SOME GENERALIZED DIFFERENCE SEQUENCE SPACES OF INVARIANT MEANS DEFINED BY ORLICZ FUNCTIONS

AHMAD H. A. BATAINEH AND LAITH E. AZAR

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We define the sequence spaces $[w^{\theta},M,p,u,\Delta]_{\sigma}$, $[w^{\theta},M,p,u,\Delta]_{\sigma}^{0}$, and $[w^{\theta},M,p,u,\Delta]_{\sigma}^{\infty}$ which are defined by combining the concepts of Orlicz functions, invariant means, and lacunary convergence. We also study some inclusion relations and linearity properties of the above-mentioned spaces. These are generalizations of those defined and studied by Savaş and Rhoades in 2002 and some others before.

1. Introduction

Let l_{∞} and c denote the Banach spaces of bounded and convergent sequences $x = (x_k)$, with $x_k \in \mathbb{R}$ or $\mathbb{C}$, normed by $\|x\| = \sup_k |x_k|$, respectively.

A paranorm on a linear topological space X is a function $g : X \to \mathbb{R}$ which satisfies the following axioms for any $x, y, x_0 \in X$ and $\lambda, \lambda_0 \in \mathbb{C}$:

(i) $g(\theta) = 0$, where $\theta = (0,0,0,\ldots)$, the zero sequence,

(ii) $g(x) = g(-x)$,

(iii) $g(x+y) \le g(x) + g(y)$ (subadditivity),

(iv) the scalar multiplication is continuous, that is,

$$\lambda \longrightarrow \lambda_0, \quad x \longrightarrow x_0 \quad \text{imply } \lambda x \longrightarrow \lambda_0 x_0; \tag{1.1}$$

in other words,

$$|\lambda - \lambda_0| \longrightarrow 0, \quad g(x - x_0) \longrightarrow 0 \quad \text{imply } g(\lambda x - \lambda_0 x_0) \longrightarrow 0. \tag{1.2}$$

A paranormed space is a linear space X with a paranorm g and is written as (X,g), (see Maddox [10, page 92]).

Any function g which satisfies all conditions (i), (ii), (iii), and (iv), together with the condition

(v) $g(x) = 0$ if and only if $x = \theta$,

is called a total paranorm on X, and the pair (X,g) is called a total paranormed space, (see Maddox [10, page 92]).

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Schaefer [20] defined the σ-convergence as follows.

Let σ be the mapping of the set of positive integers into itself. A continuous linear functional φ on l_∞ is said to be an invariant mean or σ-mean if and only if

(i) $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

(ii) $\varphi(e) = 1$,

(iii) $\varphi(x_{\sigma(n)}) = \varphi(x)$ for all $x \in l_\infty$.

In case σ is the translation mapping $n \to n+1$, a σ-mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergence sequences.

A sequence $x = (x_k) \in l_\infty$ is said to be almost convergent if all of its Banach limits coincide (see Banach [1]). Let $\hat{c}$ denote the space of all almost convergent sequences. Lorentz [8] proved that

$$\hat{c} = \left\{x \in l_\infty : \lim_{m\to\infty} t_{mn}(x) \text{ exists, uniformly in } n\right\}, \tag{1.3}$$

where $(t_{mn}(x) = x_n + x_{n+1} + \cdots + x_{n+m})/(m+1)$.

The space $[\hat{c}]$ of strongly almost convergence sequences was introduced by Maddox [11] and Freedman et al. [4] as follows:

$$[\hat{c}] = \left\{x \in l_\infty : \lim_{m\to\infty} t_{mn}\big(|x - le|\big) = 0, \text{ uniformly in } n, \text{ for some } l \in \mathbb{R}\right\}, \tag{1.4}$$

where $e = (1,1,\ldots)$.

If $x = (x_k)$, write $Tx = (Tx_k) = (x_{\sigma(k)})$. It can be shown that

$$V_\sigma = \left\{x \in l_\infty : \lim_{m\to\infty} t_{km}(x) = l, \text{ uniformly in } m\right\}, \tag{1.5}$$

$l = \sigma - \lim x$, where

$$t_{km}(x) = \frac{x_m + x_{\sigma(m)} + x_{\sigma^2(m)} + \cdots + x_{\sigma^k(m)}}{k+1}. \tag{1.6}$$

Here $\sigma^k(m)$ denotes the kth iterate of the mapping σ at m.

