THE DIFFERENTIAL FORMULA OF HASIMOTO TRANSFORMATION IN MINKOWSKI 3-SPACE

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In this work, the formula is given for the differential of the Hasimoto transformation in Minkowski 3-space.

1. Introduction

Hasimoto [10] introduced the map from vortex filament solutions of Euler's equations for incompressible fluids in the local induction approximation to solutions of the nonlinear Schrödinger equation and he showed vortex filament equation is equivalent nonlinear Schrödinger equation. After this discovering of Hasimoto, several authors [1, 5, 9, 12, 13, 14, 15, 17, 20, 21, 22, 23, 24] studied the connection between the integrable nonlinear Schrödinger equation and the nonstretching vortex filament equation. Ding and Inoguchi also presented this connection in Minkowski 3-space [6, 7, 8].

Langer and Perline derived the formula for the differential of the Hasimoto transformation in 3D spaces [16]. We also present a formula for the differential formula of Hasimoto transformation in Minkowski 3-space in this paper.

Since this construction has potential applications to further investigation using the inverse scattering scheme and finite-gap solutions, much works have been revived by several authors. In recent years, Langer and Perline found a recursion relation which generates the hierarchy of space curve equations which maps by Hasimoto transformation and nonlinear Schrödinger equation [18]. Calini and Ivey [2, 3, 4] studied finite-gap solutions of the vortex filament equation. Holm and Stechmann also investigated vortex solution motion driven by fluid helicity [11].

2. Nonlinear Schrödinger equation

Definition 2.1. The motion of very thin isolated vortex filament $X = X(s,t)$ of incompressible unbounded fluid by its own induction is described asymptotically by

$$\frac{\partial X}{\partial t} = \kappa b, \tag{2.1}$$

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where s is the length measured along the filament, t the time, κ the curvature, b the unit vector in the direction of the binormal [10].

THEOREM 2.2. *The binormal motion of timelike curves in the Minkowski* 3*-space is equivalent to the nonlinear Schrödinger equation* (NLS$^-$) *of repulsive type*

$$i\psi_t + \psi'' - \frac{1}{2}\left|\langle\psi,\psi\rangle\right|^2\psi = 0. \tag{2.2}$$

Proof. The Frenet-Serret formulas for curve γ is given by

$$\begin{bmatrix} T' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2\kappa & 0 \\ -\varepsilon_1\kappa & 0 & -\varepsilon_3\tau \\ 0 & \varepsilon_2\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ n \\ b \end{bmatrix}, \tag{2.3}$$

where $\kappa = \sqrt{|\langle T', T'\rangle|}$ is the curvature of γ, τ is the torsion, and $\langle T,T\rangle = \varepsilon_1$, $\langle n,n\rangle = \varepsilon_2$, $\langle b,b\rangle = \varepsilon_3$ are causal characters of γ. Here are the tangent vector field T, binormal vector field b, and principal normal vector field n.

We consider binormal motion of timelike curves. In this case

$$\begin{gathered} \varepsilon_1 = -1, \qquad \varepsilon_2 = 1, \qquad \varepsilon_3 = 1; \\ T \times b = -n, \quad b = T \times n; \end{gathered} \tag{2.4}$$

and the Frenet formula is

$$T' = \kappa n, \qquad n' = \kappa T - \tau b, \qquad b' = \tau n. \tag{2.5}$$

We get binormal motion vortex filament $X = X(s,t)$,

$$\begin{gathered} T = \frac{\partial X}{\partial t}(s,t) = \kappa(s,t)B(s,t), \\ \frac{\partial T}{\partial t}(s,t) = \frac{\partial X}{\partial s\partial t} = \kappa' b + \kappa\tau n, \end{gathered} \tag{2.6}$$

where a prime denotes $\partial/\partial s$.

