# ITERATIVE APPROXIMATION OF FIXED POINT FOR Φ-HEMICONTRACTIVE MAPPING WITHOUT LIPSCHITZ ASSUMPTION

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Let $E$ be an arbitrary real Banach space and let $K$ be a nonempty closed convex subset of $E$ such that $K+K\subset K$. Assume that $T:K\to K$ is a uniformly continuous and Φ-hemicontractive mapping. It is shown that the Ishikawa iterative sequence with errors converges strongly to the unique fixed point of $T$.

## 1. Introduction

Let $E$ be a real Banach space and let $E^*$ be the dual space on $E$. The normalized duality mapping $J:E\to 2^{E^*}$ is defined by

$$Jx=\{f\in E^*:\langle x,f\rangle=\|x\|\cdot\|f\|=\|f\|^2\} \tag{1.1}$$

for all $x\in E$, where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is well known that if $E$ is a uniformly smooth Banach space, then $J$ is single valued and such that $J(-x)=-J(x)$, $J(tx)=tJ(x)$ for all $x\in E$ and $t\geq 0$; and $J$ is uniformly continuous on any bounded subset of $E$. In the sequel, we shall denote single-valued normalized duality mapping by $j$ by means of the normalized duality mapping $J$. In the following, we give some concepts.

*Definition 1.1.* A mapping $T$ with domain $D(T)$ and range $R(T)$ is said to be strongly pseudocontractive if for any $x,y\in D(T)$, there exists $j(x-y)\in J(x-y)$ such that

$$\langle Tx-Ty,j(x-y)\rangle\leq k\|x-y\|^2 \tag{1.2}$$

for some constant $k\in(0,1)$. The mapping $T$ is called Φ-strongly pseudocontractive if there exists a strictly increasing function $\Phi:[0,\infty)\to[0,\infty)$ with $\Phi(0)=0$ such that the inequality

$$\langle Tx-Ty,j(x-y)\rangle\leq\|x-y\|^2-\Phi(\|x-y\|)\|x-y\| \tag{1.3}$$

holds for all $x,y\in D(T)$. Let $F(T)=\{x\in D(T):Tx=x\}$. A mapping $T$ is called Φ-hemicontractive if there exists a strictly increasing function $\Phi:[0,\infty)\to[0,\infty)$ with


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$\Phi(0) = 0$ such that the inequality

$$\langle Tx - Tq, j(x-q)\rangle \leq \|x-q\|^2 - \Phi(\|x-q\|)\|x-q\| \tag{1.4}$$

holds for all $x \in D(T)$ and $q \in F(T)$.

It is shown in [5] that the class of strongly pseudocontractive mapping is a proper subclass of $\Phi$-strongly pseudocontractive mapping. Furthermore, the example in [2] shows that the class of $\Phi$-strongly pseudocontractive mapping with the nonempty fixed point set is a proper subclass of $\Phi$-hemicontractive mapping. The classes of mappings introduced above have been studied by several authors. In [1], Chidume proved that if $E = L_p$ (or $l^p$), $p \geq 2$, $K$ is a nonempty closed convex and bounded subset of $E$, and $T : K \to K$ is a Lipschitz strongly pseudocontractive mapping, then Mann iteration process converges strongly to the unique fixed point of $T$. In [4], Deng extended the above result to the Ishikawa iteration process. After Tan and Xu [7] extended the results of both Chidume [1] and Deng [4] to $q$-uniformly smooth Banach spaces $(1 < q < 2)$, Chidume and Osilike [3] extended to real $q$-uniformly smooth Banach spaces $(1 < q < \infty)$. Recently, these results above have been extended from Lipschitz strongly pseudocontractive mapping to Lipschitz $\Phi$-strongly pseudocontractive mapping in real $q$-uniformly smooth Banach spaces $(1 < q < \infty)$. More recently, Osilike [6] proved that if $K$ is a nonempty closed convex subset of arbitrary real Banach space $E$ and $T : K \to K$ is a Lipschitzian $\Phi$-hemicontractive mapping, then Ishikawa iteration sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to the unique fixed point of $T$. It is our purpose in this paper to examine the strong convergence theorems of the Ishikawa iterative sequences with errors for $\Phi$-hemicontractive mapping in arbitrary real Banach spaces.

