MIXED-TYPE DUALITY FOR MULTIOBJECTIVE FRACTIONAL VARIATIONAL CONTROL PROBLEMS

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The concept of mixed-type duality has been extended to the class of multiobjective fractional variational control problems. A number of duality relations are proved to relate the efficient solutions of the primal and its mixed-type dual problems. The results are obtained for ρ-convex (generalized ρ-convex) functions. The results generalize a number of duality results previously obtained for finite-dimensional nonlinear programming problems under various convexity assumptions.

1. Introduction

Duality for multiobjective fractional variational problems has been of great interest in recent years [6, 7, 19]. Under different assumptions of convexity (convexity, generalized convexity, generalized ρ-convexity), Weir and Mond [16], Weir [15], and Egudo [4] have used efficiency to establish some duality results, where Wolfe [17] and Mond-Weir [11] duals are considered. Recently, Xu [18] introduced a mixed-type duality model, which contains Wolfe and Mond-Weir duality models as special cases and established various duality results by relating "efficient" solutions of his mixed-type dual pair of problems.

Preda [13] introduced the concept of generalized (F,ρ)-convexity, an extension of F-convexity defined by Hanson and Mond [5], and generalized ρ-convexity defined by Vial [14], and he used the concept to obtain some duality results. In [8], Mishra and Mukherjee discussed duality for multiobjective variational problems containing generalized (F, ρ)-convex functions. Some duality results for a class of differentiable multiobjective variational problems were studied in [3]. Mond and Hanson [10] have obtained duality theorems for control problems. Mond and Hanson [9] considered a dual formulation for a class of variational problems. Nahak and Nanda [12] used the concept of efficiency to formulate Wolfe and Mond-Weir type duals for multiobjective variational control problems and established weak and strong duality theorems under generalized (F,ρ)-convexity assumptions.

In this paper, we introduce a continuous analog of the (static) mixed-type dual of Xu [18] in a class of multiobjective fractional variational control problems and establish a large number of duality results by relating efficient solutions between this mixed-type

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dual pair. The results are obtained for differentiable ρ-convex (generalized ρ-convex) functions in their continuous version.

2. Definitions and preliminaries

We use the following notations for vector inequalities. For $x, y \in \mathbb{R}^n$,

$$\begin{aligned} x \leq y &\Longleftrightarrow x_i \leq y_i \quad \text{for } i = 1,2,\dots,n, \\ x < y &\Longleftrightarrow x_i < y_i \quad \text{for } i = 1,2,\dots,n. \end{aligned} \tag{2.1}$$

Let $I = [a,b]$ be a real interval and let $K = \{1,2,\dots,k\}$ and $M = \{1,2,\dots,m\}$. Let $\phi : I \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be continuously differentiable function. In order to consider $\phi(t,x(t),\dot{x}(t),u(t),\dot{u}(t))$, where $x(t) : I \to \mathbb{R}^n$, $u(t) : I \to \mathbb{R}^m$ are differentiable with derivatives $\dot{x}(t)$ and $\dot{u}(t)$, respectively. For notational simplicity, we write $x(t)$, $\dot{x}(t)$, $u(t)$, $\dot{u}(t)$ as x, $\dot{x}$, u, $\dot{u}$, respectively, as and when necessary. We denote the partial derivatives of ϕ by ϕ_x and $\phi_{\dot{x}}$, where

$$\begin{aligned} \phi_x &= \left[\frac{\partial \phi}{\partial x_1}, \frac{\partial \phi}{\partial x_1}, \dots, \frac{\partial \phi}{\partial x_n} \right], \\ \phi_{\dot{x}} &= \left[\frac{\partial \phi}{\partial \dot{x}_1}, \frac{\partial \phi}{\partial \dot{x}_2}, \dots, \frac{\partial \phi}{\partial \dot{x}_n} \right]. \end{aligned} \tag{2.2}$$

The partial derivatives of other functions used will be written similarly. Let $\mathrm{PS}(I,\mathbb{R}^n)$ denote the space of piecewise smooth functions x with norm $\|x\| = \|x\|_\infty + \|Dx\|_\infty$, where the differentiation operator D is given by

$$y = Dx \Longleftrightarrow x(t) = \alpha + \int_a^b y(s)ds \tag{2.3}$$

in which α is a given boundary value. Therefore, $D = d/dt$ except at discontinuities.

Consider the following multiobjective variational control programming problem.

$$\begin{gathered} \text{Minimize} \int_a^b f(t,x,\dot{x},u,\dot{u})dt = \int_a^b f^1(t,x,\dot{x},u,\dot{u})dt, \dots, \int_a^b f^k(t,x,\dot{x},u,\dot{u})dt, \\ \text{subject to } x(a) = \alpha, \quad x(b) = \beta, \\ \int_a^b h(t,x,\dot{x},u,\dot{u})dt \leq 0, \quad t \in [a,b],\ x \in \mathrm{PS}\left(I,\mathbb{R}^n\right),\ y \in \mathrm{PS}\left(I,\mathbb{R}^m\right), \\ f : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \times\mathbb{R}^k, \\ h : [a,b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \times\mathbb{R}^m \end{gathered} \tag{2.4}$$

are assumed to be continuously differentiable vector functions.

Let X denote the set of feasible solutions of (2.4).

Definition 2.1. A point (x^0,u^0) in X is said to be an efficient solution of (2.4) if for all (x,u) in X,

$$\begin{aligned}&\int_a^b f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\,dt \geq \int_a^b f^i(t,x,\dot{x},u,\dot{u})\,dt\\ &\quad\Longrightarrow \int_a^b f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\,dt = \int_a^b f^i(t,x,\dot{x},u,\dot{u})\,dt\\ &\qquad \forall i\in\{1,2,\dots,k\}.\end{aligned}\tag{2.5}$$

Definition 2.2. A point (x^0,u^0) in X is said to be a weak minimum for problem (2.4) if there exists no other (x,u) in X for which

$$\int_a^b f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\,dt > \int_a^b f^i(t,x,\dot{x},u,\dot{u})\,dt.\tag{2.6}$$

From the above two definitions, it follows that if (x,u) in X is an efficient solution for (2.4), then it is also a weak minimum for (2.4).

