INTEGRAL PROPERTIES OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS

G. S. SĂLĂGEAN

Received 20 February 2004 and in revised form 11 November 2004

We study integral properties of two classes of functions with negative coefficients defined using differential operators. The obtained results are sharp and they improve known results.

1. Introduction

Let $\mathbb{N}$ denote the set of nonnegative integers $\{0,1,\ldots,n,\ldots\}$, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, and let $\mathcal{N}_j$, $j \in \mathbb{N}^*$, be the class of functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k, \quad a_k \geq 0,\ k \in \mathbb{N},\ k \geq j+1, \tag{1.1}$$

that are analytic in the open unit disc $U = \{z : |z| < 1\}$.

Definition 1.1 [11]. The operator $D^n : \mathcal{N}_j \to \mathcal{N}_j$, $n \in \mathbb{N}$, is defined by (a) $D^0 f(z) = f(z)$; (b) $D^1 f(z) = Df(z) = zf'(z)$; (c) $D^n f(z) = D(D^{n-1} f(z))$, $z \in U$.

Definition 1.2 [4]. Let $\alpha,\lambda \in [0,1)$, $n \in \mathbb{N}$, $j,m \in \mathbb{N}^*$; a function f belonging to $\mathcal{N}_j$ is said to be in the class $T_j(n,m,\lambda,\alpha)$ if and only if

$$\operatorname{Re} \frac{D^{n+m} f(z)/D^n f(z)}{\lambda\left(D^{n+m} f(z)/D^n f(z)\right) + 1 - \lambda} > \alpha, \quad z \in U. \tag{1.2}$$

Remark 1.3. The classes $T_j(n,m,\lambda,\alpha)$ are generalizations of the classes

(i) $T_1(0,1,0,\alpha)$ and $T_1(1,1,0,\alpha)$ defined and studied by Silverman [12] (these classes are the class of starlike functions with negative coefficients and the class of convex functions with negative coefficients, resp.),
(ii) $T_j(0,1,0,\alpha)$ and $T_j(1,1,0,\alpha)$ studied by Chatterjea [7] and Srivastava et al. [13],
(iii) $T_1(n,1,0,\alpha)$ studied by Hur and Oh [10],
(iv) $T_1(0,1,\lambda,\alpha)$ and $T_1(1,1,\lambda,\alpha)$ studied by Altintas and Owa [2],
(v) $T_1(n,1,\lambda,\alpha)$ studied by Aouf and Cho [3, 8],
(vi) $T_1(n,m,0,\alpha)$ studied by Hossen et al. [9].

International Journal of Mathematics and Mathematical Sciences 2005:1 (2005) 125–131
DOI: 10.1155/IJMMS.2005.125

In [4], the next characterization theorem of the class $T_j(n,m,\lambda,\alpha)$ is given.

Theorem 1.4. *Let $n \in \mathbb{N}$, $j,m \in \mathbb{N}^*$, $\alpha,\lambda \in [0,1)$, and let $f \in \mathcal{N}_j$; then $f \in T_j(n,m,\lambda,\alpha)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^n\left[k^m(1-\alpha\lambda)-\alpha(1-\lambda)\right]a_k \leq 1-\alpha. \tag{1.3}$$

The result is sharp and the extremal functions are

$$f(z)=z-\frac{1-\alpha}{k^n\left[k^m(1-\alpha\lambda)-\alpha(1-\lambda)\right]}z^k, \quad k \in \mathbb{N},\ k \geq j+1. \tag{1.4}$$

Definition 1.5 [5]. Let $m,n \in \mathbb{N}$, $j \in \mathbb{N}^*$, $\alpha \in [0,1)$, $\lambda \in [0,1]$; a function f belonging to $\mathcal{N}_j$ is said to be in the class $L_j(n,m,\lambda,\alpha)$ if and only if

$$\operatorname{Re}\frac{(1-\lambda)D^{n+1}f(z)+\lambda D^{n+m+1}f(z)}{(1-\lambda)D^n f(z)+\lambda D^{n+m}f(z)} > \alpha, \quad z \in U. \tag{1.5}$$