Lemma 1.2. *Let $E$ be a real Banach space, then for all $x, y \in E$, there exists $j(x+y) \in J(x+y)$ such that $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle$.*

*Proof.* By definition of duality mapping, we may obtain directly the results of Lemma 1.2. □

## 2. Main results

Theorem 2.1. *Let $E$ be a real Banach space, and let $K$ be a nonempty closed convex subset of $E$ such that $K + K \subset K$. Assume that $T : K \to K$ is a uniformly continuous $\Phi$-hemicontractive mapping. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two real sequences in [0,1] satisfying the following conditions: (i) $\alpha_n, \beta_n \to 0$ as $n \to \infty$; (ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are two sequences in $K$ satisfying that $\sum_{n=0}^{\infty} \|u_n\| < \infty$ and $\sum_{n=0}^{\infty} \|v_n\| < \infty$. Define the Ishikawa iterative sequence $\{x_n\}_{n=0}^{\infty}$ with errors in $K$ by*

$$(\mathrm{IS}) \begin{cases} x_0 \in K, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, & n \geq 0, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n, & n \geq 0. \end{cases} \tag{2.1}$$

*If $\{Ty_n\}_{n=0}^{\infty}$ and $\{Tx_n\}_{n=0}^{\infty}$ are bounded, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of $T$.*

*Proof.* We first observe that the iterative sequence $\{x_n\}$ defined by (2.1) is well defined, since $K$ is convex and $T$ is a self-mapping from $K$ to itself with $K+K\subset K$. By the definition of $T$, we known that if $F(T)\neq\emptyset$, then $F(T)$ must be a singleton, let $q\in K$ denote the unique fixed point. And we also obtain that for any $x\in K$, there exists $j(x-q)\in J(x-y)$ such that

$$\langle Tx-Tq, j(x-q)\rangle \le \|x-q\|^2-\Phi(\|x-q\|)\|x-q\|. \tag{2.2}$$

Now set

$$M=\sup_{n\ge 0}\|Ty_n-q\|+\|x_0-q\|,$$
$$D=\sum_{n=0}^{\infty}\|u_n\|+M+1. \tag{2.3}$$

By using induction, we obtain $\|x_n-q\|\le M+\sum_{n=0}^{\infty}\|u_n\|, n\ge 0$, which implies that $\|x_n-q\|\le D, n\ge 0$. Using (2.1) and Lemma 1.2, we have

$$\begin{aligned}\|x_{n+1}-q\|^2&=\|(1-\alpha_n)(x_n-q)+\alpha_n(Ty_n-Tq)+u_n\|^2\\&\le\|(1-\alpha_n)(x_n-q)+\alpha_n(Ty_n-Tq)\|^2+2D\|u_n\|.\end{aligned} \tag{2.4}$$

Let $A_n=\|Ty_n-T(x_{n+1}-u_n)\|$. Then $A_n\to 0$ as $n\to\infty$. Indeed, since $T$ is uniformly continuous, we observe that $\{x_n\}_{n=0}^{\infty}$, $\{Tx_n\}_{n=0}^{\infty}$, and $\{Ty_n\}_{n=0}^{\infty}$ are all bounded and $\|y_n-(x_{n+1}-u_n)\|\to 0$ as $n\to\infty$, so that $A_n\to 0$ as $n\to\infty$. Using Lemma 1.2, (2.1), and (2.2), we have

$$\begin{aligned}&\|x_{n+1}-u_n-q\|^2\\&\quad=\|(1-\alpha_n)(x_n-q)+\alpha_n(Ty_n-Tq)\|^2\\&\quad\le(1-\alpha_n)^2\|x_n-q\|^2+2\alpha_n\langle Ty_n-Tq, j(x_{n+1}-u_n-q)\rangle\\&\quad\le(1-\alpha_n)^2\|x_n-q\|^2+2\alpha_n\langle Ty_n-T(x_{n+1}-u_n), j(x_{n+1}-u_n-q)\rangle\\&\qquad+2\alpha_n\langle T(x_{n+1}-u_n)-Tq, j(x_{n+1}-u_n-q)\rangle\\&\quad\le(1-\alpha_n)^2\|x_n-q\|^2+2\alpha_nA_n\|x_{n+1}-u_n-q\|\\&\qquad+2\alpha_n\|x_{n+1}-u_n-q\|^2-2\alpha_n\Phi(\|x_{n+1}-u_n-q\|)\|x_{n+1}-u_n-q\|\\&\quad\le(1-\alpha_n)^2\|x_n-q\|^2+2\alpha_nA_n(1-\alpha_n)\|x_n-q\|+2\alpha_n^2A_nD\end{aligned}$$