With differentiating (2.6) as to s,

$$\frac{\partial^2 T}{\partial s^2} = \kappa' n + \kappa^2 T - \kappa\tau b. \tag{2.7}$$

Then

$$\frac{\partial T}{\partial t} = T \times \frac{\partial^2 T}{\partial s^2}. \tag{2.8}$$

We will show that the binormal motion of unit speed timelike curves is equivalent to the nonlinear Schrödinger equation of repulsive type (NLS$^-$).

We get

$$\xi_1 = T, \qquad \xi_2 = (n + ib)\exp\left(-i\int_0^s \tau d\tilde{s}\right), \qquad \psi = \kappa\exp\left(-i\int_0^s \tau d\tilde{s}\right). \tag{2.9}$$

Equation (2.5) can be written as follows:

$$
\begin{aligned}
\xi_1' &= \frac{1}{2}\left(\overline{\psi}\xi_2 + \psi\overline{\xi_2}\right), & \xi_2' &= \psi\xi_1, \\
\frac{\partial \xi_1}{\partial t} &= \frac{1}{2}i\left(\psi'\overline{\xi_2} - \overline{\psi'}\xi_2\right), & \frac{\partial \xi_2}{\partial t} &= -i\psi'\xi_1 + iR\xi_2,
\end{aligned}
\tag{2.10}
$$

where R is a real function of s and t. Let $V = (\xi_1, \xi_2, \xi_3)$ be a pseudo-unitary matrix. We have

$$
\begin{aligned}
\frac{\partial}{\partial s}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \overline{\xi_2} \end{pmatrix} &= \begin{pmatrix} 0 & \frac{1}{2}\overline{\psi} & \frac{1}{2}\psi \\ \psi & 0 & 0 \\ \overline{\psi} & 0 & 0 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \overline{\xi_2} \end{pmatrix}, \qquad V_s = YV, \\
\frac{\partial}{\partial t}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \overline{\xi_2} \end{pmatrix} &= \begin{pmatrix} 0 & -\frac{1}{2}i\overline{\psi'} & \frac{1}{2}i\psi' \\ -i\psi' & iR & 0 \\ i\overline{\psi'} & 0 & -iR \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \overline{\xi_2} \end{pmatrix}, \qquad V_t = ZV.
\end{aligned}
\tag{2.11}
$$

The integrability condition $Y_t - Z_t - [Y, Z] = 0$ of (2.10) denotes

$$
R' = \frac{1}{2}\left(\overline{\psi'}\psi + \psi'\overline{\psi}\right), \tag{2.12}
$$

$$
\psi_t - i\psi'' + iR\psi = 0. \tag{2.13}
$$

From (2.12),

$$
R = \frac{1}{2}\left(\psi\overline{\psi} + A\right). \tag{2.14}
$$

Using (2.14) and (2.13), we obtain

$$
i\psi_t + \psi'' - \frac{1}{2}|\psi|^2\psi = 0. \tag{2.15}
$$

This form is equivalent to the nonlinear Schrödinger equation of repulsive type (NLS^-). □

Theorem 2.3. *The binormal motion of spacelike curves in the Minkowski* 3*-space is equivalent to the nonlinear heat system (see* [1]*)*

$$
\begin{aligned}
r_t &= r_{ss} + r^2 q, \\
q_t &= -q_{ss} - q^2 r.
\end{aligned}
\tag{2.16}
$$

3. The differential formula in Minkowski 3-space

We get the space of curves with nonvanishing curvature $\Upsilon = \{\gamma : [0, l] \to R_1^3 : \kappa \neq 0\}$, where $l = \infty$. $U = iT + jn + kb$ is vector field along γ where i, j, k are functions on $[0, l]$. U must satisfy $i' = j\kappa$ for arclength-preserving condition.

We can add on a tangential term for the resulting vector field perserving arclength parametrization. For this reason, we define the linear "normalization operator"

$$\mathcal{N}U = \varepsilon_1\left(\int_0^s j\kappa\, du\right)T + jn + kb. \tag{3.1}$$

Here vector fields are vector fields whose components are expressed as to κ, τ, and their derivatives in Minkowski 3-space.

