FIXED POINT ITERATION FOR ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Suppose that C is a nonempty closed convex subset of a real uniformly convex Banach space X. Let $T : C \to C$ be an asymptotically quasi-nonexpansive mapping. In this paper, we introduce the three-step iterative scheme for such map with error members. Moreover, we prove that if T is uniformly L-Lipschitzian and completely continuous, then the iterative scheme converges strongly to some fixed point of T.

1. Introduction

Let C be a subset of normed space X, and let T be a self-mapping on C. T is said to be *nonexpansive* provided that $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$; T is called *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that $\|T^n x - T^n y\| \leq (1 + k_n)\|x - y\|$ for all $x, y \in C$ and $n \geq 1$. T is said to be an *asymptotically quasi-nonexpansive map* if there exists a sequence $\{k_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} k_n = 0$ such that $\|T^n x - p\| \leq (1 + k_n)\|x - p\|$ for all $x \in C$ and $p \in F(T)$, and $n \geq 1$ ($F(T)$ denotes the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$).

From the above definitions, if $F(T) \neq \emptyset$, then asymptotically nonexpansive mapping must be asymptotically quasi-nonexpansive mapping.

The concept of asymptotically nonexpansiveness was introduced by Goebel and Kirk in 1972 [2]. In 2001, Noor [5, 6] introduced the three-step iterative scheme and he studied the approximate solutions of variational inclusions (inequalities) in Hilbert spaces. The three-step iterative approximation problems were studied extensively by Noor [5, 6], Glowinski and Le Tallec [1], and Haubruge et al. [3].

Recently, Xu and Noor [8] introduced the three-step iterative scheme for asymptotically nonexpansive mappings and they proved the following strong convergence theorem in Banach spaces.

Theorem 1.1 (see [8, Theorem 2.1]). *Let X be a real uniformly convex Banach space, let C be a nonempty closed, bounded convex subset of X. Let T be a completely continuous and asymptotically nonexpansive self-mapping with sequence $\{k_n\}$ satisfying $k_n \geq 0$ and*

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$\sum_{n=1}^{\infty} k_n < \infty$. *Let* $\{\alpha_n\}, \{\beta_n\}$, *and* $\{\gamma_n\}$ *be real sequences in* $[0,1]$ *satisfying*

(i) $0 < \liminf_{n\to\infty} \alpha_n \le \limsup_{n\to\infty} \alpha_n < 1$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$.

For a given $x_0 \in D$, *define*

$$\begin{aligned} z_n &= \gamma_n T^n x_n + (1-\gamma_n)x_n, \\ y_n &= \beta_n T^n z_n + (1-\beta_n)x_n, \\ x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)x_n. \end{aligned} \tag{1.1}$$

Then $\{x_n\}, \{y_n\}$, *and* $\{z_n\}$ *converge strongly to a fixed point of* T.

In this paper, we will extend the iterative scheme (1.1) to the iterative scheme of asymptotically quasi-nonexpansive mappings with error members. Moreover, we will prove the strong convergence of iterative scheme to a fixed point of T (C need not to be a bounded set), requiring T to be uniformly L-Lipschitzian and completely continuous. The results presented in this paper generalize and extend the corresponding main results of Xu and Noor [8].

2. Preliminaries

For the sake of convenience, we first recall some definitions and conclusions.

Definition 2.1 (see [2]). A Banach space X is said to be *uniformly convex* if the modulus of convexity of X

$$\delta_X(\epsilon) = \inf\left\{1 - \frac{\|x+y\|}{2} : \|x\| = \|y\| = 1,\ \|x-y\| = \epsilon\right\} > 0 \tag{2.1}$$

for all $0 < \epsilon \le 2$ (i.e., $\delta_X(\epsilon)$ is a function $(0,2] \to (0,1)$).

Definition 2.2. A mapping $T : C \to C$ is called *uniformly L-Lipschitzian* if there exists a constant $L > 0$ such that for all $x, y \in C$,

$$\|T^n x - T^n y\| \le L\|x - y\|, \quad \forall n \ge 1. \tag{2.2}$$

In what follows, we will make use of the following lemmas.