$$\begin{aligned}
&+2\alpha_n\|x_{n+1}-u_n-q\|^2-2\alpha_n\Phi(\|x_{n+1}-u_n-q\|)\|x_{n+1}-u_n-q\|\\
&\le(1-\alpha_n)^2\|x_n-q\|^2+\alpha_nA_n(1-\alpha_n)(1+\|x_n-q\|^2)+2\alpha_n^2A_nD\\
&\quad+2\alpha_n\|x_{n+1}-u_n-q\|^2-2\alpha_n\Phi(\|x_{n+1}-u_n-q\|)\|x_{n+1}-u_n-q\|\\
&\le((1-\alpha_n)^2+\alpha_nA_n)\|x_n-q\|^2+\alpha_nA_n(1+2\alpha_nD)\\
&\quad+2\alpha_n\|x_{n+1}-u_n-q\|^2-2\alpha_n\Phi(\|x_{n+1}-u_n-q\|)\|x_{n+1}-u_n-q\|,
\end{aligned}\tag{2.5}$$

which implies that

$$\begin{aligned}
\|x_{n+1}-u_n-q\|^2&\le\frac{(1-\alpha_n)^2+\alpha_nA_n}{1-2\alpha_n}\|x_n-q\|^2+\frac{\alpha_nA_n(1+2\alpha_nD)}{1-2\alpha_n}\\
&\quad-\frac{2\alpha_n}{1-2\alpha_n}\Phi(\|x_{n+1}-u_n-q\|)\|x_{n+1}-u_n-q\|\\
&\le\|x_n-q\|^2+\frac{2\alpha_n}{1-2\alpha_n}\Bigg(\frac{D^2\alpha_n+D^2A_n+A_n+2\alpha_nA_nD}{2}\\
&\qquad-\Phi(\|x_{n+1}-u_n-q\|)\|x_{n+1}-u_n-q\|\Bigg).
\end{aligned}\tag{2.6}$$

Substituting (2.6) into (2.4) yields that

$$\begin{aligned}
\|x_{n+1}-q\|^2&\le\|x_n-q\|^2+\frac{2\alpha_n}{1-2\alpha_n}\Bigg(\frac{D^2\alpha_n+D^2A_n+A_n+2\alpha_nA_nD}{2}\\
&\qquad-\Phi(\|x_{n+1}-u_n-q\|)\|x_{n+1}-u_n-q\|\Bigg)+2D\|u_n\|\\
&\le\|x_n-q\|^2+\frac{2\alpha_n}{1-2\alpha_n}(B_n-\Phi(\|x_{n+1}-u_n-q\|)\|x_{n+1}-u_n-q\|)+2D\|u_n\|,
\end{aligned}\tag{2.7}$$

where $B_n=D^2\alpha_n+D^2A_n+A_n+2\alpha_nA_nD/2$. Now we consider the following two possible cases.

*Case (i).* $\lim_{n\to\infty}\inf\|x_{n+1}-u_n-q\|=r>0$. Since $B_n\to 0,\alpha_n\to 0$ as $n\to\infty$, then there exists a positive integer $N$ such that $B_n<1/2\Phi(r)r,\alpha_n<1/2$ for all $n\ge N$. It follows from (2.7) that

$$\begin{aligned}
\|x_{n+1}-q\|^2&\le\|x_n-q\|^2+\frac{\alpha_n}{1-2\alpha_n}\Phi(r)r-\frac{2\alpha_n}{1-2\alpha_n}\Phi(r)r+2D\|u_n\|\\
&\le\|x_n-q\|^2-\frac{\alpha_n}{1-2\alpha_n}\Phi(r)r+2D\|u_n\|
\end{aligned}\tag{2.8}$$

which implies that $\Phi(r)r\sum_{n=N}^{\infty}\alpha_n/1-2\alpha_n\le\|x_N-q\|^2+2D\sum_{n=N}^{\infty}\|u_n\|<\infty$. This contradicts the assumption that $\sum_{n=0}^{\infty}\alpha_n=\infty$ and so the case (i) is impossible.