A σ-mean extends the limit functional on c in the sense that $\varphi(x) = \lim x$ for all $x \in c$ if and only if σ has no finite orbits; that is to say, if and only if for all $n \geq 0$, $j \geq 0$, $\sigma^j(n) \neq n$ (see Mursaleen [13]).

A sequence x is said to be strongly σ-convergent if there exists a number l such that

$$\big(|x_k - l|\big) \in V_\sigma \tag{1.7}$$

with the limit zero (see Mursaleen [12]).

We write $[V_\sigma]$ as the set of all strongly σ-convergent sequences. When (1.7) holds, we write $[V_\sigma] - \lim x = l$. Taking $\sigma(n) = n+1$, we obtain $[V_\sigma] = [\hat{c}]$ so that strong σ-convergence generalizes the concept of strong almost convergence. Note that $c \subset [V_\sigma] \subset V_\sigma \subset l_\infty$.

Using the concept of invariant means, the following sequence spaces have been introduced and examined by Mursaleen et al. [14] as a generalization of Das and Sahoo [3]:

$$
\begin{aligned}
w_\sigma &= \left\{x : \lim_n \frac{1}{n+1} \sum_{k=0}^{n} t_{km}(x-l) = 0, \text{ for some } l \in \mathbb{R}, \text{ uniformly in } m\right\}, \\
[w]_\sigma &= \left\{x : \lim_n \frac{1}{n+1} \sum_{k=0}^{n} \left| t_{km}(x-l) \right| = 0, \text{ for some } l \in \mathbb{R}, \text{ uniformly in } m\right\}, \\
[w_\sigma] &= \left\{x : \lim_n \frac{1}{n+1} \sum_{k=0}^{n} t_{km}(|x-l|) = 0, \text{ for some } l \in \mathbb{R}, \text{ uniformly in } m\right\}.
\end{aligned}
\tag{1.8}
$$

By a lacunary sequence $\theta = (k_r)$; $r = 0,1,2,\ldots$, where $k_0 = 0$, we will mean an increasing sequence of nonnegative integers with $k_r - k_{r-1} \to \infty$. The intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$ and we let $h_r = k_r - k_{r-1}$. The ratio k_r/k_{r-1} will be denoted by q_r. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al. [4] as

$$
N_\theta = \left\{x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in I_r} \left| x_k - l \right| = 0, \text{ for some } l\right\}. \tag{1.9}
$$

The concept of lacunary strong σ-convergence was introduced by Savaş [18] which is a generalization of the idea of lacunary strong almost convergence due to Das and Patel [2].

If $[V_\sigma^\theta]$ denotes the set of all lacunary strongly σ-convergent sequences, then Savaş [18] defined

$$
[V_\sigma^\theta] = \left\{x = (x_k) : \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} \left| x_{\sigma^k(n)} - l \right| = 0, \text{ for some } l, \text{ uniformly in } n\right\}. \tag{1.10}
$$

Recall [5, 7] that an Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$, and $M(x) \to \infty$ as $x \to \infty$.

If convexity of M is replaced by $M(x+y) \le M(x) + M(y)$, then it is called a modulus function, defined and discussed by Ruckle [16].

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to define what is called an Orlicz sequence space:

$$
l_M := \left\{x : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\right\} \tag{1.11}
$$

which is a Banach space with the norm

$$
\|x\|_M = \inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1\right\}. \tag{1.12}
$$

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that

$$M(2u) \leq KM(u) \quad (u \geq 0). \tag{1.13}$$

It is easy to see that always $K > 2$. The Δ_2-condition is equivalent to the satisfaction of the inequality

$$M(lu) \leq K(lu)M(u) \tag{1.14}$$

for every value of u and for $l > 1$ (see Krasnosel'skiĭ and Rutickiĭ [6]).