Remark 1.6. The classes $L_j(n,m,\lambda,\alpha)$ are generalizations of the classes

(1) $L_1(0,0,0,\alpha)=T_1(0,1,0,\alpha)$ and $L_1(1,0,1,\alpha)=T_1(1,1,0,\alpha)$ (the classes defined and studied by Silverman [12]),

(2) $L_j(0,0,0,\alpha)=T_j(0,1,0,\alpha)$ and $L_j(0,1,1,\alpha)=L_j(1,0,1,\alpha)=T_j(1,1,0,\alpha)$ (the classes studied by Chatterjea [7] and Srivastava et al. [13]),

(3) $L_j(0,1,\lambda,\alpha)$ studied by Altintas [1],

(4) $L_j(n,1,\lambda,\alpha)$, $L_j(n,m,0,\alpha)$, and $L_j(n,1,1,\alpha)$ studied by Aouf and Srivastava [6].

In [5], the next characterization theorem of the class $L_j(n,m,\lambda,\alpha)$ is given.

Theorem 1.7. *Let $n,m \in \mathbb{N}$, $j \in \mathbb{N}^*$, $\alpha \in [0,1)$, $\lambda \in [0,1]$, and let $f \in \mathcal{N}_j$; then $f \in L_j(n,m,\lambda,\alpha)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^n(k-\alpha)\left[1+(k^m-1)\lambda\right]a_k \leq 1-\alpha. \tag{1.6}$$

The result is sharp and the extremal functions are

$$f(z)=z-\frac{1-\alpha}{k^n(k-\alpha)\left[1+(k^m-1)\lambda\right]}z^k, \quad k \in \mathbb{N},\ k \geq j+1. \tag{1.7}$$

Let $I_c : \mathcal{N}_j \to \mathcal{N}_j$ be the integral operator defined by $g=I_c(f)$, where $c \in (-1,\infty)$, $f \in \mathcal{N}_j$, and

$$g(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}f(t)\,dt. \tag{1.8}$$

We note that if $f \in \mathcal{N}_j$ is a function of the form (1.1), then

$$g(z)=I_c(f)(z)=z-\sum_{k=j+1}^{\infty}\frac{c+1}{c+k}a_k z^k. \tag{1.9}$$

By using Theorem 1.4, in [4] it is proved that $I_c(T_j(n,m,\lambda,\alpha)) \subset T_j(n,m,\lambda,\alpha)$ and by using Theorem 1.7, in [5] it is proved that $I_c(L_j(n,m,\lambda,\alpha)) \subset L_j(n,m,\lambda,\alpha)$. In this note, these results are improved.

2. Integral properties of the class $T_j(n,m,\lambda,\alpha)$

Theorem 2.1. *Let $n \in \mathbb{N}$, $j,m \in \mathbb{N}^*$, $\alpha,\lambda \in [0,1)$, and let $c \in (-1,\infty)$; if $f \in T_j(n,m,\lambda,\alpha)$ and $g = I_c(f)$, then $g \in T_j(n,m,\lambda,\beta)$, where*

$$\begin{aligned}\beta &= \beta(m,\lambda,\alpha,c;j+1)\\ &= 1 - \frac{[(j+1)^m - 1](1-\alpha)(1-\lambda)(c+1)}{[(j+1)^m - 1][(1-\alpha\lambda)(c+j+1) - \lambda(c+1)(1-\alpha)] + (1-\alpha)j}\end{aligned} \tag{2.1}$$

and $\alpha < \beta(m,\lambda,\alpha,c;j+1) < 1$. The result is sharp.

