STRUCTURED LATTICES AND GROUND CATEGORIES OF L-SETS

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Complete lattices are considered with suitable families of lattice morphisms that provide a structure (L, Φ), useful to characterize ground categories of L-sets by means of powerset operators associated to morphisms of these categories. The construction of ground categories and powerset operators presented here extends and unifies most approaches previously considered, allowing the use of noncrisp objects and, with some restriction, the change of base. A sufficiently large category of L-sets that includes all possible ground categories on a structured lattice (L, Φ) is provided and studied, and its usefulness is justified. Many explanatory examples have been given and connection with the categories considered by J. A. Goguen and by S. E. Rodabaugh are stated.

1. Introduction

It is well known in the context of mathematics of fuzzy sets that in many disciplines and especially in fuzzy topology, it is very useful to set up the classes of objects and of morphisms to deal with (e.g., the working category, dubbed "ground category") as well as to associate to each morphism between two objects suitable operators (namely, powerset operators) between the lattices of "canonical subobjects" (namely powersets) of the considered objects.

Among papers mainly devoted to this topic, we quote [4, 8, 13, 15] (see also the survey [16]): the ground categories constructed in [13, 15], either in the fixed-basis or in the variable-basis context, contain only objects associated to (crisp) sets; the objects of the ground categories considered in [4, 8] are arbitrary L-sets (L is a suitable, fixed complete lattice).

Though not explicitly listed among the elements of the ground categories, powersets associated to objects and powerset operators associated to morphisms are fundamental in most applications, for instance, in fuzzy topology. In [13, 15], one can find a detailed and motivated justification for extending powersets and powerset operators from the traditional case of classical set theory to a more general context, including, as a first step, the Zadeh powerset operators. These latter operators are also a fundamental tool for the construction of powerset operators in [4, 8] and so they will be in this new approach.

Here an original idea of [3] is extended and developed so as to allow the construction of powerset operators to be applied in more general situations, including those considered in [4, 8] and in a special case of variable-basis fuzzy set theory extended to arbitrary *L*-sets.

The construction of powerset operators presented here uses a structure on a complete lattice *L*, consisting in a family Φ of suitable morphisms from *L* to each of its lower intervals (see Definition 3.1). So, an *L*-set $Y \in L^X$ may be viewed as a sort of "complete lattice bundle" whose base is *X* and whose fibers are the lower intervals $[\bot, Y(x)]$, $x \in X$, of the complete lattice *L*. *Y* is the maximal section of this bundle and the ordered set of all sections of the bundle determined by *Y* is the powerset of *Y*; see Definition 2.5, which is also justified in categorical terms in the final remark of Section 3.

Two powerset operators can be associated to every function between the underlying sets of any two *L*-sets (see Definition 3.5) and they are in general isotone maps between the corresponding powersets. This also allows a formal definition of ground categories of *L*-sets, characterized by means of the powerset operators associated to the considered morphisms (see Definition 3.6).

In Section 3, a category of *L*-sets (see Definition 3.16) defined by means of a preorder relation $\nearrow$ induced on *L* by the structure Φ (see Definition 3.13) is considered which contains all possible ground categories of *L*-sets on (L, Φ).

2. Preliminaries

We follow notation and terminology of [4, 13, 14, 15] unless otherwise stated, in particular for lattice-theoretic notions, categories of lattices, and ground categories for fuzzy set theories. Nevertheless, we recall and restate some definitions and results we will use later.

Arbitrary (finite, resp.) suprema, sups, or joins are synonymous and are denoted by $\bigvee$ ($\vee$, resp.); dually, arbitrary (finite, resp.) infima, infs, or meets are synonymous and are denoted by $\bigwedge$ ($\wedge$, resp.).

For any nonempty subset of a complete lattice $S \subseteq L$, we denote by $\langle S \rangle$, $\bigvee$-$\langle S \rangle$, $\bigwedge$-$\langle S \rangle$, respectively, the complete sublattice, the $\bigvee$-complete subsemilattice, the $\bigwedge$-complete subsemilattice generated by *S*.

A de Morgan frame *L* is a frame with an order-reversing involution, denoted by κ. These are objects of a concrete category **DMFrm** whose morphisms are the frame maps.

It is well known that objects of **POSet**, that is, ordered sets, can be considered as categories (called ordered categories) and morphisms of **POSet**, that is, order preserving or isotone maps, are the functors between such categories.

We recall that if

$$F : X \longrightarrow Y, \qquad G : Y \longrightarrow X \tag{2.1}$$

are functors between ordered categories, then *F* is left adjoint of *G*, and *G* is right adjoint of *F*,

$$F \dashv G \tag{2.2}$$

if and only if the adjoint inequalities hold:

(ADI) $x \leq G(F(x))$, for all $x \in X$;

(ADII) $y \geq F(G(y))$, for all $y \in Y$.

We restate the adjoint functor theorem for ordered categories and remark some consequences we will need to consider.

THEOREM 2.1 (adjoint functor theorem). *Let X, Y be ordered sets. Then the following hold.*

(1) *If $F : X \to Y$ and $G : Y \to X$ are isotone maps and*

$$F \dashv G, \tag{2.3}$$

then F preserves existing sups in X and G preserves existing infs in Y.

(2) *If X is a complete lattice and $F : X \to Y$ preserves $\bigvee$, then the function*

$$G : Y \longrightarrow X, \qquad y \longmapsto G(y) = \bigvee \{x \in X \mid F(x) \leq y\} \tag{2.4}$$

preserves order and it is the unique right adjoint of F.

(3) *If Y is a complete lattice and $G : Y \to X$ preserves $\bigwedge$, then the function*

$$F : X \longrightarrow Y, \qquad x \longmapsto F(x) = \bigwedge \{y \in Y \mid G(y) \geq x\} \tag{2.5}$$

preserves order and it is the unique left adjoint of G.

It is useful to note the following consequences that can be easily proved.

LEMMA 2.2. *Let Y, Z be complete lattices, $f \in \mathbf{Set}(Y, Z)$.*

(1) *If f is injective and preserves $\vee$ or $\wedge$, then it preserves and reflects the order.*

(2) *If f is surjective and preserves $\bigvee$ ($\bigwedge$, resp.) then the right (left, resp.) adjoint of f preserves and reflects the order.*

PROPOSITION 2.3. *Let Y, Z be complete lattices and let $f : Y \to Z$ preserve $\bigvee$ ($\bigwedge$, resp.). Then the following equivalences hold.*

(1) *f is injective if and only if the right (left, resp.) adjoint is a left inverse.*

(2) *f is surjective if and only if the right (left, resp.) adjoint is a right inverse.*

(3) *f is bijective if and only if the right (left, resp.) adjoint is the inverse.*

Of course one can note that every bijective semilattice morphism between complete lattices is a complete lattice isomorphism.

PROPOSITION 2.4. (1) *An isotone map $f : Y \to Y$ is selfadjoint if and only if it is self-inverse.*

(2) *If $f : Y \to Z$ and $g : Z \to Y$ are isotone maps and $f \dashv g$, then $f \circ g$ and $g \circ f$ are idempotent.*

From now on, we will denote the right (left, resp.) adjoint of an isotone map $f : Y \to Z$ by $f^{\vdash}$ ($f^{\dashv}$, resp.).

L always denotes a complete lattice, sometimes with further recalled properties; $\bot$ and $\top$ are its lower and upper bounds, respectively. Among others, we consider the complete lattices $\mathbf{2} = \{\bot, \top\}$ and $I = [0,1]$.

L-sets on (or *L*-subsets of) some set X are, of course, maps $A : X \to L$; crisp *L*-sets have range in $\{\bot, \top\} \subseteq L$. *L*-sets on X constitute the complete lattice L^X with the pointwise order. Every set Y can be considered in an obvious way as a crisp *L*-set on any set $X \supseteq Y$, for any complete lattice L.

The *L*-set on X with constant value $\alpha \in L$ on $Y \subseteq X$ and value $\bot$ elsewhere is denoted by α_{YX}; α_{XX} is also denoted by α_X or by $\underline{\alpha}$; in case of a singleton, we write $\alpha_{\{x\}} = \alpha_x$. The restriction of an *L*-set $A : X \to L$ to any set S, $A_{|S} : S \to L$, takes the same value as A does on each $x \in X \cap S$ and value $\bot$ elsewhere. $A_\bot = \{x \in X \mid A(x) \neq \bot\}$ is the support of A.

We state explicitly the notion of powerset already considered in [4, 8]. See the introduction and the final remark of Section 3 for a motivation of this definition.