Parashar and Choudhary [15] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function M, which generalizes the well-known Orlicz sequence space l_M and strong summable sequence spaces $[C,1,p]$, $[C,1,p]_0$, and $[C,1,p]_\infty$. It may be noted that the spaces of strongly summable sequences were discussed by Maddox [9].

The main object of this paper is to define and study the sequence spaces: $[w^\theta,M,p,u,\Delta]_\sigma$, $[w^\theta,M,p,u,\Delta]_\sigma^0$ and $[w^\theta,M,p,u,\Delta]_\sigma^\infty$, which are defined by combining the concepts of an Orlicz function, invariant mean, and lacunary convergence. We examine some linearity and inclusion relations of these spaces and establish the connection between lacunary $[w]_\sigma$-convergence and lacunary $[w]_\sigma$-convergence with respect to an Orlicz function which satisfies the Δ_2-condition.

Now, if $u = (u_k)$ is any sequence such that $u_k \neq 0 (k = 1,2,\ldots)$ and for any sequence $x = (x_n)$, the difference sequence Δx is given by $\Delta x = (\Delta x_n)_{n=1}^\infty = (x_n - x_{n-1})_{n=1}^\infty$, then we define the following sequence spaces:

$$\begin{aligned}
[w^\theta,M,p,u,\Delta]_\sigma &= \Bigg\{x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k\in I_r} \left[M\left(\frac{|t_{km}(u\Delta x - le)|}{\rho}\right)\right]^{p_k} \\
&\qquad = 0, \text{ for some } l,\ \rho > 0, \text{ uniformly in } m\Bigg\}, \\
[w^\theta,M,p,u,\Delta]_\sigma^0 &= \Bigg\{x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k\in I_r} \left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho}\right)\right]^{p_k} \\
&\qquad = 0, \text{ for some } \rho > 0, \text{ uniformly in } m\Bigg\}, \\
[w^\theta,M,p,u,\Delta]_\sigma^\infty &= \Bigg\{x = (x_k) : \sup_{r,m} \frac{1}{h_r} \sum_{k\in I_r} \left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho}\right)\right]^{p_k} \\
&\qquad < \infty \text{ for some } \rho > 0\Bigg\}.
\end{aligned} \tag{1.15}$$

If $u = e$, and $\Delta x_k = x_k$, for all k, then these spaces reduce to those defined and studied by Savaş and Rhoades [19]. Also some sequence spaces are obtained by specializing θ, M, and p. For example, if $u = e$, $\Delta x_k = x_k$, and $p_k = 1$ for all k, then we get the spaces $[w^\theta,M]_\sigma$, $[w^\theta,M]_\sigma^0$, and $[w^\theta,M]_\sigma^\infty$.

If $x \in [w^\theta, M]_\sigma$, we say that x is lacunary $[w]_\sigma$-convergence with respect to the Orlicz function M.

When $u = e$, $\Delta x_k = x_k$, for all k, and $\sigma(n) = n+1$, the spaces $[w^\theta, M, p, u, \Delta]_\sigma$, $[w^\theta, M, p, u, \Delta]_\sigma^0$, and $[w^\theta, M, p, u, \Delta]_\sigma^\infty$ reduce to the spaces $[\hat{w}_\theta, M, p]_\sigma$, $[\hat{w}_\theta, M, p]_\sigma^0$, and $[\hat{w}_\theta, M, p]_\sigma^\infty$, respectively, which are defined as

$$\begin{aligned}
[\hat{w}_\theta, M, p] &= \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{m \in I_r} \left[M\left(\frac{|t_{mn}(x - l)|}{\rho} \right) \right]^{p_k} \right. \\
&\qquad \left. = 0, \text{ uniformly in } n, \text{ for some } l,\ \rho > 0 \right\}, \\
[\hat{w}_\theta, M, p]_0 &= \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{m \in I_r} \left[M\left(\frac{|t_{mn}(x)|}{\rho} \right) \right]^{p_k} \right. \\
&\qquad \left. = 0, \text{ uniformly in } n, \text{ for some } \rho > 0 \right\}, \\
[\hat{w}_\theta, M, p]_\infty &= \left\{ x = (x_k) : \sup_{r,n} \frac{1}{h_r} \sum_{m \in I_r} \left[M\left(\frac{|t_{mn}(x)|}{\rho} \right) \right]^{p_k} \right. \\
&\qquad \left. < \infty, \text{ for some } \rho > 0 \right\}.
\end{aligned} \tag{1.16}$$

