# FIXED POINT THEORY FOR MÖNCH-TYPE MAPS DEFINED ON CLOSED SUBSETS OF FRÉCHET SPACES: THE PROJECTIVE LIMIT APPROACH

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New Leray-Schauder alternatives are presented for Mönch-type maps defined between Fréchet spaces. The proof relies on viewing a Fréchet space as the projective limit of a sequence of Banach spaces.

## 1. Introduction

This paper presents new Leray-Schauder alternatives for Mönch-type maps defined between Fréchet spaces. Two approaches [1, 2, 3, 6, 7] have recently been presented in the literature both of which are based on the fact that a Fréchet space can be viewed as a projective limit of a sequence of Banach spaces $\{E_n\}_{n\in\mathbb{N}}$ (here $\mathbb{N} = \{1,2,\ldots\}$). Both approaches are based on constructing maps $F_n$ defined on subsets of $E_n$ whose fixed points converge to a fixed point of the original operator $F$. Both approaches have advantages and disadvantages over the other [1] and in this paper, we combine the advantages of both approaches to present very general fixed point results. Our theory in particular extends and improves the theory in [3] (in [3], the single-valued case was discussed).

Finally in this section, we gather together some definitions and a fixed point result which will be needed in Section 2.

Now, let $I$ be a directed set with order $\le$ and let $\{E_\alpha\}_{\alpha\in I}$ be a family of locally convex spaces. For each $\alpha \in I, \beta \in I$ for which $\alpha \le \beta$, let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{x = (x_\alpha) \in \prod_{\alpha\in I} E_\alpha : x_\alpha = \pi_{\alpha,\beta}(x_\beta)\ \forall \alpha,\beta \in I,\ \alpha \le \beta\right\} \tag{1.1}$$

is a closed subset of $\prod_{\alpha\in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha\in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow}\{E_\alpha, \pi_{\alpha,\beta}\}$ or the generalized intersection [4, page 439] $\bigcap_{\alpha\in I} E_\alpha$).

Next, we recall a fixed point result from the literature [9] which we will use in Section 2.

Theorem 1.1. *Let $K$ be a closed convex subset of a Banach space $X$, $U$ a relatively open subset of $K$, $x_0 \in U$, and suppose that $F : \overline{U} \to CK(K)$ is an upper semicontinuous map (here $CK(K)$ denotes the family of nonempty convex compact subsets of $K$). Also assume*


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*that the following conditions hold:*

$$M \subseteq \overline{U}, \quad M \subseteq \operatorname{co}\left(\{x_0\} \cup F(M)\right) \quad \textit{with } \overline{M} = \overline{C}, \qquad (1.2)$$
$$C \subseteq M \quad \textit{countable, implies } \overline{M} \textit{ is compact,}$$

$$x \notin (1-\lambda)\{x_0\} + \lambda F x \quad \textit{for } x \in \overline{U} \setminus U,\ \lambda \in (0,1). \qquad (1.3)$$

*Then there exist a compact set* $\sum$ *of* $\overline{U}$ *and an* $x \in \sum$ *with* $x \in Fx$.

*Remark 1.2.* In [9], we see that we could take $\sum$ to be

$$\{y \in \overline{U} : y \in (1-\lambda)\{x_0\} + \lambda F y \text{ for some } \lambda \in [0,1]\}. \qquad (1.4)$$

We did not show that $\sum$ is compact in [9] but this is easy to see as we will now show. First, notice that $\sum$ is closed since $F$ is upper semicontinuous. Now let $\{y_n\}_1^\infty$ be a sequence in $\sum$. Then there exists $\{t_n\}_1^\infty$ in $[0,1]$ with $y_n \in (1-t_n)\{x_0\} + t_n F y_n$ for $n \in \mathbb{N} = \{1,2,\ldots\}$. Without loss of generality, assume that $t_n \to t \in [0,1]$. Let $C = \{y_n\}_1^\infty$. Notice that $C$ is countable and $C \subseteq \operatorname{co}(\{x_0\} \cup F(C))$. Now (1.2) with $M = C$ guarantees that $\overline{C}$ is compact (so sequentially compact). Thus there exist a subsequence $N_1$ of $\mathbb{N}$ and a $y \in \overline{C}$ with $y_n \to y$ as $n \to \infty$ in $N_1$. This together with $y_n \in (1-t_n)\{x_0\} + t_n F y_n$ and the upper semicontinuity of $F$ guarantees that $y \in (1-t)\{x_0\} + tFy$, so $y \in \overline{\sum} = \sum$. Consequently, $\sum$ is sequentially compact and hence compact. In fact, one could also of course take $\sum$ to be