Definition 2.5. The powerset of an *L*-set $Y : X \to L$ is the complete lattice

$$\mathcal{S}_Y = [\bot_X, Y] = \{A \in L^X \mid A \leq Y\}. \tag{2.6}$$

The well-known forward and backward *L*-powerset operators associated to a function $f : X \to T$ are denoted by

$$f_L^\to : L^X \longrightarrow L^T, \qquad f_L^\leftarrow : L^T \longrightarrow L^X. \tag{2.7}$$

They have been appropriately studied and justified in many papers by Rodabaugh. The fundamental properties that characterize such operators require

$$f_L^\to \in \bigvee\text{-}\mathbf{CSLat}(L^X, L^T), \quad f_L^\leftarrow \in \mathbf{CLat}(L^T, L^X), \quad f_L^\to \dashv f_L^\leftarrow. \tag{2.8}$$

In case of $L = \mathbf{2}$, they are denoted simply by $f^\to$ and $f^\leftarrow$ the classical powerset operators of f.

The origin and development of arrow notation for the powerset operators is described in detail in [16].

Set has been used as the ground category supporting fixed-basis fuzzy set theory and topology with crisp objects only, independently from the lattice basis; in fact, both the category of *L*-topological spaces *L*-**Top** and the category of *M*-fuzzy *L*-topological spaces (L, M)-**Top** are topological over **Set**.

Set $\times$ **C**, **C** a suitable subcategory of **CQML** (see [14, 15]) or, in particular, a suitable subcategory of **SLoc** (see [13]) are the ground categories used in the variable-basis *L*-topological and fuzzy *L*-topological space theory.

All along the referenced work of Rodabaugh, no ground category and no (fuzzy) topological category have been considered with objects associated to noncrisp *L*-sets. This instead has been done in some other papers (see [4, 8, 9]) in the fixed-basis case, originating from [3, 7]. Ground categories with noncrisp objects, powersets of their objects, and powerset operators have been defined in different ways all of which extend the crisp-object case. We refer mainly to the categories $\mathscr{A}L$-**Set**, $\mathscr{E}L$-**Set**, and $\mathscr{G}I$-**Set** with powersets and powerset operators introduced in [8] and reformulated and studied with more detail in [4]. Examples of their application to *L*-topological and *M*-fuzzy *L*-topological spaces are given in [5, 9].

In this paper, mainly in Section 3, we extend and generalize the construction of power-set operators already defined in $\mathscr{G}I$-**Set** [3, 4, 8], where the multiplicative structure of the unit interval was heavily used. The powerset operators, which are in any case associated to functions between sets, are defined now in a more general context depending on a fixed base lattice with a structure that determines the allowed ground categories on that lattice. Lattices with suitable structures characterize $\mathscr{A}L$-**Set** (see Example 3.9) and $\mathscr{G}L$-**Set** (see Example 3.10) (the latter one extends and slightly modifies $\mathscr{G}I$-**Set** presented in [3, 8]) as "good" ground categories. Sufficiently large structured lattices allow a variable-basis-like approach to fuzzy set theories (see Section 4).

We remark that the term "ground category" has been informally used up till now to denote categories of objects that can support a topological space theory. In Section 3, we are going to give an explicit definition of ground category that suits our work and includes most previously considered contexts.

3. Structured lattices and ground categories

Definition 3.1. A structured lattice is a pair

$$(L, \Phi), \tag{3.1}$$

where L is a complete lattice and $\Phi = \{\varphi_a\}_{a \in L}$ is a family of $\bigwedge$-complete semilattice morphisms

$$\varphi_a : L \longrightarrow [\bot, a], \quad \forall a \in L. \tag{3.2}$$

For every $a \in L$ and $\varphi_a \in \Phi$, the left adjoint of φ_a, $\varphi_a^{\dashv} : [\bot, a] \to L$, is of course a $\bigvee$-complete semilattice morphism.

Definition 3.2. Let (L, Φ) be a structured lattice and B any L-set, $B \in L^X$.

The compression operator on the powerset $\mathscr{S}_B$ (or simply on B) is the map

$$p_B : L^X \longrightarrow \mathscr{S}_B \tag{3.3}$$

defined, for all $A \in L^X$, for all $x \in X$ by

$$p_B(A)(x) = \varphi_{B(x)}(A(x)). \tag{3.4}$$

The lifting operator from the powerset $\mathscr{S}_B$ (or simply from B) is the map

$$l_B : \mathscr{S}_B \longrightarrow L^X \tag{3.5}$$

defined, for all $C \in \mathscr{S}_B$, for all $x \in X$ by

$$l_B(C)(x) = \varphi_{B(x)}^{\dashv}(C(x)). \tag{3.6}$$

Remark 3.3. Note that from $C(x) = \bot$, $x \in X$, it follows that $l_B(C)(x) = \bot$, since $\varphi_{B(x)}^{\dashv}$ preserves $\bigvee$. On the other hand, it is possible that $p_B(A)(x) \neq \bot$ for some $A \in L^X$ and $A(x) = \bot$ (see Example 3.11).

PROPOSITION 3.4. *If (L, Φ) is a structured lattice and $B \in L^X$ then,*
   (1) *p_B is a $\bigwedge$-complete semilattice morphism;*
   (2) *l_B is a $\bigvee$-complete semilattice morphism;*
   (3) *$l_B \dashv p_B$.*

Proof. For the proof, apply pointwisely the properties of φ_a and $\varphi_a^{\dashv}$, for all $a \in L$.   □

Definition 3.5. Let (L, Φ) be a structured lattice. For any two L-sets $Y \in L^X$, $Z \in L^T$ and for any

$$f : X \longrightarrow T, \tag{3.7}$$

the forward powerset operator of f from Y to Z with respect to (L, Φ)

$$f_{(L,\Phi)(Y,Z)}^{\rightarrow} : \mathscr{S}_Y \longrightarrow \mathscr{S}_Z \tag{3.8}$$

and the backward powerset operator of f from Y to Z with respect to (L, Φ)

$$f_{(L,\Phi)(Y,Z)}^{\leftarrow} : \mathscr{S}_Z \longrightarrow \mathscr{S}_Y \tag{3.9}$$

are the isotone maps defined, respectively, by

$$f_{(L,\Phi)(Y,Z)}^{\rightarrow} = p_Z \circ f_L^{\rightarrow} \circ l_Y, \qquad f_{(L,\Phi)(Y,Z)}^{\leftarrow} = p_Y \circ f_L^{\leftarrow} \circ l_Z. \tag{3.10}$$

Definition 3.6. Let (L, Φ) be a structured lattice.
   A ground category on (L, Φ) is a concrete category **C** whose objects are L-sets and morphisms $f \in \mathbf{C}(Y, Z)$, for all $Y, Z \in |\mathbf{C}|$, $Y \in L^X$, $Z \in L^T$, are maps

$$f : X \longrightarrow T \tag{3.11}$$

such that

$$f_{(L,\Phi)(Y,Z)}^{\leftarrow} \in \mathbf{CLat}(\mathscr{S}_Z, \mathscr{S}_Y), \quad f_{(L,\Phi)(Y,Z)}^{\rightarrow} \dashv f_{(L,\Phi)(Y,Z)}^{\leftarrow}. \tag{3.12}$$

   Clearly, the forward powerset operator of any morphism in any ground category on (L, Φ) is a $\bigvee$-complete semilattice morphism.
   Obviously, if **C** is a ground category and $\mathbf{A} \subseteq \mathbf{C}$ is a subcategory, then **A** is a ground category too.

Definition 3.7. Let (L, Φ) be a structured lattice. The standard ground category on (L, Φ), if it exists, is a ground category **D** on (L, Φ) that contains as a subcategory any ground category **C** on (L, Φ).

PROPOSITION 3.8. *For any structured lattice (L, Φ), the following hold.*
   (1) *A ground category exists whose objects are all the L-sets if and only if φ_a is surjective, for all $a \in L$.*
   (2) *If φ_a is surjective, for all $a \in L$, then the possible standard ground category must contain all the L-sets as objects.*

Proof. If $Y \in L^X$ is any L-set, then $(i_X)^{\rightarrow}_{(L,\Phi)(Y,Y)} = p_Y \circ l_Y = (i_X)^{\leftarrow}_{(L,\Phi)(Y,Y)}$.

Hence the concrete category whose objects are all L-sets and whose morphisms are the identity functions is a ground category if and only if $p_Y \circ l_Y$ is a complete lattice morphism and it is selfadjoint, for every L-set $Y \in L^X$.