If $u = e$, and $\Delta x_k = x_k$, for all k, $M(x) = x$, $\theta = (2^r)$, and $p_k = 1$, for all k, then $[w^\theta, M, p, u, \Delta]_\sigma = [w]_\sigma$ (see Mursaleen et al. [14]) and $[\hat{w}_\theta, M, p, u, \Delta] = [\hat{w}]$ (see Das and Sahoo [3]). When $u = e$, $\Delta x_k = x_k$, for all k, $M(x) = x$, and $p_k = 1$, for all k, then $[w^\theta, M, p, u, \Delta]_\sigma = [w^\theta]_\sigma$, $[w^\theta, M, p, u, \Delta]_\sigma^0 = [w]_\sigma^0$. If $u = e$, $\Delta x_k = x_k$, for all k, and $\theta = (2^r)$, then $[w^\theta, M, p, u, \Delta]_\sigma = [w, M, p]_\sigma$, $[w^\theta, M, p, u, \Delta]_\sigma^0 = [w, M, p]_\sigma^0$, and $[w^\theta, M, p, u, \Delta]_\sigma^\infty = [w, M, p]_\sigma^\infty$.

2. Main results

We proved the following theorems.

Theorem 2.1. *For any Orlicz function M and a bounded sequence $p = (p_k)$ of strictly positive real numbers, $[w^\theta, M, p, u, \Delta]_\sigma$, $[w^\theta, M, p, u, \Delta]_\sigma^0$, and $[w^\theta, M, p, u, \Delta]_\sigma^\infty$ are linear spaces over the set of complex numbers.*

Proof. We will prove the result only for $[w^\theta, M, p, u, \Delta]_\sigma^0$. The others can be treated similarly. Let $x, y \in [w^\theta, M, p, u, \Delta]_\sigma^0$ and $\alpha, \beta \in \mathbb{C}$, the set of complex numbers. In order to prove the result, we need to find some $\rho_3 > 0$ such that

$$\lim_r \frac{1}{h_r} \sum_{k \in I_r} \left[M\left(\frac{|t_{km}(\alpha u \Delta x + \beta u \Delta y)|}{\rho_3} \right) \right]^{p_k} = 0, \quad \text{uniformly in } m. \tag{2.1}$$

Since $x,y \in [w^\theta,M,p,u,\Delta]_\sigma^0$, there exist positive ρ_1 and ρ_2 such that

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho_1}\right)\right]^{p_k}=0,$$
$$\lim_r \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(u\Delta y)|}{\rho_2}\right)\right]^{p_k}=0, \tag{2.2}$$

uniformly in m.

Define $\rho_3=\max(2|\alpha|\rho_1,2|\beta|\rho_2)$. Since M is nondecreasing and convex,

$$\begin{aligned}
&\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(\alpha u\Delta x+\beta u\Delta y)|}{\rho 3}\right)\right]^{p_k}\\
&\quad\le \frac{1}{h_r}\sum_{k\in I_r}\frac{1}{2^{p_k}}\left[M\left(\frac{|t_{km}(\alpha u\Delta x)|}{\rho_1}\right)+M\left(\frac{|t_{km}(\beta u\Delta y)|}{\rho_2}\right)\right]^{p_k}\\
&\quad\le D\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(\alpha u\Delta x)|}{\rho_1}\right)\right]^{p_k}+D\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(\beta u\Delta y)|}{\rho_2}\right)\right]^{p_k}
\end{aligned} \tag{2.3}$$

$\to 0$, as $r\to\infty$, uniformly in m, where $D=\max(1,2^{H-1})$, $H=\sup p_k$, so that $\alpha ux+\beta uy \in [w^\theta,M,p,u,\Delta]_\sigma^0$. This completes the proof. □

Theorem 2.2. *For any Orlicz function M and a bounded sequence $p=(p_k)$ of strictly positive real numbers, $[w^\theta,M,p,u,\Delta]_\sigma^0$ is a topological linear space, totally paranormed by:*

$$g(x)=\inf\left\{\rho^{p_r/H}:\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(x)|}{\rho}\right)\right]^{p_k}\right)^{1/H}\le 1,\ r=1,2,\ m=1,2,\dots\right\}, \tag{2.4}$$

where $H=\max(1,\sup p_k)$.