Definition 2.3. The functional $F : I\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}$ is sublinear if for any $x,x^0\in\mathbb{R}^n$, $\dot{x},\dot{x}^0\in\mathbb{R}^n$, $u,u^0\in\mathbb{R}^m$, $\dot{u},\dot{u}^0\in\mathbb{R}^m$,

$$\begin{aligned}&F(t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0;\alpha_1+\alpha_2)\\ &\quad\leq F(t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0;\alpha_1)+F(t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0;\alpha_2),\\ &F(t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0;\alpha a) = \alpha F(t,x,\dot{x},u,\dot{u},x^0,\dot{x}^0,u^0,\dot{u}^0;a)\end{aligned}\tag{2.7}$$

for any $\alpha_1,\alpha_2\in\mathbb{R}^n$, $\alpha\in\mathbb{R}$, $\alpha\geq 0$ and $a\in\mathbb{R}^n$.

Definition 2.4. Let $F[x,u]=\int_a^b f(t,x,\dot{x},u,\dot{u})\,dt$ be Fréchet differentiable. Let ρ be a real number. Then the functional F at a point (x^0,u^0) in X is said to be

(a) ρ-convex if there exists a real number ρ such that

$$\begin{aligned}&\int_a^b f^i(t,x,\dot{x},u,\dot{u})\,dt - \int_a^b f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\,dt\\ &\quad\geq \int_a^b \left\{(x-x^0)f_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)+(D(x-x^0))f_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\right\}dt+\rho\|x-x^0\|^2;\end{aligned}\tag{2.8}$$

the function f is said to be strictly ρ-convex if strict inequality holds;

(b) ρ-pseudoconvex if there exists a real number ρ such that

$$\begin{aligned}&\int_a^b \left\{(x-x^0)f_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)+(D(x-x^0))f_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\right\}dt\geq -\rho\|x-x^0\|^2\\ &\quad\Longrightarrow \int_a^b f^i(t,x,\dot{x},u,\dot{u})\,dt \geq \int_a^b f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\,dt;\end{aligned}\tag{2.9}$$

the function f is said to be strictly ρ-pseudoconvex if strict inequality holds in the right-hand inequality of the above implication;

(c) ρ-quasiconvex if there exists a real number ρ such that

$$\int_a^b f^i(t,x,\dot{x},u,\dot{u})dt \leq \int_a^b f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt$$
$$\Longrightarrow \int_a^b \{(x-x^0)f_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)+(D(x-x^0))f_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}dt \leq -\rho\|x-x^0\|^2. \tag{2.10}$$

Now we use the term "generalized ρ-convexity" to indicate ρ-pseudoconvexity, ρ-quasiconvexity, and so forth. Let $l = (l^1, l^2, \dots, l^n)$ be the n-dimensional vector function and let each of its components be ρ-convex (generalized ρ-convex) at the same point (x^0, u^0). Also let $q = (q_1, q_2, \dots, q_n)$ be a vector constant such that $q_i \geq 0$ for all $i = 1, 2, \dots, n$. Then

(a) $\Sigma_N l^i(t,\cdot,\cdot,\cdot,\cdot)$ is $\Sigma_N \rho_i$-convex at (x^0,u^0);
(b) each $q_i f^i(t,\cdot,\cdot,\cdot,\cdot)$ is $q_i\rho_i$-convex at (x^0,u^0); and hence
(c) $l(t,\cdot,\cdot,\cdot,\cdot)$ is $\Sigma_N\rho_i$-convex at (x^0,u^0).

These properties will be used frequently throughout the paper. We state the continuous version of [16, Theorem 3.2] in the form of the following proposition, which will be needed in the proof of the strong duality theorem.

Proposition 2.5. *Let $(\bar{x},\bar{u})$ be a weak minimum for (2.4) at which the Kuhn-Tucker constraint qualification is satisfied. Then there exists λ in $\mathbb{R}^k$ and a piecewise smooth $\beta(\cdot): I \to \mathbb{R}^k$ such that*

$$\begin{gathered}[\lambda^T f_{\bar{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})+\beta^T h_{\bar{x}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})] = D[\lambda^T f_{\dot{\bar{x}}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{x}})+\beta^T h_{\dot{\bar{x}}}(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})],\\ \int_a^b \beta^T h(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})dt = 0, \quad \beta(t)\geq 0,\ \lambda^T e = 1,\ \lambda \geq 0,\end{gathered} \tag{2.11}$$

where e is the vector of $\mathbb{R}^k$, the components of which are all ones.

We divide the index set M of the constraint function of the problem (2.4) into two distinct subsets, namely J and Q such that $J \cup Q = M$, and let

$$\begin{gathered}\beta_j^T h^j(t,x,\dot{x},u,\dot{u}) = \sum_j \beta_j h^j(t,x,\dot{x},u,\dot{u}),\\ \beta_Q^T h^Q(t,x,\dot{x},u,\dot{u}) = \sum_Q \beta_Q h^Q(t,x,\dot{x},u,\dot{u}).\end{gathered} \tag{2.12}$$

We now consider the following multiobjective fractional variational control problem.