By Proposition 2.4, this implies that the condition $p_Y \circ l_Y \circ p_Y \circ l_Y = i_{\mathcal{S}_Y}$, for all $Y \in L^X$ and, equivalently, $\varphi_a \circ (\varphi_a^{\dashv} \circ \varphi_a \circ \varphi_a^{\dashv}) = i_{[\perp,a]}$, for all $a \in L$, hold, which means that φ_a is surjective, for all $a \in L$.

Conversely, it follows from the assumption and from Proposition 2.3 that $\varphi_a \circ \varphi_a^{\dashv} = i_{[\perp,a]}$, which ensures that $p_Y \circ l_Y = i_{\mathcal{S}_Y}$ is a complete lattice morphism and it is selfadjoint, so by considering the identity morphisms only, we get a ground category on (L,Φ).

The statement (1) is now evident and the statement (2) is then a consequence.   □

Example 3.9. If L is a complete lattice and for all $a \in L$, $\varphi_a : L \to [\perp,a]$ is defined by

$$\varphi_a(x) = a \wedge x, \quad \forall x \in L, \tag{3.13}$$

then the pair $(L, \{\varphi_a\}_{a \in L})$ is a structured lattice that we call cutting lattice.

The left adjoint morphisms $\varphi_a^{\dashv} : [\perp,a] \to L$ of φ_a are clearly the inclusion maps and if $B \in L^X$ is any L-set, the compression operator on B and the lifting operator from B are defined, respectively, by

$$p_B(A) = B \wedge A, \quad \forall A \in L^X,$$
$$l_B(C) = C, \quad \forall C \in \mathcal{S}_B. \tag{3.14}$$

This structured lattice is strictly connected with the category $\mathscr{A}L$-**Set** that has been considered in [4, 8] under the assumption that L is a completely distributive lattice; indeed, a frame structure on L has been shown in [9] to be enough. We recall that the objects of $\mathscr{A}L - \mathbf{Set}$ are the L-sets and if $Y \in L^X$, $Z \in L^T$, then

$$f \in \mathscr{A}L\text{-}\mathbf{Set}(Y,Z) \quad \text{iff } f \in \mathbf{Set}(X,T), \quad Y \leq Z \circ f. \tag{3.15}$$

It is not difficult to see that the powerset operators $f^{\rightarrow}_{(L,\Phi)(Y,Z)}$, $f^{\leftarrow}_{(L,\Phi)(Y,Z)}$ of these morphisms coincide with the powerset operators $f^{\rightarrow}_{\mathscr{A}}$, $f^{\leftarrow}_{\mathscr{A}}$ associated to f, as they are defined in [4, 8]. In fact, for all $A \in \mathcal{S}_Y$ and for all $t \in T$,

$$f^{\rightarrow}_{(L,\Phi)(Y,Z)}(A)(t) = \bigvee \{A(x) \mid x \in X : f(x) = t\} = f^{\rightarrow}_{\mathscr{A}}(A)(t). \tag{3.16}$$

On the other hand, for all $B \in \mathcal{S}_Z$ and for all $x \in X$,

$$f^{\leftarrow}_{(L,\Phi)(Y,Z)}(B)(x) = Y(x) \wedge B(f(x)) = f^{\leftarrow}_{\mathscr{A}}(B)(x). \tag{3.17}$$

Already known properties of $f^{\rightarrow}_{\mathscr{A}}$ and $f^{\leftarrow}_{\mathscr{A}}$ (see [4, 9]) show that $\mathscr{A}L$-**Set** is a ground category, according to Definition 3.6.

We will see as a consequence of Proposition 5.10 that $\mathscr{A}L$-**Set** is the standard ground category on $(L, \{a \wedge *\}_{a \in L})$ if and only if L is a frame.

Example 3.10. Let $I = [0,1]$ and for all $a \in I$, let $\varphi_a : I \to [0,a]$ be defined by

$$\varphi_a(x) = a \cdot x, \quad \forall x \in I. \tag{3.18}$$

We note that for every $a \neq 0$, φ_a is a complete lattice isomorphism, hence its left adjoint is the inverse isomorphism. We call homogeneous lattice every structured lattice such as φ_a is an isomorphism, for all $a \neq \perp$.

The structured lattice $(I, \{a \cdot *\}_{a \in I})$ is strictly connected with the category $\mathscr{G}I$-**Set** considered in [4, 8].

We recall that the objects of $\mathscr{G}I$-**Set** are all the I-sets and the morphisms $f \in \mathscr{G}I$-**Set**(Y, Z), from $Y \in L^X$ to $Z \in L^T$, are all the maps $f : Y_0 \to Z_0$ which moreover determine powerset operators

$$f_{\mathscr{G}}^{\to} : \mathscr{S}_Y \longrightarrow \mathscr{S}_Z, \qquad f_{\mathscr{G}}^{\leftarrow} : \mathscr{S}_Z \longrightarrow \mathscr{S}_Y \tag{3.19}$$

whose definition, since [3] through [4, 8], partly suggested and motivated the present paper. In fact, the compression and the lifting operators of Definition 3.2 as they are determined in case of this example were indirectly used in [3] (see also [4, 8]) and in fact, by also using notation of [4, 8], we have

$$
\begin{aligned}
p_B(A)(x) &= A(x) \cdot B(x) = (A \cdot B)(x) \quad \forall A, B \in I^X, \ \forall x \in X, \\
l_B(C)(x) &= \frac{C(x)}{B(x)} = (C \div B)(x) \quad \forall B \in I^X, \ \forall C \in \mathscr{S}_B, \ \forall x \in B_\perp.
\end{aligned}
\tag{3.20}
$$

Turning to the general approach we are now considering, we remark that $\mathscr{G}I$-**Set**, as already considered in [4, 8], is not a ground category in the sense of Definition 3.6 but the full subcategory **C** of $\mathscr{G}I$-**Set** with objects all the I-sets $Y \in I^X$, for $X \in |$**Set**$|$, satisfying the condition $Y_0 = X$ is a ground category on $(I, \{a \cdot *\}_{a \in I})$.

Example 3.11. Let L be a de Morgan frame with order-reversing involution κ, and for all $a \in L$, consider $\varphi_a : L \to [\perp, a]$ defined by

$$\varphi_a(x) = (a \wedge x) \vee (a \wedge \kappa(a)) = a \wedge (x \vee \kappa(a)), \quad \forall x \in X. \tag{3.21}$$

Then $(L, \Phi = \{\varphi_a\}_{a \in L})$ is a structured lattice, since every φ_a preserves arbitrary meets, as it can be easily seen.

The left adjoint morphisms are defined, for all $x \leq a$, by

$$\varphi_a^{\dashv}(x) = \kappa\left(\bigvee \{z \in L \mid z \wedge a \leq \kappa(x)\} \right) = \kappa(a \longrightarrow \kappa(x)), \tag{3.22}$$

where $\to$ is the implication operation of the Heyting algebra L.

We note that unlike in the Examples 3.9 and 3.10, in this case, the morphisms φ_a may not preserve the lower bound $\perp$; in fact $\varphi_a(\perp) = a \wedge \kappa(a)$. In particular, when $a \leq \kappa(a)$, φ_a is the constant map with value a.

In some sense, this example provides a pointset approach to fuzzy topology alternative to that given by Erceg [6] and further considered in [4, 5, 8] where the topological category $\mathscr{E}L$-**Set** with pointless characterization of morphisms is described. A more general approach than in [6] has been considered in [2]: structured lattices could provide a pointset version of such an approach too.

Example 3.12. There are only two ways for the trivial lattice **2** to be considered as a structured lattice. The first one gives $(\mathbf{2}, \{i_2\})$, the second one produces the structured lattice $(\mathbf{2}, \{\top_2\})$, where, of course, i_2 is the identity map and $\top_2$ is the constant map with value $\top$. In order to simplify notation, we denote these structured lattices by $(\mathbf{2}, i_2)$ and $(\mathbf{2}, \top_2)$.

If we identify any element of $\mathbf{2}^X$ with its support, then the compression and the lifting operators related to any $B \in \mathbf{2}^X$, that is, $B \subseteq X$, in $(\mathbf{2}, i_2)$ are defined for all $A \subseteq X$, for all $C \subseteq B$ by

$$p_B(A) = A \cap B, \qquad l_B(C) = C. \tag{3.23}$$

In $(\mathbf{2}, \top_2)$, the analogous operators are defined, for all $A \subseteq X$, for all $C \subseteq B$, by

$$p_B(A) = B, \qquad l_B(C) = \varnothing \tag{3.24}$$

which contradicts, in the common sense, the names of these operators.