Proof. It is easy to see that $g(x)=g(-x)$. By using Theorem 2.1, for $\alpha=\beta=1$, we get $g(x+y)\le g(x)+g(y)$. Since $M(0)=0$, we get $\inf\{\rho^{p_r/H}\}=0$ for $x=0$. Conversely, suppose $g(x)=0$, then

$$\inf\left\{\rho^{p_r/H}:\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho}\right)\right]^{p_k}\right)^{1/H}\le 1\right\}=0. \tag{2.5}$$

This implies that for a given $\epsilon>0$, there exists some $\rho_3(0<\rho_3<\epsilon)$ such that

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho_3}\right)\right]^{p_k}\right)^{1/H}\le 1. \tag{2.6}$$

Thus

$$\left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(u\Delta x)|}{\epsilon}\right)\right]^{p_k}\right)^{1/H} \le \left(\frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho_3}\right)\right]^{p_k}\right)^{1/H} \le 1, \tag{2.7}$$

for each r and m.

Suppose that $x_{\sigma^i(j)} \neq 0$ for each i and j. This implies that $t_{ij}(x) \neq 0$, for each i and j. Let $\epsilon \to 0$. Then

$$\left(\frac{|t_{ij}(x)|}{\epsilon}\right) \longrightarrow \infty. \tag{2.8}$$

It follows that

$$\left(\frac{1}{h_r}\sum_{i\in I_r}\left[M\left(\frac{|t_{ij}(u\Delta x)|}{\epsilon}\right)\right]^{p_k}\right)^{1/H} \longrightarrow \infty \tag{2.9}$$

which is a contradiction.

Therefore, $t_{ij}(x) = 0$ for each i and j, and thus $x_{\sigma^i(j)} = 0$ for each i and j.

Showing that scalar multiplication is continuous is a standard argument, so we omit it. □

Theorem 2.3. *For any Orlicz function M which satisfies the Δ_2-condition, $[w^\theta, u, \Delta]_\sigma \subseteq [w^\theta, M, u, \Delta]$.*

To prove the theorem, we need the following lemma.

Lemma 2.4. *Let M be an Orlicz function which satisfies the Δ_2-condition and let $0 < \delta < 1$. Then for each $x \ge \delta$, $M(x) < Kx\delta^{-1}M(2)$ for some constant $K > 0$.*

Proof. It follows by a straightforward calculation using the Δ_2-condition. □

Proof of Theorem 2.3. Let $x \in [w^\theta, u, \Delta]_\sigma$. Then we have

$$A_r = \frac{1}{h_r}\sum_{k\in I_r} |t_{km}(u\Delta x - le)| \longrightarrow 0, \tag{2.10}$$

as $r \to \infty$, uniformly in m, for some l.

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M(t) < \epsilon$ for $0 \le t \le \delta$.

Then we can write

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r} M(|t_{km}(u\Delta x - le)|) &= \frac{1}{h_r}\sum_{k\in I_r, |t_{km}(u\Delta x - le)|\le\delta} M(|t_{km}(u\Delta x - le)|)\\ &\quad + \frac{1}{h_r}\sum_{k\in I_r, |t_{km}(u\Delta x - le)|>\delta} M(|t_{km}(u\Delta x - le)|)\\ &< h_r^{-1}(h_r\epsilon) + h_r^{-1}K\delta^{-1}M(2)h_rA_r, \text{ by Lemma 2.4.}\end{aligned} \tag{2.11}$$

Letting $r \to \infty$, it follows that $x \in [w^\theta, M, u, \Delta]_\sigma$. □