(FP)

$$\text{Minimize } \frac{\int_a^b f(t,x,\dot{x},u,\dot{u})dt}{\int_a^b g(t,x,\dot{x},u,\dot{u})dt} = \frac{\int_a^b f^1(t,x,\dot{x},u,\dot{u})dt}{\int_a^b g^1(t,x,\dot{x},u,\dot{u})dt},\ldots,\frac{\int_a^b f^k(t,x,\dot{x},u,\dot{u})dt}{\int_a^b g^k(t,x,\dot{x},u,\dot{u})dt}, \tag{2.13}$$

$$\text{subject to } x(a)=\alpha,\quad x(b)=\beta,\quad \int_a^b h(t,x,\dot{x},u,\dot{u})dt \leq 0,\quad t\in[a,b], \tag{2.14}$$

$$x\in \mathrm{PS}(I,\mathbb{R}^n),\qquad y\in \mathrm{PS}(I,\mathbb{R}^m), \tag{2.15}$$

$$f:[a,b]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\longrightarrow\mathbb{R}^k,\qquad g:[a,b]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\longrightarrow\mathbb{R}^k, \tag{2.16}$$

$$h:[a,b]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\longrightarrow\mathbb{R}^m \tag{2.17}$$

are assumed to be continuously differentiable vector functions. Also we assume that

$$\int_a^b f(t,\cdot,\cdot,\cdot,\cdot)dt \geq 0,\qquad \int_a^b g(t,\cdot,\cdot,\cdot,\cdot)dt > 0,\quad i=1,2,\ldots,k. \tag{2.18}$$

Following Bector et al. [1], the problem $(\mathrm{FP})_v$ stated below is associated with the given problem (FP) for $v\in\mathbb{R}^k_+$, where $\mathbb{R}^k_+$ is the positive orthant of $\mathbb{R}^k$.

$(\mathrm{FP})_v$

$$\begin{gathered}\text{Minimize } \int_a^b [f(t,x,\dot{x},u,\dot{u}) - v^T g(t,x,\dot{x},u,\dot{u})]dt,\\ \text{subject to (2.14).}\end{gathered} \tag{2.19}$$

The following lemma connecting (FP) and $(\mathrm{FP})_v$ has been proved in [2].

Lemma 2.6. *Let (x^0,u^0) be an efficient solution to (FP). Then there exists $v^0\in\mathbb{R}^k_+$ such that (x^0,u^0) is efficient to the problem $(FP)_v$, where $\mathbb{R}^k_+$ is the positive orthant of $\mathbb{R}^k$.*

3. Duality

Now we introduce the continuous analog of the static mixed-type dual [8] for the primal problem (2.4).

(FD)

$$\begin{aligned}&\text{Maximize } \int_a^b \{(f(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^T g(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta_j^T(t)h^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)e\}dt,\\ &\text{subject to } x(a)=\alpha,\quad x(b)=\beta,\end{aligned} \tag{3.1}$$

$$\begin{aligned}&[\lambda^T(f_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^T g_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta^T h_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]\\ &\qquad = D[\lambda^T(f_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^T g_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta^T h_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)],\end{aligned} \tag{3.2}$$

$$\int_a^b (f(t,x,\dot{x},u,\dot{u}) - v^T g(t,x,\dot{x},u,\dot{u}))dt \geq 0, \tag{3.3}$$

$$\int_a^b \beta_Q^T(t) h^Q(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\,dt \geq 0, \tag{3.4}$$

$$\beta(t) \geq 0, \qquad \lambda^T e = 1, \quad \lambda \geq 0, \tag{3.5}$$

where e is the vector of $\mathbb{R}^k$, the components of which are all ones.

Here we present a number of duality results between $(\text{FP})_v$ and (FD) by imposing various ρ-convexity (generalized ρ-convexity) conditions upon the objective and constraint functions. We begin with a situation in which all of the functions are ρ-convex. Subsequently, we formulate more general duality criteria in which the generalized ρ-convexity requirements are placed on certain combinations of the objective and constraint functions.

Let Y denote the set of all feasible solutions of (FD).

Theorem 3.1. *Assume that for all feasible (x,u) for $(\text{FP})_v$ and for all feasible $(x^0,u^0,\lambda,v,\beta)$ for (FD),*

(a) *for each $i \in K$, $f^i(t,\cdot,\cdot,\cdot,\cdot)$, $-g^i(t,\cdot,\cdot,\cdot,\cdot)$ is ρ_i-convex, and for each $j \in M$, $h^j(t,\cdot,\cdot,\cdot,\cdot)$ is γ_j-convex.*

Further if either

(b) *for each $i \in K$, $\lambda_i > 0$ with $\Sigma_K \lambda_i \rho_i + \Sigma_M \beta_j \gamma_j \geq 0$, or*

(c) $\Sigma_K \lambda_i \rho_i + \Sigma_M \beta_j \gamma_j > 0$,

then

$$\begin{aligned}
&\int_a^b \left(f^i(t,x,\dot{x},u,\dot{u}) - v^i g^i(t,x,\dot{x},u,\dot{u})\right)dt \\
&\qquad \leq \int_a^b \left\{\left(f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^i g^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\right)\right. \\
&\qquad\qquad \left. + \beta_j^T h^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\right\}dt, \quad \forall i \in \{1,2,\ldots,k\}, \\
&\int_a^b \left(f^i(t,x,\dot{x},u,\dot{u}) - v^i g^i(t,x,\dot{x},u,\dot{u})\right)dt \\
&\qquad < \int_a^b \left\{\left(f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^i g^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\right)\right. \\
&\qquad\qquad \left. + \beta_j^T h^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\right\}dt, \quad \textit{for some } j \in \{1,2,\ldots,k\}
\end{aligned} \tag{3.6}$$

cannot hold.