In $(\mathbf{2}, i_2)$, the powerset operators of a function $f : X \to T$ relative to subsets $Y \subseteq X$ and $Z \subseteq T$ are defined, for all $A \in \mathscr{P}(Y)$ and for all $B \in \mathscr{P}(Z)$, by

$$f_{(\mathbf{2},i_2)(Y,Z)}^{\to}(A) = f^{\to}(A) \cap Z, \qquad f_{(\mathbf{2},i_2)(Y,Z)}^{\leftarrow}(B) = f^{\leftarrow}(B) \cap Y. \tag{3.25}$$

As a consequence, one can verify that a concrete category whose objects are all **2**-sets is a ground category on $(\mathbf{2}, i_2)$ if and only if the morphisms from $Y \subseteq X$ to $Z \subseteq T$ are functions from X to T that are extensions of maps from Y to Z.

We also note that the classical powerset operators of any map from Y to Z coincide with the powerset operators in $(\mathbf{2}, i_2)$ of every one of its extensions if $\mathscr{P}(Y)$, $\mathscr{P}(Z)$ are identified, via restriction, with $\mathscr{S}_Y$ and $\mathscr{S}_Z$, respectively.

In $(\mathbf{2}, \top_2)$, the powerset operators of a function $f : X \to T$ relative to subsets $Y \subseteq X$ and $Z \subseteq T$ are determined for all $A \in \mathscr{P}(Y)$ and for all $B \in \mathscr{P}(Z)$ by

$$f_{(\mathbf{2},\top_2)(Y,Z)}^{\to}(A) = Z, \qquad f_{(\mathbf{2},\top_2)(Y,Z)}^{\leftarrow}(B) = Y. \tag{3.26}$$

Evidently, both $f_{(\mathbf{2},\top_2)(Y,Z)}^{\to}$ and $f_{(\mathbf{2},\top_2)(Y,Z)}^{\leftarrow}$ preserve $\bigwedge$, but they do not necessarily preserves $\bigvee$.

Moreover, (ADI) holds but evidently (ADII) does not hold.

Let (L, Φ) be a structured lattice. Then the mapping to relation is defined in L as follows.

Definition 3.13. Let $a, b \in L$. Then say that a maps to b, and write

$$a \nearrow b \tag{3.27}$$

if $\varphi_a^{-1}(a) \leq \varphi_b^{-1}(b)$.

Remark 3.14. Clearly, $\nearrow$ is a preorder relation. The mapping to relation in the structured lattice of Example 3.9 is the order relation $\leq$ of the lattice L.

However, the mapping to relation $\nearrow$ need not be an order relation as Example 3.10 shows.

We further note that $\perp \nearrow a$, for all $a \in L$, and $a \nearrow \perp$ if and only if $\varphi_a(\perp) = a$.

The following characterization will be useful.

PROPOSITION 3.15. *If (L, Φ) is a structured lattice and $a, b \in L$, then*

$$a \nearrow b \iff \varphi_a \circ \varphi_b^{-1}(b) = a. \tag{3.28}$$

Proof. It follows from $a = \varphi_a \circ \varphi_b^{-1}(b)$ that $\varphi_a^{-1}(a) = \varphi_a^{-1} \circ \varphi_a \circ \varphi_b^{-1}(b) \leq \varphi_b^{-1}(b)$.

Conversely, if $a \nearrow b$, then $a \leq \varphi_a \circ \varphi_a^{-1}(a) \leq \varphi_a(\varphi_b^{-1}(b)) \leq a$. $\square$

The preorder relation $\nearrow$ allows the following definition to be stated.

Definition 3.16. Let (L, Φ) be a structured lattice. (L, Φ)-**Set** is the concrete category whose objects are all the L-sets and whose morphisms between the objects $Y \in L^X$ and $Z \in L^T$ are the maps

$$f : X \longrightarrow T \tag{3.29}$$

that satisfy the condition

$$Y(x) \nearrow Z(f(x)), \quad \forall x \in X. \tag{3.30}$$

Composition and identities are those of **Set**.

Remark 3.17. (1) If $\varphi_a(\perp) \neq a$, for all $a \neq \perp$, then for every $f \in (L, \Phi)$-**Set**(Y, Z), with $Y \in L^X$ and $Z \in L^T$, the following holds:

$$f^{\rightarrow}(Y_\perp) \subseteq Z_\perp. \tag{3.31}$$

In fact, it follows from $x \in Y_\perp$, $Y(x) \nearrow Z(f(x))$, and $Y(x) \neq \perp$ that $Z(f(x)) \neq \perp$ (see Remark 3.14).

(2) The mapping to relation determines, by means of Definition 3.16, all possible morphisms we can expect to be found in any ground category on a structured lattice, as the following theorem shows.

THEOREM 3.18. *Let (L, Φ) be a structured lattice. Then every ground category on (L, Φ) is a subcategory of (L, Φ)-**Set**.*

Proof. If **C** is a ground category on (L, Φ), then clearly, $|\mathbf{C}| \subseteq |(L, \Phi)$-**Set**$|$.

If $f \in \mathbf{C}(Y, Z)$, with $Y \in L^X$ and $Z \in L^T$, then $f_{(L, \Phi)(Y, Z)}^{\leftarrow} \in \mathbf{CLat}(\mathcal{S}_Z, \mathcal{S}_Y)$.

Now, for all $x \in X$,

$$Y(x) = f_{(L,\Phi)(Y,Z)}^{\leftarrow}(Z)(x) = \varphi_{Y(x)} \circ \varphi_{Z(f(x))}^{\rightarrow}(Z(f(x))). \tag{3.32}$$

Hence $Y(x) \nearrow Z(f(x))$.   □

COROLLARY 3.19. *If* (L,Φ)-**Set** *is a ground category on the structured lattice* (L,Φ)*, then* (L,Φ)-**Set** *is the standard ground category on* (L,Φ).

Remark 3.20. It may seem that a fundamental step of these constructions has been missed to be linked directly to the structured lattice; we mean the powerset of any L-set, $Y \in L^X$, which has been considered to be the interval $\mathscr{S}_Y = [\bot_X, Y]$ of the complete lattice L^X, independently of the structure Φ to be considered on the lattice L (see **Definition 2.5**).

Indeed, this is our choice motivated by the natural ordering giving a lattice structure on the set L^X and by the isotone, with respect to inclusion, correspondence mapping every L-set Y belonging to the L-powerset of X to its powerset $\mathscr{S}_Y$ as a subset of L^X. We also note that once we considered L^X as an ordered small category and $Y \in L^X$ as one of its objects, then the interval $[\bot_X, Y]$ is isomorphic to the lattice of all subobjects of Y in the category L^X and it is, moreover, a full subcategory of L^X.

Nevertheless, we note that once Definitions 3.1 and 3.2 are stated, one can see that in many cases (in particular for the well-structured lattice we will consider in the next sections) for any given L-set, $Y \in L^X$, the following equalities hold:

$$\mathscr{S}_Y = [\bot_X, Y] = [p_Y(\bot_X), p_Y(\top_X)]. \tag{3.33}$$

This would suggest an alternative definition for the powerset of $Y \in L^X$ depending on the structured lattice (L,Φ). In fact, the compression operator $p_Y : L^X \to L^X$ can be considered as it is done in **Definition 3.2**.

Then the (L,Φ)-powerset of Y can be considered to be the complete lattice

$$\Phi_Y = [p_Y(\bot_X), p_Y(\top_X)] \tag{3.34}$$

which has upper bound $p_Y(\top_X) = Y$ in any case and it coincides with $\mathscr{S}_Y$ if and only if $\varphi_a(\bot) = \bot$, for all $a \in L$.

Thanks to its isotone property, p_Y can be reduced to

$$\overline{p}_Y : L^X \longrightarrow \Phi_Y, \qquad A \longmapsto \overline{p}_Y(A) = p_Y(A) \tag{3.35}$$

and it can determine a lifting operator $\overline{l}_Y : \Phi_Y \to L^X$ defined by

$$\overline{l}_Y = \overline{p}_Y^{\rightarrow} : \Phi_Y \longrightarrow L^X. \tag{3.36}$$

It would be easily seen that $\bar{l}_Y$ could be obtained as in Definition 3.2 by using point-wisely the left adjoints $\overline{\varphi}_a^{-1}$ of the $\bigwedge$-preserving maps $\overline{\varphi}_a : L \to [\varphi_a(\bot), a]$ defined by $\overline{\varphi}_a(x) = \varphi_a(x)$, for all $x \in L$, for all $a \neq \bot$.

However, this alternative approach should somehow fulfill the requirement of allowing an isotone correspondence that associates to every L-set its powerset. For instance, this requirement would be satisfied with respect to inclusion in case of the structured lattice of Example 3.11 and in $(\mathbf{2}, \top_2)$ described in Example 3.12, the only examples we have presented in this paper for which one would have $\Phi_Y \neq \mathcal{S}_Y$.