Proof. From Theorem 1.4 and from (1.9) we have $g \in T_j(n,m,\lambda,\beta)$ if and only if

$$\sum_{k=j+1}^{\infty} \frac{k^n[k^m(1-\beta\lambda) - \beta(1-\lambda)](c+1)}{(1-\beta)(c+k)} a_k \le 1. \tag{2.2}$$

We find the largest β such that (2.2) holds. We note that the inequalities

$$\frac{k^m(1-\beta\lambda) - \beta(1-\lambda)}{1-\beta} \frac{c+1}{c+k} \le \frac{k^m(1-\alpha\lambda) - \alpha(1-\lambda)}{1-\alpha}, \quad k \ge j+1, \tag{2.3}$$

imply (2.2), because $f \in T_j(n,m,\lambda,\alpha)$ and it satisfies (1.3). But the inequalities (2.3) are equivalent to

$$A(m,\lambda,\alpha,c;k)\beta \le B(m,\lambda,\alpha,c;k), \tag{2.4}$$

where

$$\begin{aligned}A(m,\lambda,\alpha,c;k) &= (k^m - 1)[(1-\alpha\lambda)(c+k) - \lambda(c+1)(1-\alpha)] + (1-\alpha)(k-1),\\ B(m,\lambda,\alpha,c;k) &= A(m,\lambda,\alpha,c;k) - (k^m - 1)(c+1)(1-\alpha)(1-\lambda).\end{aligned} \tag{2.5}$$

Since $1 - \alpha\lambda > 1 - \alpha$ and $c + k > c + 1$, we have $A(m,\lambda,\alpha,c;k) > 0$ and from (2.4) we obtain

$$\beta \le \frac{B(m,\lambda,\alpha,c;k)}{A(m,\lambda,\alpha,c;k)} \quad \forall k \ge j+1. \tag{2.6}$$

We define $\beta(m,\lambda,\alpha,c;k) := B(m,\lambda,\alpha,c;k)/A(m,\lambda,\alpha,c;k)$. We show now that $\beta(m,\lambda,\alpha, c;k)$ is an increasing function of k, $k \geq j+1$. Indeed

$$\begin{aligned}\beta(m,\lambda,\alpha,c;k) &= 1-(1-\alpha)(1-\lambda)(c+1)\frac{k^m-1}{A(m,\lambda,\alpha,c;k)}\\ &= 1-(1-\alpha)(1-\lambda)(c+1)\frac{1}{E(m,\lambda,\alpha,c;k)},\end{aligned} \tag{2.7}$$

where $E(m,\lambda,\alpha,c;k) = A(m,\lambda,\alpha,c;k)/(k^m-1)$ and $\beta(m,\lambda,\alpha,c;k)$ increases when k increases if and only if $E(m,\lambda,\alpha,c;k)$ is also an increasing function of k.

Let $h(x) = E(m,\lambda,\alpha,c;x)$, $x \in [j+1,\infty) \subset [2,\infty)$; we have

$$\begin{aligned}h'(x) &= 1-\alpha\lambda+(1-\alpha)\frac{x^m-1-mx^m+x^{m-1}}{(x^m-1)^2}\\ &= 1-\alpha\lambda+(1-\alpha)\left[\frac{1-m}{x^m-1}+\frac{m(x^{m-1}-1)}{(x^m-1)^2}\right]\\ &> 1-\alpha\lambda-(1-\alpha) = \alpha(1-\lambda) \geq 0, \quad x \in [j+1,\infty),\end{aligned} \tag{2.8}$$

where we used the fact that

$$\frac{1-m}{x^m-1}+\frac{m(x^{m-1}-1)}{(x^m-1)^2} \geq \frac{1-m}{x^m-1} > -1. \tag{2.9}$$

We obtained $h(j+1) \leq h(k)$, $k \geq j+1$, and this implies

$$\beta = \beta(m,\lambda,\alpha,c;j+1) \leq \beta(m,\lambda,\alpha,c;k), \quad k \geq j+1. \tag{2.10}$$

The result is sharp because

$$I_c(f_\alpha) = f_\beta, \tag{2.11}$$

where

$$\begin{aligned}f_\alpha(z) &= z-\frac{1-\alpha}{(j+1)^n[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}z^{j+1},\\ f_\beta(z) &= z-\frac{1-\beta}{(j+1)^n[(j+1)^m(1-\beta\lambda)-\beta(1-\lambda)]}z^{j+1}\end{aligned} \tag{2.12}$$

are the extremal functions of $T_j(n,m,\lambda,\alpha)$ and $T_j(n,m,\lambda,\beta)$, respectively, and $\beta = \beta(m,\lambda, \alpha,c;j+1)$.