Proof. If $x = x^0$, then a weak duality theorem holds trivially, so we assume that $x \neq x^0$. From (3.2), we have

$$\begin{aligned}
&\int_a^b (x-x^0)\left[\lambda^T\left(f_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^T g_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\right) + \beta^T h_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\right]dt \\
&\quad = \int_a^b (x-x^0) D\left[\lambda^T\left(f_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^T g_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\right)\right. \\
&\qquad\qquad \left. + \beta^T h_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\right]dt.
\end{aligned} \tag{3.7}$$

Contrary to the result, we assume that (3.6) holds. Then in view of feasibility of (x,u) for $(\mathrm{FP})_v$, we have

$$\begin{aligned}
&\int_a^b \{(f^i(t,x,\dot{x},u,\dot{u}) - v^i g^i(t,x,\dot{x},u,\dot{u})) + \beta_j^T h^j(t,x,\dot{x},u,\dot{u})\}dt \\
&\qquad \le \int_a^b \{(f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^i g^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) \\
&\qquad\qquad + \beta_j^T h^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}dt, \quad \forall i \in K, \\
&\int_a^b \{(f^i(t,x,\dot{x},u,\dot{u}) - v^i g^i(t,x,\dot{x},u,\dot{u})) + \beta_j^T h^j(t,x,\dot{x},u,\dot{u})\}dt \\
&\qquad < \int_a^b \{(f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^i g^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) \\
&\qquad\qquad + \beta_j^T h^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}dt, \quad \text{for some } i \in K.
\end{aligned} \tag{3.8}$$

From the strict positivity of each component λ_i of λ and the fact that $\lambda^T e = 1$, it follows that

$$\begin{aligned}
&\int_a^b \{\lambda^T (f^i(t,x,\dot{x},u,\dot{u}) - v^i g^i(t,x,\dot{x},u,\dot{u})) + \beta_j^T h^j(t,x,\dot{x},u,\dot{u})\}dt \\
&\qquad < \int_a^b \{\lambda^T (f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^i g^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta_j^T h^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}dt.
\end{aligned} \tag{3.9}$$

Using the definitions of ρ_i-convexity of $(f^i(t,\cdot,\cdot,\cdot,\cdot) - v^i g^i(t,\cdot,\cdot,\cdot,\cdot))$, $i \in K$, and γ_j-convexity of $h^j(t,\cdot,\cdot,\cdot,\cdot)$, $j \in M$, we have

$$\begin{aligned}
&\int_a^b \{(f^i(t,x,\dot{x},u,\dot{u}) - v^i g^i(t,x,\dot{x},u,\dot{u})) - (f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^i g^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0))\}dt \\
&\quad \ge \int_a^b \{(x-x^0)(f^i_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^i g^i_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) \\
&\qquad\quad + (D(x-x^0))(f^i_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^i g^i_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0))\}dt \\
&\quad + \rho_i \|x-x^0\|^2, \quad \forall i \in K,
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&\int_a^b \{h^j(t,x,\dot{x},u,\dot{u}) - h^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}dt \\
&\quad \ge \int_a^b \{(x-x^0)h^j_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) + (D(x-x^0))h^j_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}dt \\
&\quad + \gamma \|x-x^0\|^2, \quad \forall j \in M.
\end{aligned} \tag{3.11}$$

Now multiplying (3.10) by λ_i, (3.11) by β_j and adding the resulting inequalities, we get

$$\begin{aligned}
&\int_a^b \{\lambda^T(f(t,x,\dot{x},u,\dot{u}) - \nu^T g(t,x,\dot{x},u,\dot{u})) + \beta^T h^j(t,x,\dot{x},u,\dot{u}) \\
&\qquad - \lambda^T(f(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^T g(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta^T h^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}dt \\
&\quad \geq \int_a^b \{(x-x^0)[\lambda^T(f_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^T g_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta^T h_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)] \\
&\qquad\quad + (D(x-x^0))[\lambda^T(f_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^T g_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) \\
&\qquad\qquad\qquad + \beta^T h_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]\}dt + \left(\sum_K \alpha_i\rho_i + \sum_M \beta_j\gamma_j\right)\|x-x^0\|^2.
\end{aligned} \tag{3.12}$$

By integration by parts, the right-hand side reduces to the following via (b):

$$\begin{aligned}
&\int_a^b \{(x-x^0)[\lambda^T(f_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^T g_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta^T h_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]\}dt \\
&\quad + \{[\lambda^T(f_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^i g_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta^T h_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)](x-x^0)\}_{t=\alpha}^{t=\beta} \\
&\quad - \int_a^b (x-x^0)D[\lambda^T(f_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^T g_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) \\
&\qquad\qquad + \beta^T h_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]dt.
\end{aligned} \tag{3.13}$$

On using the boundary conditions (3.7), it yields

$$\begin{aligned}
&\int_a^b \{\lambda^T(f(t,x,\dot{x},u,\dot{u}) - \nu^T g(t,x,\dot{x},u,\dot{u})) + \beta^T h(t,x,\dot{x},u,\dot{u}) \\
&\qquad - \lambda^T(f(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^T g(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) - \beta^T h(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}dt \geq 0.
\end{aligned} \tag{3.14}$$

Since $M = J \cup Q$, so

$$\beta^T h(t,\cdot,\cdot,\cdot,\cdot) = \beta_j^T h^j(t,\cdot,\cdot,\cdot,\cdot) + \beta_Q^T h^j(t,\cdot,\cdot,\cdot,\cdot), \tag{3.15}$$

and hence the above inequality implies, along with (3.9), that

$$\int_a^b \{\beta_Q^T(t)h^Q(t,x,\dot{x},u,\dot{u}) - \beta_Q^T(t)h^Q(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}dt > 0. \tag{3.16}$$

Now since $(x^0,u^0,\lambda,\nu,\beta) \in Y$, from (3.4),

$$\int_a^b \beta_Q^T(t)h^Q(t,x,\dot{x},u,\dot{u})dt > 0, \tag{3.17}$$

which is a contradiction to the fact that (x,u) is feasible for $(\mathrm{FP})_\nu$, and therefore (3.6) cannot hold.

(c) In this case, the multipliers λ_i of the objective function $(f^i(t,\cdot,\cdot,\cdot,\cdot) - v^i g^i(t,\cdot,\cdot,\cdot,\cdot))$ need not be strictly positive, and it gives $\leq$ in place of $<$ of (3.9). If we assume the condition in (c), we get $>$ in place of $\geq$ of (3.14). So, we get (3.16) and we conclude the theorem as in the case of (b). And it completes the proof. $\square$

The above theorem has a number of important special cases, which can be identified by the properties of ρ-convex functions. Here we state some of them as corollaries.