THEOREM 2.5. *Let $\theta = (k_r)$ be a lacunary sequence with* $\liminf_r q_r > 1$. *Then for any Orlicz function M, $[w,M,p,u,\Delta]_\sigma \subset [w^\theta,M,p,u,\Delta]_\sigma$, $[w,M,p,u,\Delta]_\sigma^0 \subset [w^\theta,M,p,u,\Delta]_\sigma^0$, and $[w,M,p,u,\Delta]_\sigma^\infty \subset [w^\theta,M,p,u,\Delta]_\sigma^\infty$, where*

$$\begin{aligned}[w,M,p,u,\Delta]_\sigma = \Bigg\{ x = (x_k) : \lim_n \frac{1}{n+1}\sum_{k=0}^{n}\left[M\left(\frac{|t_{km}(u\Delta x - le)|}{\rho}\right)\right]^{p_k} \\ = 0, \text{ uniformly in } m, \text{ for some } l, \rho > 0 \Bigg\},\end{aligned}$$

$$[w,M,p,u,\Delta]_\sigma^\infty = \left\{ x = (x_k) : \sup_n \frac{1}{n+1}\sum_{k=0}^{n}\left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho}\right)\right]^{p_k} < \infty, \text{ for some } \rho > 0 \right\}. \tag{2.12}$$

(In case $l = 0$, write $[w,M,p,u,\Delta]_\sigma = [w,M,p,u,\Delta]_\sigma^0$).

Proof. We will prove $[w,M,p,u,\Delta]_\sigma \subset [w^\theta,M,p,u,\Delta]_\sigma$ only. The others can be treated similarly. It is sufficient to show that $[w,M,p,u,\Delta]_\sigma^0 \subset [w^\theta,M,p,u,\Delta]_\sigma^0$; the general inclusion follows by linearity. Suppose $\liminf_r q_r > 1$, then there exists $\delta > 0$ such that $q_r = (k_r/k_{r-1}) \ge 1+\delta$ for all $r \ge 1$. Then for $x \in [w,M,p,u,\Delta]_\sigma^0$, we write

$$\begin{aligned}A_r &= \frac{1}{h_r}\sum_{k\in I_r}\left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho}\right)\right]^{p_k} \\ &= \frac{1}{h_r}\sum_{k=1}^{k_r}\left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho}\right)\right]^{p_k} - \frac{1}{h_r}\sum_{k=1}^{k_{r-1}}\left[M\left(\frac{|t_{km}(u\Delta x - le)|}{\rho}\right)\right]^{p_k} \\ &= \frac{k_r}{h_r}\left(k_r^{-1}\sum_{k=1}^{k_r}\left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho}\right)\right]^{p_k}\right) - \frac{k_{r-1}}{h_r}\left(k_{r-1}^{-1}\sum_{k=1}^{k_{r-1}}\left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho}\right)\right]^{p_k}\right).\end{aligned} \tag{2.13}$$

Since $h_r = k_r - k_{r-1}$, we have

$$\frac{k_r}{h_r} \le \frac{1+\delta}{\delta}, \qquad \frac{k_{r-1}}{h_r} \le \frac{1}{\delta}. \tag{2.14}$$

The terms $k_r^{-1}\sum_{k=1}^{k_r}[M(|t_{km}(u\Delta x)|/\rho)]^{p_k}$ and $k_{r-1}^{-1}\sum_{k=1}^{k_{r-1}}[M(|t_{km}(u\Delta x)|/\rho)]^{p_k}$ both converge to zero uniformly in m, and it follows that A_r converges to zero as $r \to \infty$, uniformly in m, that is, $x \in [w^\theta,M,p,u,\Delta]_\sigma^0$. This completes the proof. □

THEOREM 2.6. *Let $\theta = (k_r)$ be a lacunary sequence with* $\limsup_r q_r < \infty$. *Then for any Orlicz function M, $[w^\theta,M,p,u,\Delta]_\sigma \subset [w,M,p,u,\Delta]_\sigma$, $[w^\theta,M,p,u,\Delta]_\sigma^0 \subset [w,M,p,u,\Delta]_\sigma^0$, and $[w^\theta,M,p,u,\Delta]_\sigma^\infty \subset [w,M,p,u,\Delta]_\sigma^\infty$.*