3.1. The differential of the Hasimoto transformation for timelike curves. For the first time in literature, conclusions of the formula of the differential of the Hasimoto transformation were presented by Langer and Perline [16]. In this paper, we also state conclusions and this formula for the first time in Minkowski 3-space.

Hasimoto transformation will be written as

$$\mathcal{H}(\gamma) = \psi = \kappa\rho, \tag{3.2}$$

where

$$\rho(s) = e^{-i\int_0^s \tau du}. \tag{3.3}$$

The differential of $\mathcal{H}$ can be expressed as

$$d\mathcal{H}(U) = \varepsilon_1\langle\zeta_2, \mathcal{R}^2U\rangle + ic\psi. \tag{3.4}$$

ζ_2 is the complex vector field

$$\zeta_2 = (n + ib)\rho, \tag{3.5}$$

$\mathcal{R}$ is the linear "recursion operator" as given by

$$\mathcal{R}U = \mathcal{N}(T \times U'), \tag{3.6}$$

$\times$ is the Minkowski cross product, and c is a real constant involving boundary terms. Considering brevity, we write the differential formula as follows:

$$d\mathcal{H}(U) \equiv \mathcal{M}(U) = \varepsilon_1\rho\langle(n + ib), \mathcal{R}^2U\rangle. \tag{3.7}$$

We compute differential formula to the field $U = \kappa b$. Thus

$$\begin{aligned} U' &= \kappa' b + \kappa\tau N, \\ T \times U' &= \kappa' T \times b + \kappa\tau T \times n. \end{aligned} \tag{3.8}$$

From (2.4),

$$\begin{aligned} T \times U' &= -\kappa' n + \kappa\tau b, \\ \mathcal{R}U = \mathcal{N}(T \times U') &= \frac{1}{2}\kappa^2 T - \kappa' n + \kappa\tau b. \end{aligned} \tag{3.9}$$

Continuing,

$$\begin{aligned}(\mathcal{R}U)' &= \left(\frac{1}{2}\kappa^3 - \kappa'' + \kappa\tau^2\right)n + (2\kappa'\tau + \kappa\tau')b,\\ T \times (\mathcal{R}U)' &= \left(\frac{1}{2}\kappa^3 - \kappa'' + \kappa\tau^2\right)T \times n + (2\kappa'\tau + \kappa\tau')T \times b.\end{aligned} \tag{3.10}$$

From (2.4),

$$\begin{aligned}T \times (\mathcal{R}U)' &= \left(\frac{1}{2}\kappa^3 - \kappa'' + \kappa\tau^2\right)b - (2\kappa'\tau + \kappa\tau')n,\\ \mathcal{R}^2 U &= -\frac{1}{2}\kappa^2\tau T - (2\kappa'\tau + \kappa\tau')n + \left(\frac{1}{2}\kappa^3 + \kappa'' - \kappa\tau^2\right)b,\end{aligned} \tag{3.11}$$

and as result

$$\mathcal{M}(U) = \rho\left[(2\kappa'\tau + \kappa\tau') - i\left(\frac{1}{2}\kappa^3 - \kappa'' + \kappa\tau^2\right)\right]. \tag{3.12}$$

We can give some results of this formula. First, differentiating $\psi = \kappa\rho$, one gets

$$\begin{aligned}\psi' &= \rho(\kappa' - i\kappa\tau),\\ \psi'' &= \rho\left[(\kappa'' - \kappa\tau^2) - i(2\kappa'\tau + \kappa\tau')\right].\end{aligned} \tag{3.13}$$

The filament flow $\gamma_t = U$ induces a flow on ψ satisfying

$$i\psi_t + \psi'' - \frac{1}{2}\left|\langle\psi,\psi\rangle\right|^2\psi = 0. \tag{3.14}$$

This form is equivalent to (2.15), the nonlinear Schrödinger equation of repulsive type.