We remark that this alternative approach would produce nothing different and nothing new in the class of well-structured lattices we are going to consider.

4. Well-structured lattices

Definition 4.1. A structured lattice $(L, \Phi = \{\varphi_a\}_{a \in L})$ is said to be well-structured if the following conditions are satisfied.

(a) $a \neq \bot \neq b$, $a \nearrow b \Rightarrow \varphi_a \circ \varphi_b^{-1}$ is a complete lattice morphism.

(b) $a \nearrow b \Rightarrow \varphi_b \circ \varphi_a^{-1}$ is the left adjoint of $\varphi_a \circ \varphi_b^{-1}$.

PROPOSITION 4.2. *If (L, Φ) is a structured lattice that satisfies condition* (a), *then* $\varphi_a(\bot) = \bot$, *for all $a \in L$.*

Proof. Clearly $\varphi_\bot^{-1}(\bot) = \bot$; if $a \neq \bot$, it follows from $a \nearrow a$ and condition (a) of Definition 4.1 that

$$\bot \leq \varphi_a(\bot) \leq \varphi_a\left(\varphi_a^{-1}(\bot)\right) = \bot. \tag{4.1}$$

$\square$

Example 4.3. If $I = [0, 1]$, for all $a \in I$, the map $\psi_a : I \to [0, a]$ defined for all $x \in I$ by

$$\psi_a(x) = \begin{cases} 2ax & \text{if } x \in \left[0, \dfrac{1}{2}\right), \\ a & \text{if } x \in \left[\dfrac{1}{2}, 1\right] \end{cases} \tag{4.2}$$

is a complete lattice morphism. For all $a, b \in I$, $a \neq 0 \neq b$, the composition $\psi_a \circ \psi_b^{-1}$ is defined for all $x \in [0, b]$ by

$$\psi_a \circ \psi_b^{-1}(x) = \frac{a}{b} x \tag{4.3}$$

and evidently it is a complete lattice isomorphism. This shows that every pair (a, b) with $a \neq 0 \neq b$ is in the relation $\nearrow$ and for these pairs, the condition (a) of well-structured lattices is satisfied.

Moreover, $\psi_b \circ \psi_a^{-1}$ is the inverse isomorphism, and consequently the left adjoint morphism of $\psi_a \circ \psi_b^{-1}$.

Since the condition (b) is satisfied in the trivial cases, too, we see that $(I, \Psi = \{\psi_a\}_{a \in I})$ is a well-structured lattice.

Example 4.4. Once more, let $I = [0,1]$ and for all $a \in I$, consider the map $\varphi_a : I \to [0,a]$ defined for all $x \in I$ by

$$\varphi_a(x) = \begin{cases} a & \text{if } x = 1, \\ \dfrac{a \cdot x}{2} & \text{if } x \neq 1. \end{cases} \tag{4.4}$$

It is clear that for all $a \in I$, φ_a preserves $\bigwedge$.

Let $a, b \in I$, $b \neq 0$, then $\varphi_a \circ \varphi_b^{\dashv}$ is defined for all $x \in [0,b]$ by

$$\varphi_a \circ \varphi_b^{\dashv}(x) = \begin{cases} \varphi_a\left(\dfrac{a \cdot x}{b}\right) = \dfrac{2x}{b} \cdot \dfrac{a}{2} = \dfrac{a}{b} \cdot x & \text{if } x \in \left[0, \dfrac{b}{2}\right), \\ \varphi_a(1) = a & \text{if } x \in \left[\dfrac{b}{2}, b\right], \end{cases} \tag{4.5}$$

and it preserves $\bigwedge$, but it does not preserve $\bigvee$.

Nevertheless $\varphi_a \circ \varphi_b^{\dashv}(b) = a$, hence $a \nearrow b$, for all $b \neq 0$.

So, the structured lattice $(I, \Phi = \{\varphi_a\}_{a \in I})$ is not well structured; indeed neither the condition (a) nor (b) is satisfied, as it can be easily verified.

Example 4.5. Let L be a complete lattice and let $\overline{\varphi}_a : [\bot, a] \to [\bot, a]$ be a complete lattice isomorphism, for all $a \in L$, $a \neq \bot$, and let $\overline{\varphi}_\bot(x) = \bot$, for all $x \in L$.

For all $a \in L$, let $\varphi_a : L \to [\bot, a]$ be defined for all $x \in L$ by

$$\varphi_a(x) = \overline{\varphi}_a(a \wedge x). \tag{4.6}$$

For all $a \in L$, φ_a preserves $\bigwedge$, and if $a \neq \bot$, then $\varphi_a^{\dashv}(x) = \overline{\varphi}_a^{-1}(x)$, for all $x \in [\bot, a]$. Consequently, we see that $a \nearrow b \Leftrightarrow a \leq b$, for all $a, b \in L$.

Moreover one can see that for all $a \leq b$, the condition $\varphi_b \circ \varphi_a^{\dashv} \dashv \varphi_a \circ \varphi_b^{\dashv}$ is satisfied.

As for the condition (a) of Definition 4.1, we note that for all $\{b_j \mid j \in J\} \subseteq L$, the equalities

$$\varphi_a\left(\bigvee_{j \in J} b_j\right) = \overline{\varphi}_a\left(\left(\bigvee_{j \in J} b_j\right) \wedge a\right), \qquad \bigvee_{j \in J} \varphi_a(b_j) = \overline{\varphi}_a\left(\bigvee_{j \in J}(b_j \wedge a)\right) \tag{4.7}$$

hold, so clearly φ_a preserves $\bigvee$, for all $a \neq \bot$, if and only if L is a frame.

As a consequence $\varphi_a \circ \varphi_b^{\dashv}$ is a complete lattice morphism for all $\bot < a \leq b$ if and only if L is a frame.

We conclude that $(L, \Phi = \{\varphi_a\}_{a \in I})$ is well-structured if and only if L is a frame.

We also note that the structured lattice of Example 3.9 is a well-structured lattice if and only if L is a frame, too. In fact, it can be obtained as a special case of this one, when $\overline{\varphi}_a = i_{[\bot, a]}$, for all $a \neq \bot$.

LEMMA 4.6. *If (L, Φ) is a structured lattice, $f \in (L, \Phi)$-$\mathbf{Set}(Y, Z)$, with $Y \in L^X$ and $Z \in L^T$, and $\varphi_{Y(x)} \circ \varphi_{Z(f(x))}^{\dashv}$ is a complete lattice morphism, for all $x \in Y_\bot$, then $f_{(L,\Phi)(Y,Z)}^{\leftarrow}$ is a complete lattice morphism.*

Proof. If $\{B_j\}_{j \in J} \subseteq \mathscr{S}_Z$ and $x \in X$, then if $x \in Y_\perp$,

$$f_{(L,\Phi)(Y,Z)}^{\leftarrow}\left(\bigvee_{j \in J} B_j\right)(x) = \varphi_{Y(x)} \circ \varphi_{Z(f(x))}^{\rightarrow}\left(\bigvee_{j \in J} B_j(f(x))\right)$$

$$= \bigvee_{j \in J}\left(\varphi_{Y(x)} \circ \varphi_{Z(f(x))}^{\rightarrow}(B_j(f(x)))\right) \tag{4.8}$$

$$= \left(\bigvee_{j \in J} f_{(L,\Phi)(Y,Z)}^{\leftarrow}(B_j)\right)(x).$$

If $x \notin Y_\perp$, then trivially

$$f_{(L,\Phi)(Y,Z)}^{\leftarrow}\left(\bigvee_{j \in J} B_j\right)(x) = \perp = \left(\bigvee_{j \in J} f_{(L,\Phi)(Y,Z)}^{\leftarrow}(B_j)\right)(x). \tag{4.9}$$

Similarly, it can be proved that $f_{(L,\Phi)(Y,Z)}^{\leftarrow}$ preserves $\bigwedge$. □

Lemma 4.7. *If (L,Φ) is a structured lattice, $f \in (L,\Phi)$-$\mathbf{Set}(Y,Z)$, with $Y \in L^X$ and $Z \in L^T$, and $\varphi_{Z(f(x))} \circ \varphi_{Y(x)}^{\rightarrow} \dashv \varphi_{Y(x)} \circ \varphi_{Z(f(x))}^{\rightarrow}$, for all $x \in X$, then the following adjoint inequality (ADI) holds:*
(ADI)

$$f_{(L,\Phi)(Y,Z)}^{\leftarrow} \circ f_{(L,\Phi)(Y,Z)}^{\rightarrow} \geq i_{\mathscr{S}_Y}. \tag{4.10}$$

