'\right). \tag{2.8}$$

To estimate the Weyl number of T_k in $F^{\tau}_{qv}(\Omega)'$, we use the factorization

$$F^{\tau}_{qv}(\Omega)' \xrightarrow{T_k} F^{\sigma}_{pu}(\Omega) \overset{\mathrm{id}}{\hookrightarrow} F^{\tau}_{qv}(\Omega)'. \tag{2.9}$$

By Proposition 1.3, $k \in F^{\sigma}_{pu}(\Omega; X)$ implies that $T_k : X' \hookrightarrow F^{\sigma}_{pu}(\Omega)$ is p-summing. By Corollary 1.4, we have

$$x_n\left(T_k : F^{\tau}_{qv}(\Omega)' \longrightarrow F^{\sigma}_{pu}(\Omega)\right) \le \pi_p(T_k) n^{-1/\max(p,2)} \le c_1 \|k\|_{F^{\sigma}_{pu}(X)} n^{-1/\max(p,2)}, \tag{2.10}$$

that is,

$$T_k \in S^{(x)}_{s,\infty}\left(F^{\sigma}_{pu}(\Omega), F^{\tau}_{qv}(\Omega)'\right), \quad s = \max(p,2). \tag{2.11}$$

We conclude from the multiplication theorem that $\mathrm{id} \circ T_k \in S^{(x)}_{r,\infty}(F^{\tau}_{qv}(\Omega)')$, where $1/r = \rho + 1/s$.

In the case when $p > q'$, then we have

$$k \in F^{\sigma}_{pu}(\Omega; F^{\tau}_{qv}(\Omega)) \Longrightarrow k \in F^{\sigma}_{q'u}(\Omega; F^{\tau}_{qv}(\Omega)). \tag{2.12}$$

In this way the second case is reduced to the first one.

So, we have shown that the map $k \to \mathrm{id} \circ T_k$, which assigns to every kernel the corresponding operator, acts as follows:

$$\mathrm{op} : F^{\sigma}_{pu}(F^{\tau}_{qv}) \longrightarrow S^{(x)}_{r,\infty}(F^{\tau}_{qv}(\Omega)'). \tag{2.13}$$

This result can be improved by interpolation. To this end, choose p_0, p_1, and θ such that $1/p = 1-\theta/p_0 + \theta/p_1$. We now apply the formula

$$(F^{\sigma}_{p_0u}(E), F^{\sigma}_{p_1u}(E))_{\theta,p} = F^{\sigma}_{pu}(E), \quad E = F^{\tau}_{qv}. \tag{2.14}$$

Then, using interpolation as in [2], where $1/r = 1-\theta/r_0 + \theta/r_1$, hence

$$(S^{(x)}_{r_0\infty}, S^{(x)}_{r_1\infty})_{\theta,p} \subseteq S^{(x)}_{rp}. \tag{2.15}$$

Hence the interpolation property yields

$$\mathrm{op} : F^{\sigma}_{pu}(F^{\tau}_{qv}) \longrightarrow S^{(x)}_{r,p}(F^{\tau}_{qv}(\Omega)'). \tag{2.16}$$

By the eigenvalue theorem (Theorem 1.7), we therefore obtain $(\lambda_n(k)) \in l_{r,p}$. □

Theorem 2.2 (eigenvalue theorem for Sobolev kernels)**.** *Let $1 \le p < \infty$, $1 < q < \infty$, $1/r = m+n+1/q^+$, and $w = \min(q,2)$.*

Then

$$k \in [W^m_p(0,1), W^n_q(0,1)] \Longrightarrow (\lambda_n(k)) \in l_{r,w}. \tag{2.17}$$

Proof. See [4]. □

The following example proves that our result improves the previous theorem of [4].

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $1 \le p, q, v < \infty$, and $\tau > 0$ with $\tau > N(1/p + 1/q - 1)$, $p \le v$, and $1/r := \tau/N + 1/\max(2, q')$. Then the eigenvalues of any kernel $k \in L_p(F^{\tau}_{qv})$ belong to the Lorentz sequence space $l_{r,v}$ with*

$$||(\lambda_n(k))_{n\in N}||_{l_{r,v}} \le c\|k\|_{L_p(F^{\tau}_{qv})}. \tag{2.18}$$

Proof. We may assume that $p \le q'$. Then, reasoning similarly as in the proof of Theorem 2.1, it follows that the map $k \to T_k$, which assigns to every kernel the corresponding operator, acts as follows:

$$\mathrm{op} : L_p(F^{\tau}_{qv}) \longrightarrow S^{(x)}_{r,\infty}(L_p(\Omega)). \tag{2.19}$$

This result can be improved by interpolation. To this end, we apply the imbedding

$$(L_p,(E_0,E_1)_{\theta,m}) \subseteq ((L_p,E_0),(L_p,E_1))_{\theta,m}, \quad p < m, \tag{2.20}$$

to the interpolation couple $(F^{\tau_0}_{q,v_0}, F^{\tau_1}_{q,v_1})$. The interpolation property now implies that

$$\text{op} : L_p(F^{\tau}_{qv}) \longrightarrow S^{(x)}_{r,v}(L_p(\Omega)). \tag{2.21}$$

By the eigenvalue theorem (Theorem 1.7), we therefore obtain $(\lambda_n(k)) \in l_{r,v}$. □

Example 2.4. (1) In this example, we will indicate a special case of the Lizorkin space $F^{\sigma}_{pu}(\mathbb{R}^N)$. When $1 < p < \infty$ and $s \in \mathbb{N}_0$, then

$$F^{s}_{p,2}(\mathbb{R}^N) = W^{s}_{p}(\mathbb{R}^N) \tag{2.22}$$

are the classical Sobolev spaces.

We compare this case with Theorem 2.2. We find that

$$k \in W^{\sigma}_{p}(W^{\tau}_{q}) \Longrightarrow (\lambda_n(k)) \in l_{r,w}, \tag{2.23}$$

where $w = \min(q,2)$, and

$$k \in F^{\sigma}_{pu}(F^{\tau}_{qv}) \Longrightarrow (\lambda_n(k)) \in l_{r,p}. \tag{2.24}$$

We conclude that if $p < w = \min(q,2)$, $2 \le q < \infty$, $1 < p < 2$, then

$$l_{r,w} \subset l_{r,p}, \tag{2.25}$$

that is,

$$\|(\lambda_n(k))_{n\in\mathbb{N}}\|_{r,p} \le \|(\lambda_n(k))_{n\in\mathbb{N}}\|_{r,w}. \tag{2.26}$$

(2) We compare

$$k \in W^{\sigma}_{p}(W^{\tau}_{q}) \Longrightarrow (\lambda_n(k)) \in l_{r,w}, \tag{2.27}$$

where $w = \min(q,2)$, with

$$k \in L_p(F^{\tau}_{qv}) \Longrightarrow (\lambda_n(k)) \in l_{r,v}, \tag{2.28}$$

where $p \le v$. We conclude that if $v < w = \min(q,2)$, $2 \le q < \infty$, $1 < p < 2$, then

$$l_{r,w} \subset l_{r,v}, \tag{2.29}$$

that is,

$$\|(\lambda_n(k))_{n\in N}\|_{r,v} \le \|(\lambda_n(k))_{n\in N}\|_{r,w}. \tag{2.30}$$

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