Lemma 2.3 (see [4]). *Let the nonnegative number sequences* $\{a_n\}, \{b_n\}$, *and* $\{d_n\}$ *satisfy that*

$$a_{n+1} \le (1+b_n)a_n + d_n, \quad \forall n = 1,2,\ldots, \sum_{n=1}^{\infty} b_n < \infty,\ \sum_{n=1}^{\infty} d_n < \infty. \tag{2.3}$$

Then,

(1) $\lim_{n\to\infty} a_n$ *exists;*

(2) *if* $\liminf_{n\to\infty} a_n = 0$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 2.4 ([7], J. Schu's Lemma). *Let X be a real uniformly convex Banach space, $0 < \alpha \leq t_n \leq \beta < 1$, $x_n, y_n \in X$, $\limsup_{n\to\infty} \|x_n\| \leq a$, $\limsup_{n\to\infty} \|y_n\| \leq a$, and $\lim_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = a$, $a \geq 0$. Then, $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Main results

In this section, we prove our main theorem. First of all, we will need the following lemmas.

Lemma 3.1. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let T be an asymptotically quasi-nonexpansive mapping with sequence $\{k_n\}_{n\geq 1}$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \neq \emptyset$. Let $x_0 \in C$ and*

$$\begin{aligned} z_n &= \alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n, \\ y_n &= \alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n, \\ x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n, \end{aligned} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$, and $\{\gamma_n''\}$ are real sequences in $[0,1]$ and $\{u_n\},\{v_n\}$, and $\{w_n\}$ are three bounded sequences in C such that

(i) $\alpha_n + \beta_n + \gamma_n = \alpha_n' + \beta_n' + \gamma_n' = \alpha_n'' + \beta_n'' + \gamma_n'' = 1$,

(ii) $\sum_{n=1}^{\infty} \gamma_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n' < \infty$, $\sum_{n=1}^{\infty} \gamma_n'' < \infty$.

If $p \in F(T)$, then $\lim_{n\to\infty} \|x_n - p\|$ exists.

Proof. Let $p \in F(T)$. Since $\{u_n\},\{v_n\}$, and $\{w_n\}$ are bounded sequences in C, put

$$M = \sup_{n\geq 1} ||u_n - p|| \vee \sup_{n\geq 1} ||v_n - p|| \vee \sup_{n\geq 1} ||w_n - p||. \tag{3.2}$$

Then M is a finite number. So for each $n \geq 1$, we note that

$$\begin{aligned} ||x_{n+1} - p|| &= ||\alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n - p|| \\ &\leq \alpha_n ||T^n y_n - p|| + \beta_n ||x_n - p|| + \gamma_n ||w_n - p|| \\ &\leq \alpha_n (1+k_n) ||y_n - p|| + \beta_n ||x_n - p|| + \gamma_n ||w_n - p||, \end{aligned} \tag{3.3}$$

$$\begin{aligned} ||y_n - p|| &= ||\alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n - p|| \\ &\leq \alpha_n' ||T^n z_n - p|| + \beta_n' ||x_n - p|| + \gamma_n' ||v_n - p|| \\ &\leq \alpha_n' (1+k_n) ||z_n - p|| + \beta_n' ||x_n - p|| + \gamma_n' ||v_n - p||, \end{aligned} \tag{3.4}$$

$$||z_n - p|| \leq \alpha_n'' (1+k_n) ||x_n - p|| + \beta_n'' ||x_n - p|| + \gamma_n'' ||u_n - p||. \tag{3.5}$$

Substituting (3.5) into (3.4),

$$
\begin{aligned}
\|y_n - p\| &\le \alpha'_n \alpha''_n (1+k_n)^2 \|x_n - p\| \\
&\quad + \alpha'_n \beta''_n (1+k_n) \|x_n - p\| + \alpha'_n \gamma''_n (1+k_n) \|u_n - p\| + \beta'_n \|x_n - p\| + \gamma'_n \|v_n - p\| \\
&\le (1-\beta'_n-\gamma'_n)\alpha''_n (1+k_n)^2 \|x_n - p\| + \beta'_n \|x_n - p\| \\
&\quad + (1-\beta'_n-\gamma'_n)\beta''_n \|x_n - p\| + m_n \\
&\le \beta'_n (1+k_n)^2 \|x_n - p\| + (1-\beta'_n)\alpha''_n (1+k_n)^2 \|x_n - p\| \\
&\quad + (1-\beta'_n)\beta''_n (1+k_n)^2 \|x_n - p\| + m_n \\
&= \beta'_n (1+k_n)^2 \|x_n - p\| + (1-\beta'_n)(\alpha''_n + \beta''_n)(1+k_n)^2 \|x_n - p\| + m_n \\
&\le \beta'_n (1+k_n)^2 \|x_n - p\| + (1-\beta'_n)(1+k_n)^2 \|x_n - p\| + m_n \\
&= (1+k_n)^2 \|x_n - p\| + m_n,
\end{aligned}
\tag{3.6}
$$