$$\{y \in \overline{U} : y \in F y\} \qquad (1.5)$$

for the compact set in Theorem 1.1.

## 2. Projective limit approach

Let $E = (E, \{|\cdot|_n\}_{n\in\mathbb{N}})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n : n \in \mathbb{N}\}$. We assume that the family of seminorms satisfies

$$|x|_1 \le |x|_2 \le |x|_3 \le \cdots \quad \text{for every } x \in E. \qquad (2.1)$$

To $E$, we associate a sequence of Banach spaces $\{(\mathbf{E}_n, |\cdot|_n)\}$ described as follows. For every $n \in \mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by

$$x \sim_n y \quad \text{iff } |x-y|_n = 0. \qquad (2.2)$$

We denote by $\mathbf{E}^n = (E/\sim_n, |\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n, |\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n : E \to \mathbf{E}_n$. Now since (2.1) is satisfied, the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m \ge n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m} : \mathbf{E}_m \to \mathbf{E}_n$ since $\mathbf{E}_m/\sim_n$ can be regarded as a subset of $\mathbf{E}_n$. We now assume that the following condition holds:

$$\begin{aligned}&\text{for each } n \in \mathbb{N}, \text{ there exist a Banach space } (E_n, |\cdot|_n)\\&\text{and an isomorphism (between normed spaces) } j_n : \mathbf{E}_n \longrightarrow E_n.\end{aligned} \tag{2.3}$$

*Remark 2.1.* (i) For convenience, the norm on $E_n$ is denoted by $|\cdot|_n$.

(ii) In our applications, $\mathbf{E}_n = \mathbf{E}^n$ for each $n \in \mathbb{N}$.

(iii) Note that if $x \in \mathbf{E}_n$ (or $\mathbf{E}^n$), then $x \in E$. However if $x \in E_n$, then $x$ is not necessarily in $E$ and in fact $E_n$ is easier to use in applications as we will see in Theorem 2.3 (even though $E_n$ is isomorphic to $\mathbf{E}_n$).

Finally, we assume that

$$E_1 \supseteq E_2 \supseteq \cdots \qquad \text{and for each } n \in \mathbb{N}, \quad |x|_n \le |x|_{n+1} \quad \forall x \in E_{n+1}. \tag{2.4}$$

Let $\lim_{\leftarrow} E_n$ (or $\bigcap_1^\infty E_n$, where $\bigcap_1^\infty$ is the generalized intersection [4]) denote the projective limit of $\{E_n\}_{n\in\mathbb{N}}$ (note that $\pi_{n,m} = j_n \mu_{n,m} j_m^{-1} : E_m \to E_n$ for $m \ge n$) and note that $\lim_{\leftarrow} E_n \cong E$, so for convenience we write $E = \lim_{\leftarrow} E_n$.

For each $X \subseteq E$ and each $n \in \mathbb{N}$, we set $X_n = j_n \mu_n(X)$ and we let $\overline{X_n}$ and $\partial X_n$ denote, respectively, the closure and the boundary of $X_n$ with respect to $|\cdot|_n$ in $E_n$. Also the pseudointerior of $X$ is defined by [2]

$$\text{pseudo} - \text{int}(X) = \{x \in X : j_n \mu_n(x) \in \overline{X_n} \setminus \partial X_n \ \text{ for every } n \in \mathbb{N}\}. \tag{2.5}$$

Our main result in this paper is the extension of Theorem 1.1 to an applicable result in the Fréchet space setting (we refer the reader to [1]; in applications, usually the set $U$ is bounded and as a result has *empty* interior in the nonnormable situation).