Corollary 3.2. *Assume that for all feasible (x,u) for (FP)$_v$ and for all feasible $(x^0,u^0,\lambda,v,\beta)$ for (FD),*

(a) *for each $i \in K$, $f^i(t,\cdot,\cdot,\cdot,\cdot)$, $-g^i(t,\cdot,\cdot,\cdot,\cdot)$ is ρ_i-convex, and for each $j \in M$, $\beta_j^T h^j(t,\cdot,\cdot,\cdot,\cdot)$ is γ_j-convex.*

Further if either

(b) *for each $i \in K$, $\lambda_i > 0$ with $\Sigma_K \lambda_i \rho_i + \Sigma_M \gamma_j \geq 0$, or*
(c) $\Sigma_K \lambda_i \rho_i + \Sigma_M \gamma_j > 0$,

then (3.6) cannot hold.

Proof. Since $h^j(t,\cdot,\cdot,\cdot,\cdot)$ is γ_j-convex, whenever $\beta_j^T h^j(t,\cdot,\cdot,\cdot,\cdot)$ is $\beta_j\gamma_j$-convex and $\beta_j \geq 0$, the proof is similar to Theorem 3.1. $\square$

Corollary 3.3. *For all feasible (x,u) for (FP)$_v$ and for all feasible $(x^0,u^0,\lambda,v,\beta)$ for (FD), assume that Corollary 3.2 holds, except that instead of $\beta_j^T h^j(t,\cdot,\cdot,\cdot,\cdot)$ being γ_j-convex, the function $(t,x^0,\dot{x}^0,u^0,\dot{u}^0) \to \Sigma_M \beta_j^T h^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)$ is γ-convex, and instead of Corollary 3.2(b) and (c), the following conditions hold, respectively,*

(b) *for each $i \in K$, $\lambda_i > 0$ with $\Sigma_K \lambda_i \rho_i + \gamma \geq 0$,*
(c) $\Sigma_K \lambda_i \rho_i + \gamma > 0$.

Then (3.6) cannot hold.

We note that, in Theorem 3.1, each constraint function $h^j(t,\cdot,\cdot,\cdot,\cdot)$ is assumed to be γ_j-convex whereas, in Corollary 3.3, all constraint functions are aggregated into one γ-convex function. So, it is possible to consider a situation intermediate between these two extreme cases (keeping in view the partition of the constraint function in the objective function of the dual problem (FD)) in which some of the constraint functions can be combined into γ-convex function while the rest are individually γ-convex. Situations of this type are presented in the next two corollaries.

Corollary 3.4. *Assume that for all feasible (x,u) for (FP)$_v$ and for all feasible $(x^0,u^0,\lambda,v,\beta)$ for (FD),*

(a) *for each $i \in K$, $f^i(t,\cdot,\cdot,\cdot,\cdot)$, $-g^i(t,\cdot,\cdot,\cdot,\cdot)$ is ρ_i-convex and $\beta_j^T h^j(t,\cdot,\cdot,\cdot,\cdot)$ is γ_j-convex, whereas $\beta_Q^T h^j(t,\cdot,\cdot,\cdot,\cdot)$ is γ_Q-convex.*

Further if either

(b) *for each $i \in K$, $\lambda_i > 0$ with $\Sigma_K \lambda_i \rho_i + \gamma_j + \Sigma_Q \gamma_j \geq 0$, or*
(c) $\Sigma_K \lambda_i \rho_i + \gamma_j + \Sigma_Q \gamma_j > 0$,

then (3.6) cannot hold.

Corollary 3.5. *Assume that for all feasible* (x,u) *for (FP)*$_v$ *and for all feasible* $(x^0,u^0,\lambda,v,\beta)$ *for (FD),*

(a) *for each* $i \in K$, $f^i(t,\cdot,\cdot,\cdot,\cdot)$, $-g^i(t,\cdot,\cdot,\cdot,\cdot)$ *is* ρ_i*-convex and* $\beta_j^T h^j(t,\cdot,\cdot,\cdot,\cdot)$ *is* γ_j*-convex whereas* $\beta_Q^T h^j(t,\cdot,\cdot,\cdot,\cdot)$ *is* γ_Q*-convex.*

Further if either

(b) *for each* $i \in K$, $\lambda_i > 0$ *with* $\Sigma_K \lambda_i \rho_i + \gamma_j + \gamma_Q \geq 0$, *or*
(c) $\Sigma_K \lambda_i \rho_i + \gamma_j + \gamma_Q > 0$,

then (3.6) cannot hold.

Corollary 3.6. *Assume that for all feasible* (x,u) *for (FP)*$_v$ *and for all feasible* $(x^0,u^0,\lambda,v,\beta)$ *for (FD),*

(a) $\lambda^T[f(t,\cdot,\cdot,\cdot,\cdot) - v^T g(t,\cdot,\cdot,\cdot,\cdot)] + \beta^T h(t,\cdot,\cdot,\cdot,\cdot)$ *is* γ*-convex.*

Further if either

(b) *for each* $i \in K$, $\lambda_i > 0$ *with* $\rho \geq 0$, *or*
(c) $\rho > 0$,

then (3.6) cannot hold.

In the rest of this section, we use the generalized ρ-convexity. So, we restrict ourselves in most of the cases to situations in which only scalarizations of the objective and constraint functions are considered. The related corollaries can also be seen as in the case of Theorem 3.1. Therefore, we do not state these corollaries explicitly.