where $m_n = \gamma''_n (1+k_n)M + \gamma'_n M$. Substituting (3.6) into (3.3) again, we have

$$
\begin{aligned}
\|x_{n+1} - p\| &\le \alpha_n (1+k_n)\left((1+k_n)^2 \|x_n - p\| + m_n\right) + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\
&= \alpha_n (1+k_n)^3 \|x_n - p\| + \alpha_n (1+k_n) m_n + \beta_n \|x_n - p\| + \gamma_n \|w_n - p\| \\
&\le (\alpha_n + \beta_n)(1+k_n)^3 \|x_n - p\| + (1+k_n) m_n + \gamma_n \|w_n - p\| \\
&\le (1+k_n)^3 \|x_n - p\| + (1+k_n) m_n + \gamma_n \|w_n - p\| \\
&\le (1+k_n)^3 \|x_n - p\| + (1+k_n) m_n + \gamma_n M \\
&= (1+d_n) \|x_n - p\| + b_n,
\end{aligned}
\tag{3.7}
$$

where $d_n = 3k_n + 3k_n^2 + k_n^3$ and $b_n = (1+k_n)m_n + \gamma_n M$. Since $\sum_{n=1}^{\infty} d_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, by Lemma 2.3, we have that $\lim_{n\to\infty} \|x_n - p\|$ exists. This completes the proof. □

Lemma 3.2. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let T be an asymptotically quasi-nonexpansive mapping with sequence $\{k_n\}_{n\ge 1}$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \ne \emptyset$. Let $x_0 \in C$ and for each $n \ge 0$,*

$$
\begin{aligned}
z_n &= \alpha''_n T^n x_n + \beta''_n x_n + \gamma''_n u_n, \\
y_n &= \alpha'_n T^n z_n + \beta'_n x_n + \gamma'_n v_n, \\
x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n,
\end{aligned}
\tag{3.8}
$$

where $\{u_n\}, \{v_n\}$, and $\{w_n\}$ are three bounded sequences in C and $\{\alpha_n\}$, $\{\alpha'_n\}$, $\{\alpha''_n\}$, $\{\beta_n\}$, $\{\beta'_n\}$, $\{\beta''_n\}$, $\{\gamma_n\}$, $\{\gamma'_n\}$, and $\{\gamma''_n\}$ are real sequences in $[0,1]$ which satisfy the same assumptions as Lemma 3.1 and the additional assumption that $0 \le \alpha < \alpha_n$, $\beta_n, \alpha'_n, \beta'_n \le \beta < 1$ for some α, β in $(0,1)$. Then $\lim_{n\to\infty} \|T^n y_n - x_n\| = 0 = \lim_{n\to\infty} \|T^n z_n - x_n\|$.

Proof. For any $p \in F(T)$, it follows from Lemma 3.1, that $\lim_{n\to\infty} \|x_n - p\|$ exists. Let $\lim_{n\to\infty} \|x_n - p\| = a$ for some $a \geq 0$. From (3.6), we have

$$\|y_n - p\| \leq (1+k_n)^2 \|x_n - p\| + m_n. \tag{3.9}$$

Taking $\limsup_{n\to\infty}$ in both sides, we obtain

$$\limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|x_n - p\| = \lim_{n\to\infty} \|x_n - p\| = a. \tag{3.10}$$

Note that

$$\begin{aligned} \limsup_{n\to\infty} \|T^n y_n - p\| &\leq \limsup_{n\to\infty} (1+k_n)\|y_n - p\| = \limsup_{n\to\infty} \|y_n - p\| \leq a, \\ a = \lim_{n\to\infty} \|x_{n+1} - p\| &= \lim_{n\to\infty} \|\alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n - p\| \\ &= \lim_{n\to\infty} \left\| \alpha_n \left[T^n y_n - p + \frac{\gamma_n}{2\alpha_n}(w_n - p) \right] + \beta_n \left[x_n - p + \frac{\gamma_n}{2\beta_n}(w_n - p) \right] \right\| \\ &= \lim_{n\to\infty} \left\| \alpha_n \left[T^n y_n - p + \frac{\gamma_n}{2\alpha_n}(w_n - p) \right] + (1-\alpha_n) \left[x_n - p + \frac{\gamma_n}{2\beta_n}(w_n - p) \right] \right\|. \end{aligned} \tag{3.11}$$