*Case (ii).* $\lim_{n\to\infty}\inf\|x_{n+1}-u_n-q\|=0$. In this case, there exists a subsequence $\{x_{n_j+1}-u_{n_j}-q\}$ such that $x_{n_j+1}-u_{n_j}-q\to 0$ as $j\to\infty$. Hence, for any $0<\varepsilon<1$, there exists a positive integer $n_j$ such that $\|x_{n_j+1}-u_{n_j}-q\|<\varepsilon$ and $B_n<\Phi(\varepsilon)\varepsilon$, $2D\sum_{k=n_j+1}^{\infty}\|u_k\|<\varepsilon$ for all $n\geq n_j$ for all $n\geq n_j$. Now we show that $\|x_{n_j+m}\|<\varepsilon$ for all $m\geq 1$. First, by (2.4), we have $\|x_{n_j+1}-q\|^2\leq\varepsilon^2+2D\|u_{n_j}\|$. Again consider the following two possible cases.
*Case(ii-1).* $\|x_{n_j+2}-u_{n_j+1}-q\|<\varepsilon$. Using (2.4), we obtain

$$\begin{aligned}\|x_{n_j+2}-q\|^2&=\left\|(1-\alpha_{n_j+1})(x_{n_j+1}-q)+\alpha_{n_j+1}(Ty_{n_j+1}-Tq)+u_{n_j+1}\right\|^2\\&\leq\left\|x_{n_j+2}-u_{n_j+1}-q\right\|^2+2D\left\|u_{n_j+1}\right\|\\&\leq\varepsilon^2+2D\left\|u_{n_j+1}\right\|.\end{aligned}\tag{2.9}$$

*Case(ii-2).* $\|x_{n_j+2}-u_{n_j+1}-q\|\geq\varepsilon$. Then using (2.7) yields that

$$\left\|x_{n_j+2}-q\right\|^2\leq\varepsilon^2+2D\left(\left\|u_{n_j}\right\|+\left\|u_{n_j+1}\right\|\right).\tag{2.10}$$

For all $m\geq 1$, using induction, we have $\|x_{n_j+m}-q\|^2\leq\varepsilon^2+2D\sum_{k=n_j}^{n_j+m-1}\|u_k\|<2\varepsilon$. Thus we prove that $x_n\to q$ as $n\to\infty$. This completes the proof. □

*Remark 2.2.* The assumption $K+K\subset K$ only is used to guarantee that the iterative sequence $\{x_n\}_{n=0}^{\infty}$ is well defined. We can drop this assumption in Theorem 2.1 by using a revised iterative scheme.

Corollary 2.3. *Let $E$ be a real Banach space, and let $K$ be a nonempty bounded and convex subset of $E$. Assume that $T:K\to K$ is a uniformly continuous $\Phi$-hemicontractive mapping. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, $\{\hat{\alpha}\}_{n=0}^{\infty}$, $\{\hat{\beta}\}_{n=0}^{\infty}$, and $\{\hat{\gamma}\}_{n=0}^{\infty}$ be six real sequences in [0,1] satisfying the following conditions: (i) $\beta_n\to 0,\hat{\beta}_n\to 0,\hat{\gamma}_n\to 0$ as $n\to\infty$; (ii) $\sum_{n=0}^{\infty}\beta_n=\infty,\sum_{n=0}^{\infty}\gamma_n<\infty$; (iii) $\alpha_n+\beta_n+\gamma_n=\hat{\alpha}_n+\hat{\beta}_n+\hat{\gamma}_n=1$. Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two bounded sequences in $K$. Define iteratively the Ishikawa sequence $\{x_n\}_{n=0}^{\infty}$ with errors in $K$ as follows:*

$$\begin{gathered}x_0\in K,\\y_n=\hat{\alpha}_nx_n+\hat{\beta}_nTx_n+\hat{\gamma}_nv_n,\quad n\geq 0,\\x_{n+1}=\alpha_nx_n+\beta_nTy_n+\gamma_nu_n,\quad n\geq 0.\end{gathered}\tag{2.11}$$

*Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (2.11) converges strongly to the unique fixed point of $T$.*

*Proof.* We observe that (2.11) can be rewritten as follows:

$$\begin{gathered}x_0\in K,\\y_n=(1-\hat{\beta}_n)x_n+\hat{\beta}_nTx_n+\hat{\gamma}_n(v_n-x_n),\quad n\geq 0,\\x_{n+1}=(1-\beta_n)x_n+\beta_nTy_n+\gamma_n(u_n-x_n),\quad n\geq 0.\end{gathered}\tag{2.12}$$