Proof. If $A \in \mathscr{S}_Y$ and $x \in X$, then

$$\left(f_{(L,\Phi)(Y,Z)}^{\leftarrow} \circ f_{(L,\Phi)(Y,Z)}^{\rightarrow}\right)(A)(x)$$

$$= \varphi_{Y(x)} \circ \varphi_{Z(f(x))}^{\rightarrow} \circ \varphi_{Z(f(x))}\left(\bigvee\{l_Y(A)(x') \mid x' \in Y_\perp : f(x') = f(x)\}\right) \tag{4.11}$$

$$\geq \varphi_{Y(x)} \circ \varphi_{Z(f(x))}^{\rightarrow} \circ \varphi_{Z(f(x))} \circ \varphi_{Y(x)}^{\rightarrow}(A(x)) \geq A(x). \qquad \square$$

Lemma 4.8. *If (L,Φ) is a structured lattice, $f \in (L,\Phi)$-$\mathbf{Set}(Y,Z)$, with $Y \in L^X$ and $Z \in L^T$, $\varphi_{Z(f(x))} \circ \varphi_{Y(x)}^{\rightarrow} \dashv \varphi_{Y(x)} \circ \varphi_{Z(f(x))}^{\rightarrow}$, for all $x \in X$ and moreover the conditions*

$$\varphi_{Z(f(x))}|_{\bigvee -<\bigcup\{(\varphi_{Y(x')}^{\rightarrow})^{\rightarrow}([\perp,Y(x')])|x' \in Y_\perp : f(x')=f(x)\}>} \text{ preserves } \bigvee, \quad \forall x \in Y_\perp,$$

$$\varphi_{Z(t)}(\perp) = \perp, \quad \forall t \in Z_\perp \tag{4.12}$$

are satisfied, then the following adjoint inequality (ADII) holds:
(ADII)

$$f_{(L,\Phi)(Y,Z)}^{\rightarrow} \circ f_{(L,\Phi)(Y,Z)}^{\leftarrow} \leq i_{\mathscr{S}_Z}. \tag{4.13}$$

Proof. If $B \in \mathscr{S}_Z$ and $t \in T$, then

$$\left(f_{(L,\Phi)(Y,Z)}^{\rightarrow} \circ f_{(L,\Phi)(Y,Z)}^{\leftarrow}\right)(B)(t) = \varphi_{Z(t)}(f_L^{\leftarrow}(l_Y \circ p_Y \circ f_L^{\leftarrow} \circ l_Z(B))(t))$$

$$= \varphi_{Z(t)}\left(\bigvee\{l_Y \circ p_Y \circ f_L^{\leftarrow} \circ l_Z(B)(x') \mid x' \in Y_\perp : f(x') = t\}\right) \tag{4.14}$$

if $t \in f^{\rightarrow}(Y_\perp)$, then $x \in Y_\perp$ exists such that $f(x) = t$ and by the assumption, it follows

$$
\begin{aligned}
\left(f_{(L,\Phi)(Y,Z)}^{\rightarrow} \circ f_{(L,\Phi)(Y,Z)}^{\leftarrow} \right)(B)(t) \\
= \varphi_{Z(f(x))}\left(\bigvee \{ \varphi_{Y(x')}^{\rightarrow} \circ \varphi_{Y(x')} \circ \varphi_{Z(f(x'))}^{\rightarrow}(B(f(x'))) \mid x' \in Y_\perp : f(x') = t \} \right) \\
= \bigvee \left\{ \left(\varphi_{Z(f(x))} \circ \varphi_{Y(x')}^{\rightarrow} \right) \circ \left(\varphi_{Y(x')} \circ \varphi_{Z(f(x'))}^{\rightarrow} \right)(B(f(x'))) \mid x' \in Y_\perp : f(x') = t \right\} \\
\le \bigvee \{ B(t) \} = B(t)
\end{aligned}
$$

(4.15)

if $t \notin f^{\rightarrow}(Y_\perp)$, since $\varphi_{Z(t)}(\perp) = \perp$,

$$
\left(f_{(L,\Phi)(Y,Z)}^{\rightarrow} \circ f_{(L,\Phi)(Y,Z)}^{\leftarrow} \right)(B)(t) = \varphi_{Z(t)}\left(\bigvee \varnothing \right) = \varphi_{Z(t)}(\perp) = \perp \le B(t). \qquad (4.16)
$$

$\square$

As a trivial consequence of the preceding lemmas, thanks to the adjoint functor theorem, the following lemma holds.

LEMMA 4.9. *If (L,Φ) is a structured lattice, $f \in (L,\Phi)$-$\mathbf{Set}(Y,Z)$, with $Y \in L^X$ and $Z \in L^T$ and the assumptions of Lemmas 4.6 and 4.8 are satisfied, then $f_{(L,\Phi)(Y,Z)}^{\leftarrow}$ preserves $\bigvee$.*

PROPOSITION 4.10. *If (L,Φ) is a well-structured lattice, $\mathbf{C}$ is a subcategory of (L,Φ)-$\mathbf{Set}$ and for every $f \in \mathbf{C}(Y,Z)$, with $Y \in L^X$ and $Z \in L^T$, the condition*

$$
\varphi_{Z(f(x))}|_{\bigvee - <\bigcup \{ (\varphi_{Y(x')}^{\rightarrow})^{\rightarrow}([\perp, Y(x')]) \mid x' \in Y_\perp : f(x') = f(x) \} >} \text{ preserves } \bigvee, \quad \forall x \in Y_\perp \qquad (4.17)
$$

is satisfied, then $\mathbf{C}$ is a ground category on (L,Φ).

Proof. The proof follows from Lemmas 4.6, 4.7, 4.8 by also considering Proposition 4.2. $\square$

COROLLARY 4.11. *If (L,Φ) is a well-structured lattice and if for all $a \in L$, $a \ne \perp$, φ_a preserves $\bigvee$, then (L,Φ)-$\mathbf{Set}$ is the standard ground category on (L,Φ).*

Proof. Clearly, since for all $a \in L$, $a \ne \perp$, φ_a preserves $\bigvee$, then for all $f \in (L,\Phi)$-$\mathbf{Set}(Y,Z)$, with $Y \in L^X$ and $Z \in L^T$, the conditions of Proposition 4.10 are satisfied; then the statement follows from Corollary 3.19. $\square$

We recall (see Proposition 3.8) that for every structured lattice (L,Φ), the surjectivity condition of every $\varphi_a \in \Phi$ ensures the existence of a ground category on (L,Φ) with the largest class of objects.

In case (L,Φ) is well structured, the surjectivity condition on the maps $\varphi_a \in \Phi$ allows the largest choice (in the sense of Theorem 3.18) of morphisms too, as the following theorem shows.

THEOREM 4.12. *If (L,Φ) is a well-structured lattice and if for all $a \in L$, φ_a is surjective, then (L,Φ)-$\mathbf{Set}$ is the standard ground category on (L,Φ).*

Proof. For all $f \in (L,\Phi)\text{-}\mathbf{Set}(Y,Z)$, with $Y \in L^X$ and $Z \in L^T$, since (L,Φ) is a well-structured lattice, by Lemmas 4.6 and 4.7, it follows that $f^{-}_{(L,\Phi)(Y,Z)}$ is a complete lattice morphism and the adjoint inequality (ADI) holds. To prove the adjoint inequality (ADII), we consider any $B \in \mathscr{S}_Z$ and any $t \in T$.

Then

$$
\left(f^{\rightarrow}_{(L,\Phi)(Y,Z)} \circ f^{-}_{(L,\Phi)(Y,Z)}\right)(B)(t)
$$
$$
= \varphi_{Z(t)}\left(\bigvee\{l_Y \circ p_Y \circ f_L^{-} \circ l_Z(B)(x) \mid x \in Y_{\perp} : f(x) = t\}\right).
$$

(4.18)

If $t \in f^{\rightarrow}(Y_{\perp})$ and $t = f(x)$, with $x \in Y_{\perp}$, then

$$
\left(f^{\rightarrow}_{(L,\Phi)(Y,Z)} \circ f^{-}_{(L,\Phi)(Y,Z)}\right)(B)(t)
$$
$$
= \varphi_{Z(f(x))}\left(\bigvee\{\varphi^{\rightarrow}_{Y(x')} \circ \varphi_{Y(x')} \circ \varphi^{\rightarrow}_{Z(f(x'))}(B(f(x'))) \mid x' \in Y_{\perp} : f(x') = t\}\right)
$$
$$
\leq \varphi_{Z(f(x))}\left(\bigvee\{\varphi^{\rightarrow}_{Z(f(x'))}(B(f(x'))) \mid x' \in Y_{\perp} : f(x') = t\}\right)
$$
$$
= \varphi_{Z(t)} \circ \varphi^{\rightarrow}_{Z(t)}(B(t)) = B(t).
$$

(4.19)

If $t \notin f^{\rightarrow}(Y_{\perp})$, since $\varphi_{Z(t)}(\perp) = \perp$, then

$$
\left(f^{\rightarrow}_{(L,\Phi)(Y,Z)} \circ f^{-}_{(L,\Phi)(Y,Z)}\right)(B)(t) = \varphi_{Z(t)}\left(\bigvee \varnothing\right) = \varphi_{Z(t)}(\perp) = \perp \leq B(t).
$$

(4.20)

Then $(L,\Phi)\text{-}\mathbf{Set}$ is a ground category and the statement follows from Corollary 3.19.