Theorem 3.7. *Assume that for all feasible* (x,u) *for (FP)*$_v$ *and for all feasible* $(x^0,u^0,\lambda,v,\beta)$ *for (FD),*

(a) $\beta_Q^T h^Q(t,\cdot,\cdot,\cdot,\cdot)$ *is* ρ*-quasiconvex,*
(b) *for* $i \in K$, $\lambda_i > 0$ *and* $(f^i(t,\cdot,\cdot,\cdot,\cdot) - v^i g^i(t,\cdot,\cdot,\cdot,\cdot)) + \beta_j^T h^j(t,\cdot,\cdot,\cdot,\cdot)$ *is both* γ_i*-quasiconvex and* γ_i*-pseudoconvex with* $\Sigma_K \lambda_i \gamma_i + \rho \geq 0$.

Then (3.6) cannot hold.

Proof. If $x = x^0$, then a weak duality theorem holds trivially, so we assume that $x \neq x^0$. Since $(x,u) \in X$ and $(x^0,u^0,\lambda,v,\beta) \in Y$, we have

$$\int_a^b \beta_Q^T h^Q(t,x,\dot{x},u,\dot{u})dt \leq 0 \leq \int_a^b \beta_Q^T(t) h^Q(t,x^0,\dot{x}^0,u^0,\dot{u}^0)dt. \tag{3.18}$$

ρ-quasiconvexity in (a), in view of the above theorem, implies that

$$\begin{aligned}\int_a^b &\{(x-x^0)\beta_Q^T h_{x^0}^Q(t,x^0,\dot{x}^0,u^0,\dot{u}^0) + [D(x-x^0)\beta_Q^T h_{\dot{x}^0}^Q(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]\}dt \\ &\leq -\rho\|x-x^0\|^2.\end{aligned} \tag{3.19}$$

The substitution of the duality constraint (3.2) in the first term of the above implication gives us, along with (3.15),

$$\begin{aligned}\int_a^b [&(x-x^0)\{D[\lambda^T(f_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)-\nu^T g_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0))+\beta_j^T h_{\dot{x}^0}^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\\ &+\beta_Q^T h_{\dot{x}^0}^Q(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]-\lambda^T(f_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)-\nu^T g_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0))\\ &+\beta_j^T h_{\dot{x}^0}^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}]dt+\int_a^b\{D(x-x^0)\beta_Q^T h_{\dot{x}^0}^Q(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}dt\\ &\le -\rho\|x-x^0\|^2.\end{aligned} \tag{3.20}$$

By integration by parts and making use of boundary conditions, we get

$$\begin{aligned}\int_a^b &(x-x^0)\{\lambda^T(f_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)-\nu^T g_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0))+\beta_j^T h_{x^0}^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\\ &+(D(x-x^0))[\lambda^T(f_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)-\nu^T g_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0))\\ &+\beta_j^T h_{\dot{x}^0}^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]\}dt\\ &\ge \rho\|x-x^0\|^2.\end{aligned} \tag{3.21}$$

So making use of the condition $\Sigma_K \lambda_i \gamma_i + \rho \ge 0$ and as $\lambda^T e = 1$, we get

$$\begin{aligned}\sum_K \lambda_i \int_a^b &\{(x-x^0)[(f_{x^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)-\nu^i g_{x^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0))+\beta_j^T h_{x^0}^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]\\ &+[D(x-x^0)]\lambda_i[(f_{\dot{x}^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)-\nu^i g_{\dot{x}^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0))\\ &+\beta_j^T h_{\dot{x}^0}^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]\}dt \ge -\left(\sum_K \lambda_i\gamma_i\right)\|x-x^0\|^2.\end{aligned} \tag{3.22}$$

Since $\lambda_i > 0$, $i \in K$, it follows from the above that

$$\begin{aligned}\int_a^b &\{(x-x^0)[(f_{x^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)-\nu^i g_{x^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0))+\beta_j^T h_{x^0}^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]\\ &+[D(x-x^0)][(f_{\dot{x}^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)-\nu^i g_{\dot{x}^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0))+\beta_j^T h_{\dot{x}^0}^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]\}dt\\ &\ge -\gamma_i\|x-x^0\|^2, \quad \forall i \in K,\end{aligned} \tag{3.23}$$

$$\begin{aligned}\int_a^b &\{(x-x^0)[(f_{x^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)-\nu^i g_{x^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0))+\beta_j^T h_{x^0}^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]\\ &+[D(x-x^0)][(f_{\dot{x}^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)-\nu^i g_{\dot{x}^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0))+\beta_j^T h_{\dot{x}^0}^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]\}dt\\ &> -\gamma_i\|x-x^0\|^2, \quad \text{for some } i \in K.\end{aligned} \tag{3.24}$$

Suppose (3.23) holds, then the γ_i-pseudoconvexity assumption in (b) gives along with the feasibility of (x,u) for $(\text{FP})_v$, for all $i \in K$,

$$\begin{aligned}&\int_a^b (f^i(t,x,\dot{x},u,\dot{u}) - v^i g^i(t,x,\dot{x},u,\dot{u}))\,dt\\&\qquad\geq \int_a^b \{(f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^i g^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta_j^T h^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}\,dt.\end{aligned}\tag{3.25}$$

Now, suppose (3.24) holds, then the equivalent form of the γ_i-quasiconvexity assumption in (b) gives, along with the feasibility of (x,u) for $(\text{FP})_v$, for some $i \in K$,

$$\begin{aligned}&\int_a^b (f^i(t,x,\dot{x},u,\dot{u}) - v^i g^i(t,x,\dot{x},u,\dot{u}))\,dt\\&\qquad> \int_a^b \{(f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^i g^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta_j^T h^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}\,dt.\end{aligned}\tag{3.26}$$

So (3.25) and (3.26) show that (3.6) cannot hold, and this completes the proof. □

The following Theorem 3.8 is stated without proof.