Indeed, we have

$$I_c(f_\alpha)(z) = z-\frac{(1-\alpha)(c+1)}{(j+1)^n(c+j+1)[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}z^{j+1}. \tag{2.13}$$

But if we use the notations $A = A(m,\lambda,\alpha,c;j+1)$ and $B = B(m,\lambda,\alpha,c;j+1)$, we deduce

$$\begin{aligned}
&\frac{1-\beta}{(j+1)^m(1-\beta\lambda)-\beta(1-\lambda)}\\
&\quad=\frac{A-B}{(j+1)^m(A-B\lambda)-B(1-\lambda)}\\
&\quad=\frac{[(j+1)^m-1](1-\alpha)(1-\lambda)(c+1)}{(1-\lambda)\{A(j+1)^m+[(j+1)^m-1]\lambda(j+1)^m(1-\alpha)(c+1)-B\}}\\
&\quad=\frac{[(j+1)^m-1](1-\alpha)(c+1)}{[(j+1)^m-1][(j+1)^m\lambda(1-\alpha)(1+c)+A+(1-\alpha)(c+1)(1-\lambda)]}\\
&\quad=\frac{(1-\alpha)(c+1)}{(c+j+1)[(j+1)^m(1-\alpha\lambda)-\alpha(1-\lambda)]}
\end{aligned} \tag{2.14}$$

and this implies (2.11).

From $\beta = 1-[(j+1)^m-1](1-\alpha)(1-\lambda)(c+1)/A$ and because $A > 0$, we obtain $\beta < 1$. We also have $\beta > \alpha$; indeed

$$\begin{aligned}
\beta-\alpha &= (1-\alpha)\left\{1-\frac{[(j+1)^m-1](c+1)(1-\lambda)}{[(j+1)^m-1][(1-\alpha\lambda)(c+j+1)-\lambda(c+1)(1-\alpha)]+(1-\alpha)j}\right\}\\
&> (1-\alpha)\left\{1-\frac{(c+1)(1-\lambda)}{(1-\alpha\lambda)(c+j+1)-\lambda(c+1)(1-\alpha)}\right\}\\
&= \frac{(1-\alpha)(1-\alpha\lambda)j}{j(1-\alpha\lambda)+(c+1)(1-\lambda)} > 0.
\end{aligned} \tag{2.15}$$

□

3. Integral properties of the class $L_j(n,m,\lambda,\alpha)$

Theorem 3.1. *Let $n,m \in \mathbb{N}$, $j \in \mathbb{N}^*$, $\alpha \in [0,1)$, $\lambda \in [0,1]$, and let $c \in (-1,\infty)$; if $f \in L_j(n,m,\lambda,\alpha)$ and $g = I_c(f)$, then $g \in L_j(n,m,\lambda,\gamma)$, where*

$$\gamma = \gamma(\alpha,c;j+1) = 1-\frac{(1-\alpha)(c+1)}{2-\alpha+c+j} \tag{3.1}$$

and $\alpha < \gamma(\alpha,c;j+1) < 1$. The result is sharp.

Proof. From Theorem 1.7 and from (1.9) we have $g \in L_j(n,m,\lambda,\beta)$ if and only if

$$\sum_{k=j+1}^{\infty}\frac{k^n(k-\gamma)[1+(k^m-1)\lambda](c+1)}{(1-\gamma)(c+k)}a_k \le 1. \tag{3.2}$$

We find the largest γ such that (3.2) holds. We note that the inequalities

$$\frac{(k-\gamma)(c+1)}{(1-\gamma)(c+k)} \le \frac{k-\alpha}{1-\alpha}, \quad k \ge j+1, \tag{3.3}$$

imply (3.2), because $f \in L_j(n,m,\lambda,\alpha)$. But the inequalities (3.3) are equivalent to

$$(k-1)(k+c+1-\alpha)\gamma \le (k-1)(k+\alpha c), \quad k \ge j+1. \tag{3.4}$$

Since $(k+c+1-\alpha) > 0$ and $k-1 \ge j \ge 1$, we deduce

$$\gamma \le \frac{k+\alpha c}{k+c+1-\alpha} \quad \forall k \ge j+1. \tag{3.5}$$

We define $\gamma(\alpha,c;k) := 1-(1-\alpha)(c+1)/(k+c+1-\alpha)$. Obviously, $\gamma(\alpha,c;j+1) \le \gamma(\alpha,c;k)$ for $k \ge j+1$, hence we obtain that $\gamma = \gamma(\alpha,c;j+1)$.