3.2. The differential of the Hasimoto transformation for spacelike curves. The Hasimoto transformation is given by

$$\mathcal{H}_i(\gamma) = \kappa\rho_i, \quad i = 1,2, \tag{3.15}$$

where the differential of $\mathcal{H}$ can be expressed as

$$\begin{gathered} d\mathcal{H}_i(U) = \varepsilon_1\langle\zeta_{i+1}, \mathcal{R}^2U\rangle, \quad i = 1,2,\\ \zeta_1 = T, \qquad \zeta_2 = (n+b)\rho_1, \qquad \zeta_3 = (n-b)\rho_2,\end{gathered} \tag{3.16}$$

where

$$\rho_1(s) = \exp\left(-\int_0^s \tau\, du \right),$$
$$\rho_2(s) = \exp\left(\int_0^s \tau\, du \right), \tag{3.17}$$

and finally we formula can be written as

$$d\mathcal{H}_i(U) \equiv \mathcal{M}_i(U) = \varepsilon_1 \langle \zeta_{i+1}, \mathcal{R}^2 U \rangle, \quad i = 1,2. \tag{3.18}$$

We compute the differential formula for vector field $U = \kappa b$. Thus

$$U' = \kappa' b + \kappa\tau n,$$
$$T \times U' = \kappa' T \times b + \kappa\tau T \times n. \tag{3.19}$$

Since

$$T \times b = -\varepsilon_3 n \quad b = \varepsilon_2 T \times n,$$
$$T \times U' = \kappa' n + \kappa\tau b,$$
$$\mathcal{R}U = \mathcal{N}(T \times U') = \frac{1}{2}\kappa^2 T + \kappa' n + \kappa\tau b,$$
$$(\mathcal{R}U)' = \left(\frac{1}{2}\kappa^3 + \kappa'' + \kappa\tau^2\right) n + (2\kappa'\tau + \kappa\tau') b,$$
$$T \times (\mathcal{R}U)' = \left(\frac{1}{2}\kappa^3 + \kappa'' + \kappa\tau^2\right) T \times n + (2\kappa'\tau + \kappa\tau') T \times b. \tag{3.20}$$

From (3.20),

$$\mathcal{R}^2 U = T \times (\mathcal{R}U)' = \cdots + \left(\frac{1}{2}\kappa^3 + \kappa'' + \kappa\tau^2\right) b + (2\kappa'\tau + \kappa\tau') n, \tag{3.21}$$

and as result

$$\mathcal{M}_1(U) = \varepsilon_1 \rho_1 \langle (n+b), \mathcal{R}^2 U \rangle = \rho_1 \left(2\kappa'\tau + \kappa\tau' - \kappa'' - \kappa\tau^2 - \frac{1}{2}\kappa^3\right),$$
$$\mathcal{M}_2(U) = \varepsilon_1 \rho_2 \langle (n-b), \mathcal{R}^2 U \rangle = \rho_2 \left(2\kappa'\tau + \kappa\tau' + \kappa'' + \kappa\tau^2 + \frac{1}{2}\kappa^3\right). \tag{3.22}$$

We can give some results of this formula : with differentianting $q = \kappa\rho_1$ and $r = \kappa\rho_2$, we obtain

$$r' = r_s = \rho_2(\kappa' + \kappa\tau),$$
$$q' = q_s = \rho_1(\kappa' - \kappa\tau),$$
$$r'' = r_{ss} = \rho_2[(\kappa'' + \kappa\tau^2 + 2\kappa'\tau + \kappa\tau')],$$
$$q'' = q_{ss} = \rho_1[(\kappa'' + \kappa\tau^2 - 2\kappa'\tau - \kappa\tau')]. \tag{3.23}$$

We conclude that the filament flow $\gamma_t = U$ induces a flow on q and r satisfying

$$\begin{aligned} r_t &= d\mathcal{H}_1(U) = r_{ss} + r^2 q, \\ q_t &= d\mathcal{H}_2(U) = -q_{ss} - q^2 r. \end{aligned} \tag{3.24}$$

This form is equivalent to the nonlinear heat equation (2.16).

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