By J. Schu's Lemma 2.4, we have

$$\lim_{n\to\infty} \left\| T^n y_n - x_n + \left(\frac{\gamma_n}{2\alpha_n} - \frac{\gamma_n}{2\beta_n} \right)(w_n - p) \right\| = 0. \tag{3.12}$$

Since $\lim_{n\to\infty} \|(\gamma_n/2\alpha_n - \gamma_n/2\beta_n)(w_n - p)\| = 0$, it follows that

$$\lim_{n\to\infty} \|T^n y_n - x_n\| = 0. \tag{3.13}$$

Finally, we will prove that $\lim_{n\to\infty} \|T^n z_n - x_n\| = 0$. To this end, we note that for each $n \geq 1$,

$$\|x_n - p\| \leq \|T^n y_n - x_n\| + \|T^n y_n - p\| \leq \|T^n y_n - x_n\| + (1+k_n)\|y_n - p\|. \tag{3.14}$$

Since $\lim_{n\to\infty} \|T^n y_n - x_n\| = 0 = \lim_{n\to\infty} k_n$, we obtain that

$$a = \lim_{n\to\infty} \|x_n - p\| \leq \liminf_{n\to\infty} \|y_n - p\|. \tag{3.15}$$

It follows that

$$a \leq \liminf_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|y_n - p\| \leq a. \tag{3.16}$$

This implies that

$$\lim_{n\to\infty} \|y_n - p\| = a. \tag{3.17}$$

On the other hand, we note that

$$\begin{aligned}
\|z_n - p\| &= \|\alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n - p\| \\
&\le \alpha_n''(1+k_n)\|x_n - p\| + \beta_n''\|x_n - p\| + \gamma_n''\|u_n - p\| \\
&\le \alpha_n''(1+k_n)\|x_n - p\| + (1-\alpha_n'')(1+k_n)\|x_n - p\| + \gamma_n''\|u_n - p\| \\
&\le (1+k_n)\|x_n - p\| + \gamma_n''\|u_n - p\|.
\end{aligned} \tag{3.18}$$

By boundedness of the sequence $\{u_n\}$ and $\lim_{n\to\infty} k_n = 0 = \lim_{n\to\infty} \gamma_n''$, we have

$$\limsup_{n\to\infty} \|z_n - p\| \le \limsup_{n\to\infty} \|x_n - p\| = a, \tag{3.19}$$

and so

$$\begin{aligned}
&\limsup_{n\to\infty} \|T^n z_n - p\| \le \limsup_{n\to\infty} (1+k_n)\|z_n - p\| \le a, \\
a &= \lim_{n\to\infty} \|y_n - p\| = \lim_{n\to\infty} \|\alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n - p\| \\
&= \lim_{n\to\infty} \left\| \alpha_n' \left[T^n z_n - p + \frac{\gamma_n'}{2\alpha_n'}(v_n - p) \right] + \beta_n' \left[x_n - p + \frac{\gamma_n'}{2\beta_n'}(v_n - p) \right] \right\| \\
&= \lim_{n\to\infty} \left\| \alpha_n' \left[T^n z_n - p + \frac{\gamma_n'}{2\alpha_n'}(v_n - p) \right] + (1-\alpha_n') \left[x_n - p + \frac{\gamma_n'}{2\beta_n'}(v_n - p) \right] \right\|.
\end{aligned} \tag{3.20}$$

By J. Schu's Lemma 2.4, we have

$$\lim_{n\to\infty} \left\| T^n z_n - x_n + \left(\frac{\gamma_n'}{2\alpha_n'} - \frac{\gamma_n'}{2\beta_n'} \right)(v_n - p) \right\| = 0. \tag{3.21}$$