It is easily seen that under the assumptions of Corollary 2.3, the sequence $\{x_n\}_{n=0}^{\infty}$ is bounded. Now the conclusion follows from Theorem 2.1. This completes the proof. □

THEOREM 2.4. *Let $E$ be a real Banach space, and let $K$ be a nonempty closed convex subset of $E$ such that $K+K \subset K$. Assume that $T: K \to K$ is a uniformly continuous $\Phi$-hemicontractive mapping. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two real sequences in [0,1] satisfying the following conditions: (i) $\alpha_n, \beta_n \to 0$ as $n \to \infty$; (ii)$\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are two sequences in $K$ satisfying $\|u_n\|, \|v_n\| \to 0$ as $n \to \infty$, where $\|u_n\| = o(\alpha_n)$. Define the Ishikawa iterative sequence $\{x_n\}_{n=0}^{\infty}$ with errors in $K$ by*

$$\text{(IS)} \begin{cases} x_0 \in K, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, & n \geq 0, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n, & n \geq 0. \end{cases} \tag{2.13}$$

*If $\{Ty_n\}_{n=0}^{\infty}$ and $\{Tx_n\}_{n=0}^{\infty}$ are bounded, then the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of $T$.*

*Proof.* Since $K+K \subset K$ and $K$ is convex, we see that the sequence $\{x_n\}_{n=0}^{\infty}$ is well defined. By the definition of $T$, $T$ has a unique fixed point in $K$. Let $q$ denote the unique fixed point. Now we shall show that $\{x_n\}_{n=0}^{\infty}$ is bounded. In fact, we may set $\|u_n\| = \varepsilon_n \alpha_n$, where $\varepsilon_n \to 0$ as $n \to \infty$. Set $D = \sup_{n \geq 0}\{\|Ty_n - q\| + \varepsilon_n\} + \|x_0 - q\|$, by induction, we can show that $\|x_n - q\| \leq D$ for all $n \geq 0$, so that $\{y_n\}$ is bounded. And we have

$$\langle Tx - Tq, j(x-q)\rangle \leq \|x-q\|^2 - \Phi(\|x-q\|)\|x-q\| \tag{2.14}$$

for each $x \in K$. By using Lemma 1.2 and (2.7), we have

$$||x_{n+1} - q||^2 \leq ||(1-\alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq)||^2 + 2D||u_n||. \tag{2.15}$$

After repeating the usage of the proof of Theorem 2.1, we obtain

$$\begin{aligned} &||(1-\alpha_n)(x_n-q) + \alpha_n(Ty_n - Tq)||^2 \\ &\quad \leq ((1-\alpha_n)^2 + \alpha_n A_n)||x_n - q||^2 + \alpha_n A_n(1 + 2\alpha_n D) \\ &\qquad + 2\alpha_n||x_{n+1} - u_n - q||^2 - 2\alpha_n \Phi(||x_{n+1} - u_n - q||)||x_{n+1} - u_n - q||. \end{aligned} \tag{2.16}$$

Thus, we have

$$\begin{aligned} &\|x_{n+1} - q\|^2 \\ &\quad \leq \|x_n - q\|^2 + \frac{2\alpha_n}{1-2\alpha_n}\Bigg(\frac{D^2\alpha_n + D^2 A_n + A_n + 2\alpha_n A_n D}{2} \\ &\qquad\qquad - \Phi(||x_{n+1} - u_n - q||)||x_{n+1} - u_n - q||\Bigg) + 2D\|u_n\| \\ &\quad \leq \|x_n - q\|^2 + \frac{2\alpha_n}{1-2\alpha_n}\Big(B_n + C_n - \Phi(\|x_{n+1} - u_n - q\|)\|x_{n+1} - u_n - q\|\Big) + 2D\|u_n\|, \end{aligned} \tag{2.17}$$