Proof. We will prove $[w^\theta,M,p,u,\Delta]_\sigma \subset [w,M,p,u,\Delta]_\sigma$ only. The others can be treated similarly. It is sufficient to show that $[w,M,p,u,\Delta]_\sigma^0 \subset [w^\theta,M,p,u,\Delta]_\sigma^0$; the general inclusion follows by linearity. Suppose $\limsup_r q_r < \infty$, then there exists $B > 0$ such that $q_r < B$ for all $r \ge 1$. Let $x \in [w^\theta,M,p,u,\Delta]_\sigma^0$ and $\epsilon > 0$. Then there exists $L > 0$ such that for every $j \geq L$ and all m,

$$A_j = \frac{1}{h_j} \sum_{k \in I_j} \left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho} \right) \right]^{p_k} < \epsilon. \tag{2.15}$$

We can also find $K > 0$ such that $A_j < K$ for all $j = 1, 2, \ldots$.

Now let n be any integer with $k_{r-1} < n \leq k_r$, where $r > L$. Then

$$\begin{aligned}
&\frac{1}{n} \sum_{k=1}^{n} \left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho} \right) \right]^{p_k} \\
&\quad \leq k_{r-1}^{-1} \sum_{k=1}^{k_r} \left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho} \right) \right]^{p_k} \\
&\quad = k_{r-1}^{-1} \left\{ \sum_{k \in I_1} \left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho} \right) \right]^{p_k} + \sum_{k \in I_2} \left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho} \right) \right]^{p_k} \right. \\
&\qquad\qquad \left. + \cdots + \sum_{k \in I_r} \left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho} \right) \right]^{p_k} \right\} \\
&\quad = \frac{k_1}{k_{r-1}} k_1^{-1} \sum_{k \in I_1} \left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho} \right) \right]^{p_k} + \frac{k_2 - k_1}{k_{r-1}} (k_2 - k_1)^{-1} \sum_{k \in I_2} \left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho} \right) \right]^{p_k} \\
&\qquad + \cdots + \frac{k_R - k_{R-1}}{k_{r-1}} (k_R - k_{R-1})^{-1} \sum_{k \in I_2} \left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho} \right) \right]^{p_k} \\
&\qquad + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}} (k_r - k_{r-1})^{-1} \sum_{k \in I_2} \left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho} \right) \right]^{p_k} \\
&\quad = \frac{k_1}{k_{r-1}} A_1 + \frac{k_2 - k_1}{k_{r-1}} A_2 + \cdots + \frac{k_R - k_{R-1}}{k_{r-1}} A_R \\
&\qquad + \frac{k_{R+1} - k_R}{k_{r-1}} A_{R+1} + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}} A_r \\
&\quad \leq \left(\sup_{j \geq 1} A_j \right) \frac{k_R}{k_{r-1}} + \left(\sup_{j \geq R} A_j \right) \frac{k_r - k_R}{k_{r-1}} < K \frac{k_R}{k_{r-1}} + \epsilon B.
\end{aligned} \tag{2.16}$$

Since $k_{r-1} \to \infty$ as $n \to \infty$, it follows that

$$\frac{1}{n} \sum_{k=1}^{n} \left[M\left(\frac{|t_{km}(u\Delta x)|}{\rho} \right) \right]^{p_k} \longrightarrow 0, \tag{2.17}$$

uniformly in m, and consequently $x \in [w, M, p, u, \Delta]_{\sigma}^{0}$. This completes the proof of the theorem. □

Theorem 2.7. *Let $\theta = (k_r)$ be a lacunary sequence with $1 < \liminf_r q_r < \limsup_r q_r < \infty$. Then for any Orlicz function M, $[w, M, p, u, \Delta]_{\sigma} = [w^{\theta}, M, p, u, \Delta]_{\sigma}$.*

Proof. Theorem 2.7 follows from Theorems 2.5 and 2.6. □

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Ahmad H. A. Bataineh: Department of Mathematics, Al al-Bayt University, Mafraq 25113, Jordan
E-mail address: ahabf2003@yahoo.ca

Laith E. Azar: Department of Mathematics, Al al-Bayt University, Mafraq 25113, Jordan
E-mail address: azar_laith@yahoo.com