Since $\lim_{n\to\infty} \|(\gamma_n'/2\alpha_n' - \gamma_n'/2\beta_n')(v_n - p)\| = 0$, it follows that

$$\lim_{n\to\infty} \|T^n z_n - x_n\| = 0. \tag{3.22}$$

This completes the proof. □

Theorem 3.3. *Let X be a real uniformly convex Banach space, C a nonempty closed convex subset of X. Let T be uniformly L-Lipschitzian, completely continuous, and an asymptotically quasi-nonexpansive mapping with sequence $\{k_n\}_{n\ge 1}$ such that $\sum_{n=1}^{\infty} k_n < \infty$ and $F(T) \ne \emptyset$. Let $x_0 \in C$ and for each $n \ge 0$,*

$$\begin{aligned}
z_n &= \alpha_n'' T^n x_n + \beta_n'' x_n + \gamma_n'' u_n, \\
y_n &= \alpha_n' T^n z_n + \beta_n' x_n + \gamma_n' v_n, \\
x_{n+1} &= \alpha_n T^n y_n + \beta_n x_n + \gamma_n w_n,
\end{aligned} \tag{3.23}$$

where $\{u_n\}, \{v_n\}$, and $\{w_n\}$ are three bounded sequences in C and $\{\alpha_n\}$, $\{\alpha_n'\}$, $\{\alpha_n''\}$, $\{\beta_n\}$, $\{\beta_n'\}$, $\{\beta_n''\}$, $\{\gamma_n\}$, $\{\gamma_n'\}$, and $\{\gamma_n''\}$ are real sequences in $[0,1]$ which satisfy the same assumptions as Lemma 3.1 and the additional assumption that $0 \le \alpha < \alpha_n, \beta_n, \alpha_n', \beta_n' \le \beta < 1$ for some α, β in $(0,1)$. Then $\{x_n\}, \{y_n\}$, and $\{z_n\}$ converge strongly to a fixed point of T.

Proof. It follows from Lemma 3.2 that

$$\lim_{n\to\infty} ||T^n y_n - x_n|| = 0 = \lim_{n\to\infty} ||T^n z_n - x_n|| \tag{3.24}$$

and this implies that

$$||x_{n+1} - x_n|| \le \alpha_n ||T^n y_n - x_n|| + \gamma_n ||w_n - x_n|| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.25}$$

We note that

$$\begin{aligned} ||T^n x_n - x_n|| &\le ||T^n x_n - T^n y_n|| + ||T^n y_n - x_n|| \le L||x_n - y_n|| + ||T^n y_n - x_n|| \\ &\le \alpha'_n L||x_n - T^n z_n|| + \gamma'_n L||v_n - x_n|| + ||T^n y_n - x_n|| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \end{aligned} \tag{3.26}$$

$$\begin{aligned} ||x_n - Tx_n|| &\le ||x_{n+1} - x_n|| + ||x_{n+1} - T^{n+1}x_{n+1}|| + ||T^{n+1}x_{n+1} - T^{n+1}x_n|| + ||T^{n+1}x_n - Tx_n|| \\ &\le ||x_{n+1} - x_n|| + ||x_{n+1} - T^{n+1}x_{n+1}|| + (1 + k_{n+1})||x_{n+1} - x_n|| + L||T^n x_n - x_n||. \end{aligned} \tag{3.27}$$

It follows from (3.25), (3.26), and the above inequality that

$$\lim_{n\to\infty} ||x_n - Tx_n|| = 0. \tag{3.28}$$

By Lemma 3.1, $\{x_n\}$ is bounded. It follows from our assumption that T is completely continuous and that there exists a subsequence $\{Tx_{n_k}\}$ of $\{Tx_n\}$ such that $Tx_{n_k} \to p \in C$ as $k \to \infty$. Moreover, by (3.28), we have $\|Tx_{n_k} - x_{n_k}\| \to 0$ which implies that $x_{n_k} \to p$ as $k \to \infty$. By (3.28) again, we have

$$||p - Tp|| = \lim_{k\to\infty} ||x_{n_k} - Tx_{n_k}|| = 0. \tag{3.29}$$

This shows that $p \in F(T)$. Furthermore, since $\lim_{n\to\infty} \|x_n - p\|$ exists, we have $\lim_{n\to\infty} \|x_n - p\| = 0$, that is, $\{x_n\}$ converges to some fixed point of T. It follows that

$$\begin{aligned} ||y_n - x_n|| &\le \alpha'_n ||T^n z_n - x_n|| + \gamma'_n ||v_n - x_n|| \longrightarrow 0, \\ ||z_n - x_n|| &\le \alpha''_n ||T^n x_n - x_n|| + \gamma''_n ||u_n - x_n|| \longrightarrow 0. \end{aligned} \tag{3.30}$$

Therefore, $\lim_{n\to\infty} y_n = p = \lim_{n\to\infty} z_n$. This completes the proof. □

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