$\square$

5. Comparison with other approaches

Definition 5.1. Let (L,Φ) be a structured lattice. Denote by $(L,\Phi)\text{-}\mathbf{Flat}$ the full subcategory of $(L,\Phi)\text{-}\mathbf{Set}$ whose objects are all the constant L-sets.

Remark 5.2. (1) Let $\alpha_X, \beta_T \in |(L,\Phi)\text{-}\mathbf{Flat}|$. Then $(L,\Phi)\text{-}\mathbf{Flat}(\alpha_X, \beta_T) = \mathbf{Set}(X,T)$ if $\alpha \nearrow \beta$, otherwise $(L,\Phi)\text{-}\mathbf{Flat}(\alpha_X, \beta_T) = \varnothing$.

(2) If $\alpha_X \in |(L,\Phi)\text{-}\mathbf{Flat}|$, then for all $B \in L^X$, for all $C \in \mathscr{S}_{\alpha_X}$, for all $x \in X$, the following hold:

$$
p_{\alpha_X}(B)(x) = \varphi_{\alpha}(B(x)), \qquad l_{\alpha_X}(C)(x) = \varphi^{\rightarrow}_{\alpha}(C(x)).
$$

(5.1)

The following result and Remark 5.4 seem to be a good justification of the notion of well-structured lattice.

THEOREM 5.3. $(L,\Phi)\text{-}\mathbf{Flat}$ *is a ground category on the structured lattice* (L,Φ) *if and only if* (L,Φ) *is well structured.*

Proof. If (L,Φ) is well structured and a morphism $f \in (L,\Phi)\text{-}\mathbf{Flat}(\alpha_X, \beta_T)$ exists, then $\alpha \nearrow \beta$ and the conditions required in Proposition 4.10 are satisfied.

In particular, the restriction $\varphi_{\beta|(\varphi_\alpha^\leftarrow)^\rightarrow([\bot,\alpha])}$ preserves $\bigvee$, since $\varphi_\alpha^\leftarrow$ preserves $\bigvee$ and so $\varphi_\beta \circ \varphi_\alpha^\leftarrow$ does, by condition (b) of Definition 4.1. Now the claimed sufficiency follows from Proposition 4.10.

Conversely, assume that (L,Φ)-**Flat** is a ground category on (L,Φ). We consider any singleton $\{x\}$ and any element $\alpha \in L$, $\alpha \neq \bot$; then $\alpha_x \in |(L,\Phi)-\textbf{Flat}|$.

Consider the complete lattice isomorphism

$$\chi_\alpha : \mathscr{S}_{\alpha_x} \longrightarrow [\bot,\alpha] \tag{5.2}$$

defined for all $\delta_x \in \mathscr{S}_{\alpha_x}$ by $\chi_\alpha(\delta_x) = \delta_x(x) = \delta$.

If $\alpha \neq \bot$ and $\alpha \nearrow \beta$, then the identity map $i : \{x\} \to \{x\}$ is a morphism from α_x to β_x in (L,Φ)-**Flat**, and the following equalities hold:

$$\begin{aligned}
\chi_\beta \circ i^\rightarrow_{(L,\Phi)(\alpha_x,\beta_x)} \circ \chi_\alpha^{-1} &= \varphi_\beta \circ \varphi_\alpha^\leftarrow, \\
\chi_\alpha \circ i^\leftarrow_{(L,\Phi)(\alpha_x,\beta_x)} \circ \chi_\beta^{-1} &= \varphi_\alpha \circ \varphi_\beta^\leftarrow.
\end{aligned} \tag{5.3}$$

Since (L,Φ)-**Flat** is a ground category, the consequent properties of the powerset operators of the morphism i allow to see that $\varphi_\alpha \circ \varphi_\beta^\leftarrow$ is a complete lattice morphism and that the adjoint inequalities

$$\begin{aligned}
(\varphi_\alpha \circ \varphi_\beta^\leftarrow) \circ (\varphi_\beta \circ \varphi_\alpha^\leftarrow)(\gamma) &\geq \gamma, \\
(\varphi_\beta \circ \varphi_\alpha^\leftarrow) \circ (\varphi_\alpha \circ \varphi_\beta^\leftarrow)(\delta) &\leq \delta
\end{aligned} \tag{5.4}$$

are satisfied. So (L,Φ) is a well-structured lattice. $\qquad\square$

Remark 5.4. The above theorem shows that any well-structured lattice (L,Φ) provides a ground category (L,Φ)-**Flat** that is isomorphic to a subcategory $\textbf{Set} \times \textbf{C}$ of the category $\textbf{Set} \times \textbf{SLoc}$ considered by Rodabaugh (see in particular [13]) as a ground category for (fuzzy) L-topological space theory.

Of course, one need to reduce $|\textbf{C}|$ to be the set $\{[\bot,a] \mid a \in L, a \neq \bot\}$ and $\textbf{C}([\bot,a],[\bot,b])$ to be $\{\varphi_a \circ \varphi_b^\leftarrow\}$ if $a \nearrow b$ and to be $\varnothing$ if $a \nearrow\!\!\!\!/\ b$.

It is worth noting that naturally corresponding morphisms (by means of the identification of $f : X \to T$ in (L,Φ)-**Flat**(a_X,b_T) with

$$(f,\varphi^{\mathrm{op}}) \in \textbf{Set} \times \textbf{SLoc}((X,[\bot,a]),(T,[\bot,b])), \tag{5.5}$$

where $\varphi^{\mathrm{op}} = \varphi_a \circ \varphi_b^\leftarrow)$ in (L,Φ)-**Flat** and in $\textbf{Set} \times \textbf{C}$ have the same powerset operators in the respective context, thanks to conditions (a) and (b) of Definition 4.1 and [13, Theorem 7.10].

Example 5.5. We consider $P,Q \in |\textbf{CLat}|$, $\varphi \in \textbf{CLat}^{\mathrm{op}}(P,Q)$, hence $\varphi^{\mathrm{op}} : Q \to P$ is a complete lattice morphism. Consider the disjoint union of P and Q ordered by the disjoint union of order relations in P and Q; then identify the lower bounds of P and Q getting a lower bound $\bot$ for both P and Q. Denote by a the upper bound of P, by b the upper bound of Q, and eventually add a new upper bound $\top = a \vee b$. Denote by L the new obtained complete lattice. P and Q can be identified with the intervals $[\bot,a]$ and $[\bot,b]$, respectively, of L.