Theorem 3.8. *Assume that for all feasible (x,u) for $(FP)_v$ and for all feasible $(x^0,u^0,\lambda,v,\beta)$ for (FD),*

(a) *$\beta_Q^T h^Q(t,\cdot,\cdot,\cdot,\cdot)$ is ρ-quasiconvex,*
(b) *for each $i \in K$, $\lambda_i > 0$ and $(f^i(t,\cdot,\cdot,\cdot,\cdot) - v^i g^i(t,\cdot,\cdot,\cdot,\cdot)) + \beta_j^T h^j(t,\cdot,\cdot,\cdot,\cdot)$ is γ_i-quasiconvex and there exists some $q \in K$ such that it is strictly γ_q-pseudoconvex (with the corresponding component λ_q of λ positive) with $\Sigma_K \lambda_i \gamma_i + \rho \geq 0$.*

Then (3.6) cannot hold.

Theorem 3.9. *Assume that for all feasible (x,u) for $(FP)_v$ and for all feasible $(x^0,u^0,\lambda,v,\beta)$ for (FD),*

(a) *$\beta_Q^T h^Q(t,\cdot,\cdot,\cdot,\cdot)$ is ρ-quasiconvex,*
(b) *for each $i \in K$, $\lambda_i > 0$ and $\lambda^T(f(t,\cdot,\cdot,\cdot,\cdot) - v^T g(t,\cdot,\cdot,\cdot,\cdot)) + \beta_j^T h^j(t,\cdot,\cdot,\cdot,\cdot)$ is γ-pseudoconvex with $\rho + \gamma \geq 0$.*

Then (3.6) cannot hold.

Proof. As in the case of Theorem 3.7, we assume that $x \neq x^0$ and get (3.21). Now using the condition $\rho + \gamma \geq 0$ and by our ρ-pseudoconvexity assumption in (b), we get

$$\begin{aligned}&\int_a^b \{\lambda^T(f(t,x,\dot{x},u,\dot{u}) - v^T g(t,x,\dot{x},u,\dot{u}))\,dt + \beta_j^T h^j(t,x,\dot{x},u,\dot{u})\}\,dt\\&\qquad\geq \int_a^b \{\lambda^T(f(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^T g(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta_j^T h^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}\,dt.\end{aligned}\tag{3.27}$$

The feasibility of (x,u) for $(\mathrm{FP})_\nu$ and the fact that $\lambda^T e = 1$ imply

$$\lambda^T \int_a^b (f(t,x,\dot{x},u,\dot{u}) - \nu^T g(t,x,\dot{x},u,\dot{u}))dt$$
$$\geq \lambda^T \int_a^b \{(f(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^T g(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta_j^T h^j(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\}dt. \tag{3.28}$$

This concludes the theorem, since $\lambda_i \geq 0$ for each $i \in K$. □

Theorem 3.10. *Assume that for all feasible (x,u) for $(FP)_\nu$ and for all feasible $(x^0,u^0,\lambda,\nu,\beta)$ for (FD),*

(a) *for each $i \in K$, $\lambda_i > 0$ and $(f^i(t,\cdot,\cdot,\cdot,\cdot) - \nu^i g^i(t,\cdot,\cdot,\cdot,\cdot)) + \beta^T h(t,\cdot,\cdot,\cdot,\cdot)$ is both ρ-pseudoconvex and ρ-quasiconvex with $\Sigma_K \lambda_i \rho_i \geq 0$.*

Then (3.6) cannot hold.

Proof. We assume that $x \neq x^0$. From the duality constraint (3.2), we get (3.7). Now by integration by parts,

$$\int_a^b (x - x^0)[\lambda^T(f_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^T g_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta^T h_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]dt$$
$$= \{(x-x^0)[\lambda^T(f_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^T g_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta^T h_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]\}_{t=\alpha}^{t=\beta}$$
$$- \int_a^b [D(x - x^0)][\lambda^T(f_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^T g_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0))$$
$$+ \beta^T h_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]dt. \tag{3.29}$$

Since $\lambda_i > 0$, $i \in K$, and by the fact that $\lambda^T e = 1$, we get

$$\sum_k \lambda_i \int_a^b \{(x - x^0)[(f_{x^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^i g_{x^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta^T h_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]$$
$$+ [D(x - x^0)][(f_{\dot{x}^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^i g_{\dot{x}^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0))$$
$$+ \beta^T h_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]\}dt = 0. \tag{3.30}$$

Given that $\Sigma_K \lambda_i \rho_i \geq 0$ and $\|x - x^0\|^2$ is always positive,

$$\sum_k \lambda_i \int_a^b \{(x - x^0)[(f_{x^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^i g_{x^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) + \beta^T h_{x^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]$$
$$+ [D(x - x^0)][(f_{\dot{x}^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - \nu^i g_{\dot{x}^0}^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0))$$
$$+ \beta^T h_{\dot{x}^0}(t,x^0,\dot{x}^0,u^0,\dot{u}^0)]\}dt \geq -\Sigma_K \lambda_i \rho_i ||x - x^0||^2. \tag{3.31}$$

Again using the nonnegativity of each λ_i, $i \in K$, and ρ-pseudoconvexity and the equivalent form of ρ-quasiconvexity in Theorem 3.10(a), it follows from the above inequality that

$$\begin{aligned}
&\int_a^b \{(f^i(t,x,\dot{x},u,\dot{u}) - v^i g^i(t,x,\dot{x},u,\dot{u})) + \beta^T h(t,x,\dot{x},u,\dot{u})\} dt \\
&\qquad \geq \int_a^b \{(f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^i g^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) \\
&\qquad\qquad + \beta^T h(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\} dt, \quad \forall i \in K, \\
&\int_a^b \{(f^i(t,x,\dot{x},u,\dot{u}) - v^i g^i(t,x,\dot{x},u,\dot{u})) + \beta^T h(t,x,\dot{x},u,\dot{u})\} dt \\
&\qquad > \int_a^b \{(f^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0) - v^i g^i(t,x^0,\dot{x}^0,u^0,\dot{u}^0)) \\
&\qquad\qquad + \beta^T h(t,x^0,\dot{x}^0,u^0,\dot{u}^0)\} dt, \quad \text{for some } i \in K.
\end{aligned} \tag{3.32}$$

Now using the feasibility of (x,u) for $(\text{FP})_v$ and $(x^0,u^0,\lambda,v,\beta)$ for (FD) provides us with the desired conclusion that (3.6) cannot hold. □

Next we state the last weak duality theorem. The proof is on similar lines as earlier.