We have $\gamma < 1$ because $(1-\alpha)(c+1)/(k+c+1-\alpha) > 0$ and $\gamma > \alpha$ because

$$\gamma - \alpha = (1-\alpha)\frac{1-\alpha+j}{2-\alpha+c+j} > 0. \tag{3.6}$$

The result is sharp. Indeed, we consider the function

$$\varphi_\alpha(z) = z - \frac{1-\alpha}{(j+1)^n(j+1-\alpha)[1-\lambda+\lambda(j+1)^m]}z^{j+1} \tag{3.7}$$

that belongs to $L_j(n,m,\lambda,\alpha)$. Then

$$I_c(\varphi_\alpha)(z) = z - \frac{(1-\alpha)(c+1)}{(j+1)^n(j+1-\alpha)[1-\lambda+\lambda(j+1)^m](c+j+1)}z^{j+1}, \tag{3.8}$$

and because

$$\frac{(1-\alpha)(c+1)}{(j+1-\alpha)(c+j+1)} = \frac{1-\gamma}{j+1-\gamma}, \tag{3.9}$$

we deduce that $I_c(\varphi_\alpha) = \varphi_\gamma$ belongs to $L_j(n,m,\lambda,\gamma)$. □

References

[1] O. Altintas, *On a subclass of certain starlike functions with negative coefficients*, Math. Japon. **36** (1991), no. 3, 489–495.

[2] O. Altintas and S. Owa, *On a subclass of univalentfunctions with negative coefficients*, Pusan Kyongnam Math. J. **4** (1988), no. 4, 41–46.

[3] M. K. Aouf and N. E. Cho, *On a certain subclass of analytic functions with negative coefficients*, Turkish J. Math. **22** (1998), no. 1, 15–32.

[4] M. K. Aouf, H. E. Darwish, and A. A. Attiya, *Generalization of certain subclasses of analytic functions with negative coefficients*, Studia Univ. Babeş-Bolyai Math. **45** (2000), no. 1, 11–22.

[5] M. K. Aouf, H. M. Hossen, and A. Y. Lashin, *On certain families of analytic functions with negative coefficients*, Indian J. Pure Appl. Math. **31** (2000), no. 8, 999–1015.

[6] M. K. Aouf and H. M. Srivastava, *Some families of starlike functions with negative coefficients*, J. Math. Anal. Appl. **203** (1996), no. 3, 762–790.

[7] S. K. Chatterjea, *On starlike functions*, J. Pure Math. **1** (1981), 23–26.

[8] N. E. Cho and M. K. Aouf, *Some applications of fractional calculus operators to a certain subclass of analytic functions with negative coefficients*, Turkish J. Math. **20** (1996), no. 4, 553–562.

[9] H. M. Hossen, G. S. Sălăgean, and M. K. Aouf, *Notes on certain classes of analytic functions with negative coefficients*, Mathematica (Cluj) **39(62)** (1997), no. 2, 165–179.

[10] M. D. Hur and G. H. Oh, *On certain class of analytic functions with negative coefficients*, Pusan Kyongnam Math. J. **5** (1989), 69–80.

[11] G. S. Sălăgean, *Subclasses of univalent functions*, Complex Analysis—Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), Lecture Notes in Math., vol. 1013, Springer-Verlag, Berlin, 1983, pp. 362–372.

[12] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. **51** (1975), 109–116.

[13] H. M. Srivastava, S. Owa, and S. K. Chatterjea, *A note on certain classes of starlike functions*, Rend. Sem. Mat. Univ. Padova **77** (1987), 115–124.

G. S. Sălăgean: Faculty of Mathematics and Computer Science, Babes-Bolyai University, 1 M. Kogalniceanu Street, 3400 Cluj-Napoca, Romania

E-mail address: salagean@math.ubbcluj.ro