Now for every $a' \leq a$, we denote by $\varphi_{a'} : L \to [\bot, a']$ the map defined, for all $x \in L$, by

$$\varphi_{a'}(x) = \begin{cases} a' \wedge \varphi^{\mathrm{op}}(x) & \text{if } x \leq b, \\ a' & \text{if } x = \top, \\ \bot & \text{if } x \leq a. \end{cases} \tag{5.6}$$

Clearly, every $\varphi_{a'}$ preserves $\bigwedge$ and its left adjoint is determined, for all $p \in [\bot, a']$, by

$$\varphi_a^{\dashv}(p) = (\varphi^{\mathrm{op}})^{\dashv}(p). \tag{5.7}$$

For all $b' \leq b$, let $\varphi_{b'} : L \to [\bot, b']$ be defined, for all $x \in L$, by

$$\varphi_{b'}(x) = b' \wedge x. \tag{5.8}$$

For all $b' \in Q$, $\varphi_{b'}$ preserves $\bigwedge$ and its left adjoint is the inclusion map.
Eventually, we consider $\varphi_\top = i_L$.
The pair $(L, \{\varphi_\gamma\}_{\gamma \in L})$ is a structured lattice.
Clearly, $a \nearrow b$, and evidently $\varphi_a \circ \varphi_b^{\dashv} = \varphi^{\mathrm{op}}$ is a complete lattice morphism.
Moreover, the following equalities hold:

$$(\varphi_a \circ \varphi_b^{\dashv})^{\dashv} = (\varphi^{\mathrm{op}})^{\dashv} = \varphi_b \circ \varphi_a^{\dashv}. \tag{5.9}$$

Remark 5.6. The above example shows how structured lattices allow to "capture" unchanged all "pieces" of the category $\mathbf{Set} \times \mathbf{CLat}^{\mathrm{op}}$, the main effective ground category for variable-basis (fuzzy) L-topological space theory considered by Rodabaugh, where powerset operators can be univocally associated to every morphism (see [13, Theorem 7.10]).
    More precisely, with notation of Example 5.5 and of [13], we consider the full subcategory $\mathbf{A}$ of (L, Φ)-$\mathbf{Flat}$ (where (L, Φ) is the structured lattice of Example 5.5) with objects

$$|\mathbf{A}| = \{a_X, b_X \mid X \in |\mathbf{Set}|\}. \tag{5.10}$$

If $f \in \mathbf{A}(a_X, b_T)$, then for all $B \in \mathscr{S}_{b_T}$, for all $x \in X$,

$$f_{(L,\Phi)(a_X,b_T)}^{\leftarrow}(B)(x) = \varphi_a \circ \varphi_b^{\dashv}(B(f(x))) = \varphi^{\mathrm{op}}(B(f(x))) = (f,\varphi)^{\leftarrow}(B)(x) \tag{5.11}$$

which implies the equality

$$f_{(L,\Phi)(a_X,b_T)}^{\leftarrow} = (f,\varphi)^{\leftarrow}. \tag{5.12}$$

Since φ^{op} is a complete lattice morphism, it follows that the backward powerset operator $f_{(L,\Phi)(a_X,b_T)}^{\leftarrow}$ is a complete lattice morphism. (The proof runs in a similar way as in Theorem 5.3.)
    Once more, as in the proof of Theorem 5.3, we can see that the adjunction $\varphi_b \circ \varphi_a^{\dashv} \dashv \varphi_a \circ \varphi_b^{\dashv}$ obtained in Example 5.5 produces the adjunction

$$f_{(L,\Phi)(a_X,b_T)}^{\rightarrow} \dashv f_{(L,\Phi)(a_X,b_T)}^{\leftarrow}. \tag{5.13}$$

By combining this result with the analogous adjunction $(f,\varphi)^{\rightharpoonup} \dashv (f,\varphi)^{\leftharpoonup}$ obtained in [13], thanks to the adjoint functor theorem, we can say that

$$f^{\rightarrow}_{(L,\Phi)(a_X,b_T)} = (f,\varphi)^{\rightharpoonup}. \tag{5.14}$$

We turn now on the cutting lattices we considered in Example 3.9 that produce Goguen's category of L-sets and we give the following characterization of these lattices in terms of properties of the structure Φ.

PROPOSITION 5.7. *A structured lattice* $(L,\Phi = \{\varphi_a\}_{a\in L})$ *is a cutting lattice if and only if it satisfies the following conditions.*
(a) $a \neq \perp \neq b$, $a \nearrow b \Rightarrow \varphi_a \circ \varphi_b^{\dashv}$ *is a complete lattice morphism.*
(b′) $a \nearrow b \Rightarrow \varphi_a \circ \varphi_b^{\dashv} = (\varphi_a)_{|[\perp,b]}$ *and* $\varphi_b \circ \varphi_a^{\dashv}(d) = \varphi_a^{\dashv}(d)$, *for all* $d \in [\perp,a]$.

Proof. The proof is a consequence of the following two lemmas. □

LEMMA 5.8. *If* (L,Φ) *is a structured lattice that satisfies the above conditions* (a) *and* (b′), *then*
(1) φ_a *is surjective, for all* $a \in L$;
(2) $\varphi_a(a) = a = \varphi_a^{\dashv}(a)$, *for all* $a \in L$;
(3) $a \nearrow b \Leftrightarrow a \leq b$, *for all* $a,b \in L$.

Proof. (1) The statement is trivial for $a = \perp$. Let $a \neq \perp$ and let $\overline{\varphi}_a : [\perp,a] \to [\perp,a]$ be the domain restriction of φ_a to $[\perp,a]$ and let $\varphi_a' : [\perp,a] \to [\perp,a]$ be the range restriction of $\varphi_a^{\dashv}$ to $[\perp,a]$. Then, since $a \nearrow a$, it follows from (b′) that $\overline{\varphi}_a = \varphi_a'$; moreover $\varphi_a' \dashv \overline{\varphi}_a$, and consequently $\overline{\varphi}_a = \varphi_a'$ is selfadjoint. By Proposition 2.4, $\overline{\varphi}_a$ is self-inverse hence it is bijective and consequently φ_a is surjective.
(2) The proof is trivial for $a = \perp$. So, let $a \neq \perp$. Since $a \nearrow a$, it follows from (a) and (b′) that

$$\begin{aligned} a &= \varphi_a \circ \varphi_a^{\dashv}(a) = \varphi_a(a), \\ a &= \varphi_a \circ \varphi_a^{\dashv}(a) = \varphi_a^{\dashv}(a). \end{aligned} \tag{5.15}$$

(3) It is a trivial consequence of (2). □

LEMMA 5.9. *Let* (L,Φ) *be a structured lattice satisfying the conditions* (a) *and* (b′). *Then, for all* $a \in L$ *and for all* $x \in L$,

$$\varphi_a(x) = a \wedge x. \tag{5.16}$$

Proof. By excluding the trivial case when $a = \perp$, we consider $\perp \neq a$. After noting that, by also using the preceding lemma

$$\varphi_b(a) = \varphi_b \circ \varphi_a^{\dashv}(a) = \varphi_a^{\dashv}(a) = a, \quad \forall b \geq a, \tag{5.17}$$

one can see that for all $x \in L$, since $x \wedge a \leq a$ and $\varphi_a(x) \leq a$, then

$$\varphi_a(x) = \varphi_a(x) \wedge a = \varphi_a(x) \wedge \varphi_a(a) = \varphi_a(x \wedge a) = x \wedge a. \tag{5.18}$$

□

PROPOSITION 5.10. *If* (L, Φ) *is a cutting lattice, then the category* (L, Φ)-**Set** *is a ground category, hence the standard ground category on* (L, Φ) *if and only if L is a frame.*

Proof. The required condition on L is sufficient (see Example 4.5 and Theorem 4.12). Conversely, let $\{a_j\}_{j \in J} \subseteq L$ and $b \in L$. Let $X = T = \{x\}$, $Y = b_x$, $Z = \top_x$, and let $i : \{x\} \to \{x\}$ be the identity map; since $b = Y(x) \leq Z(x) = \top$, then i is a morphism in the ground category (L, Φ)-**Set** from Y to Z and by using its backward powerset operator described in Example 3.9, one has

$$
\begin{aligned}
\bigvee_{j \in J} (a_j \wedge b) &= \bigvee_{j \in J} \left((a_j)_x \wedge b_x \right)(x) = \bigvee_{j \in J} \left(i_{(L,\Phi)(Y,Z)}^{\leftarrow} (a_j)_x \right)(x) \\
&= i_{(L,\Phi)(Y,Z)}^{\leftarrow} \left(\bigvee_{j \in J} (a_j)_x \right)(x) = \left(\bigvee_{j \in J} (a_j)_x \wedge b_x \right)(x) \\
&= \left(\bigvee_{j \in J} (a_j)_x \right)(x) \wedge b_x(x) = \left(\bigvee_{j \in J} a_j \right) \wedge b.
\end{aligned}
\tag{5.19}
$$

Hence, L satisfies the first infinite distributive law, that is, L is a frame. $\square$

Remark 5.11. Since $(L, \{a \wedge *\}_{a \neq \perp})$-**Set** clearly coincides with $\mathcal{A}L$-**Set** already described in Example 3.9 and the powerset operators of any morphism f with respect to $(L, \{a \wedge *\}_{a \neq \perp})$ coincide with $f_{\mathcal{A}}^{\rightarrow}$ and $f_{\mathcal{A}}^{\leftarrow}$ if and only if L is a frame, as already shown in Example 3.9, now we can say that the "set-theoretic context" or "ground" used for constructing the topological categories $\mathcal{A}L$-**Top** in [5, 8] and $\mathcal{A}(L, M)$-**Top** in [9] can be formally considered to be the standard ground category on the cutting lattice $(L, \{a \wedge *\}_{a \neq \perp})$.

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