Theorem 3.11. *Assume that for all feasible (x,u) for $(\text{FP})_v$ and for all feasible $(x^0,u^0,\lambda,v,\beta)$ for (FD),*

(a) *for each $i \in K$, $\lambda_i > 0$ and $\lambda^T(f(t,\cdot,\cdot,\cdot,\cdot) - v^T g(t,\cdot,\cdot,\cdot,\cdot)) + \beta^T h(t,\cdot,\cdot,\cdot,\cdot)$ is ρ-convex with $\rho \geq 0$, or*
(b) *$\lambda^T(f(t,\cdot,\cdot,\cdot,\cdot) - v^T g(t,\cdot,\cdot,\cdot,\cdot)) + \beta^T h(t,\cdot,\cdot,\cdot,\cdot)$ is strictly ρ-convex with $\rho \geq 0$.*

Then (3.6) cannot hold.

Assumption (b) above that $\lambda^T(f(t,\cdot,\cdot,\cdot,\cdot) - v^T g(t,\cdot,\cdot,\cdot,\cdot)) + \beta^T h(t,\cdot,\cdot,\cdot,\cdot)$ is strictly ρ-convex can be replaced by much weaker conditions. This leads to the following corollary.

Corollary 3.12. *For all feasible (x,u) for $(\text{FP})_v$ and for all feasible $(x^0,u^0,\lambda,v,\beta)$ for (FD), assume that Theorem 3.11 holds, except that instead of (b), for each $i \in K$, $(f^i(t,\cdot,\cdot,\cdot,\cdot) - v^i g^i(t,\cdot,\cdot,\cdot,\cdot)) + \beta^T h(t,\cdot,\cdot,\cdot,\cdot)$ is ρ_i-convex, and for at least one $q \in K$, $(f^q(t,\cdot,\cdot,\cdot,\cdot) - v^q g^q(t,\cdot,\cdot,\cdot,\cdot)) + \beta^T h(t,\cdot,\cdot,\cdot,\cdot)$ is strictly ρ_q-convex (with the corresponding component λ_q of λ positive) with $\Sigma_K \lambda_i \rho_i \geq 0$. Then (3.6) cannot hold.*

Before proving strong duality theorem, we give the following lemma.

Lemma 3.13. *Assume that weak duality (any of the Theorems 3.1–3.11 or any of the Corollaries 3.2–3.12) holds between $(\text{FP})_v$ and (FD). If $(\bar{x},\bar{u},\bar{\lambda},\bar{v},\bar{\beta})$ is feasible for (FD) with $\int_a^b \bar{\beta}^T h(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) dt = 0$ and $(\bar{x},\bar{u})$ is feasible for $(\text{FP})_v$, then $(\bar{x},\bar{u})$ is efficient for $(\text{FP})_v$ and $(\bar{x},\bar{u},\bar{\lambda},\bar{v},\bar{\beta})$ is efficient for (FD).*

Proof. On the contrary, we assume that $(\bar{x},\bar{u})$ is not efficient for $(\mathrm{FP})_v$, then there exists a feasible (x,u) for $(\mathrm{FP})_v$ such that

$$\begin{aligned}
&\int_a^b \left(f^i(t,x,\dot{x},u,\dot{u}) - v^i g^i(t,x,\dot{x},u,\dot{u})\right)dt \\
&\qquad \leq \int_a^b \left(f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - v^i g^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)dt, \quad \forall i \in K, \\
&\int_a^b \left(f^i(t,x,\dot{x},u,\dot{u}) - v^i g^i(t,x,\dot{x},u,\dot{u})\right)dt \\
&\qquad < \int_a^b \left(f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - v^i g^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})\right)dt, \quad \text{for some } i \in K.
\end{aligned} \tag{3.33}$$

Since $\int_a^b \bar{\beta}^T h(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})dt = 0$ $(\Rightarrow \int_a^b \bar{\beta}_j^T h^j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})dt = 0)$, we can write $(f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \bar{v}^i f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})) + \bar{\beta}_j^T h^j(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})$ in place of $(f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}) - \bar{v}^i f^i(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}}))$ in the right-side terms of (3.33). □

Now using the feasibility of (x,u) for $(\mathrm{FP})_v$ and $(\bar{x},\bar{u},\bar{\lambda},\bar{v},\bar{\beta})$ for (FD), we get a contradiction to the weak duality. So, $(\bar{x},\bar{u})$ is efficient for $(\mathrm{FP})_v$. Similarly, we can show that $(\bar{x},\bar{u},\bar{\lambda},\bar{v},\bar{\beta})$ is efficient for (FD).

Now using this lemma in conjunction with the necessary optimality conditions (Proposition 2.5) of Section 2, we establish the following strong duality theorem.

Theorem 3.14. *Let $(\bar{x},\bar{u})$ be an efficient solution for $(\mathrm{FP})_v$ and assume that $(\bar{x},\bar{u})$ satisfies the Kuhn-Tucker constraint qualification for $(\mathrm{FP})_v$. Then there exists $\bar{\lambda} \in \mathbb{R}^k$ and a piecewise smooth function $\bar{\beta} : I \to \mathbb{R}^m$ such that $(\bar{x},\bar{u},\bar{\lambda},\bar{v},\bar{\beta})$ is feasible for (FD), along with the condition $\int_a^b \bar{\beta}^T h(t,\bar{x},\dot{\bar{x}},\bar{u},\dot{\bar{u}})dt = 0$. Further, if weak duality (Theorems 3.1–3.11 or Corollaries 3.2–3.12) also holds between $(\mathrm{FP})_v$ and (FD), then $(\bar{x},\bar{u},\bar{\lambda},\bar{v},\bar{\beta})$ is efficient for (FD).*

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