where $B_n = D^2\alpha_n + D^2 A_n + A_n + 2\alpha_n A_n D/2 \to 0, C_n = 1 - 2\alpha_n/\alpha_n D\|u_n\| \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} \inf \|x_{n+1} - u_n - q\| = 0$. If it is not the case, then there exist $\delta > 0$ and positive integer $N$ such that $B_n + C_n < 1/2\Phi(r)r, \alpha_n < 1/2$ for all $n \geq N$. It follows that $\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n/1 - 2\alpha_n\Phi(r)r$, which leads to $\Phi(r)r\sum_{n=N}^{\infty}\alpha_n \leq \|x_N - q\|^2 < \infty$, a contradiction. Hence, there exists a subsequence $\{x_{n_j} + 1\}$ such that $x_{n_j} + 1 \to q$ as $j \to \infty$. At this point, we can choose a positive integer $n_j$ such that $\|x_{n_j+1} - q\| < \varepsilon$ and $B_n + C_n < \Phi(\varepsilon/2)\varepsilon/4$, $\|u_n\| < \varepsilon/2$ for all $n \geq n_j$. We show that $\|x_{n_j+2} - q\| < \varepsilon$. If not, we assume that $\|x_{n_j+2} - q\| \geq \varepsilon$, then $\|x_{n_j+2} - u_{n_j+1} - q\| \geq \|x_{n_j+2} - q\| - \|u_{n_j+1} \geq \varepsilon/2$ so that $\Phi(x_{n_j+2} - u_{n_j+1} - q) \geq \Phi(\varepsilon/2)$. Thus, using (2.17), we have

$$||x_{n_j+2} - q||^2 \leq ||x_{n_j+1} - q||^2 - \frac{\alpha_{n_j+1}}{1 - 2\alpha_{n_j+1}}\Phi\left(\frac{\varepsilon}{2}\right)\frac{\varepsilon}{2} < \varepsilon^2, \tag{2.18}$$

this is a contradiction and so $\|x_{n_j+2} - q\| < \varepsilon$. By induction, $\|x_{n_j+m} - q\| < \varepsilon$ for all $m \geq 1$. □

Corollary 2.5. *Let $E$ be a real Banach space, and let $K$ be a nonempty bounded and convex subset of $E$. Assume that $T : K \to K$ is a uniformly continuous $\Phi$-hemicontractive mapping. Let $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty}$, $\{\hat{\alpha}\}_{n=0}^{\infty}$, $\{\hat{\beta}\}_{n=0}^{\infty}$, and $\{\hat{\gamma}\}_{n=0}^{\infty}$ be six real sequences in [0,1] satisfying the following conditions: (i) $\beta_n \to 0, \hat{\beta}_n \to 0, \hat{\gamma}_n \to 0$ as $n \to \infty$; (ii) $\sum_{n=0}^{\infty}\beta_n = \infty, \gamma_n = o(\beta_n)$; (iii) $\alpha_n + \beta_n + \gamma_n = \hat{\alpha}_n + \hat{\beta}_n + \hat{\gamma}_n = 1, n \geq 0$. Let $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ be two bounded sequences in $K$. Define iteratively the Ishikawa sequence $\{x_n\}_{n=0}^{\infty}$ with errors in $K$ as follows:*

$$\begin{aligned} x_0 &\in K, \\ y_n &= \hat{\alpha}_n x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n v_n, \quad n \geq 0, \\ x_{n+1} &= \alpha_n x_n + \beta_n T y_n + \gamma_n u_n, \quad n \geq 0. \end{aligned} \tag{2.19}$$

*Then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by (2.11) converges strongly to the unique fixed point of $T$.*

*Proof.* We observe that (2.11) can be rewritten as follows:

$$\begin{aligned} x_0 &\in K, \\ y_n &= (1 - \hat{\beta}_n)x_n + \hat{\beta}_n T x_n + \hat{\gamma}_n(v_n - x_n), \quad n \geq 0, \\ x_{n+1} &= (1 - \beta_n)x_n + \beta_n T y_n + \gamma_n(u_n - x_n), \quad n \geq 0. \end{aligned} \tag{2.20}$$

It is easily to obtain the conclusion from Theorem 2.4. This completes the proof. □

*Remark 2.6.* Theorems 2.1 and 2.4 extend the results of [5] from real $q$-uniformly smooth Banach spaces to arbitrary real Banach spaces. It is also easy to see that our results are significant extensions of the results of [1, 2, 3, 4, 7] to arbitrary real Banach spaces and to the more general classes of mapping ($\Phi$-hemicontractive mapping) considered here. Moreover, our iteration schemes extend from the usual iterative sequences to the iterative sequences with errors.

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