Theorem 2.2. *Let $E$ and $E_n$ be as described above and let $F : X \to 2^E$, where $X \subseteq E$ (here $2^E$ denotes the family of nonempty subsets of $E$). Suppose that the following conditions are satisfied:*

$$x_0 \in \text{pseudo} - \text{int}(X), \tag{2.6}$$

$$\textit{for each } n \in \mathbb{N}, \quad F : \overline{X_n} \longrightarrow CK(E_n) \textit{ is an upper semicontinuous map}, \tag{2.7}$$

$$\begin{aligned}&\textit{for each } n \in \mathbb{N}, \quad M \subseteq \overline{X_n} \textit{ with } M \subseteq \operatorname{co}(\{j_n\mu_n(x_0)\} \cup F(M)),\\&\textit{with } \overline{M} = \overline{C} \textit{ and } C \subseteq M \textit{ countable, implies that } \overline{M} \textit{ is compact}\end{aligned} \tag{2.8}$$

$$\textit{for each } n \in \mathbb{N}, \quad y \notin (1-\lambda) j_n\mu_n(x_0) + \lambda F y \textit{ in } E_n \quad \forall \lambda \in (0,1),\ y \in \partial X_n, \tag{2.9}$$

$$\begin{aligned}&\textit{for each } n \in \{2,3,\ldots\} \textit{ if } y \in \overline{X_n} \textit{ solves } y \in Fy \textit{ in } E_n, \textit{ then } y \in \overline{X_k},\\&\textit{for } k \in \{1,\ldots,n-1\}.\end{aligned} \tag{2.10}$$

*Then $F$ has a fixed point in $X$.*

*Proof.* Fix $n \in \mathbb{N}$. Let $\sum_n = \{x \in \overline{X_n} : x \in Fx \text{ in } E_n\}$. Now Theorem 1.1 (note that (2.6) implies that $j_n\mu_n(x_0) \in \overline{X_n} \setminus \partial X_n$) guarantees that there exists $y_n \in \sum_n$ with $y_n \in Fy_n$. We look at $\{y_n\}_{n\in\mathbb{N}}$. Now $y_1 \in \sum_1$. Also $y_k \in \sum_1$ for $k \in \mathbb{N} \setminus \{1\}$ since $y_k \in \overline{X_1}$ from (2.10) (see also (2.4)). As a result, $y_n \in \sum_1$ for $n \in \mathbb{N}$ and since $\sum_1$ is compact (see Remark 1.2), there exist a subsequence $N_1^\star$ of $\mathbb{N}$ and a $z_1 \in \sum_1$ with $y_n \to z_1$ in $E_1$ as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Now $y_n \in \sum_2$ for $n \in N_1$ so there exist a subsequence $N_2^\star$ of $N_1$ and a $z_2 \in \sum_2$ with $y_n \to z_2$ in $E_2$ as $n \to \infty$ in $N_2^\star$. Note from (2.4) that $z_2 = z_1$ in $E_1$ since $N_2^\star \subseteq N_1$. Let $N_2 = N_2^\star \setminus \{2\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \cdots, \quad N_k^\star \subseteq \{k, k+1, \ldots\} \tag{2.11}$$

and $z_k \in \sum_k$ with $y_n \to z_k$ in $E_k$ as $n \to \infty$ in $N_k^\star$. Note that $z_{k+1} = z_k$ in $E_k$ for $k \in \{1,2,\ldots\}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in \mathbb{N}$. Let $y = z_k$ in $E_k$. Notice that $y$ is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in Fy_n$ in $E_n$ for $n \in N_k$ and $y_n \to y$ in $E_k$ as $n \to \infty$ in $N_k$ (since $y = z_k$ in $E_k$) together with the fact that $F : \overline{X_k} \to CK(E_k)$ is upper semicontinuous (note that $y_n \in \sum_k$ for $n \in N_k$) imply that $y \in Fy$ in $E_k$. We can do this for each $k \in \mathbb{N}$ so as a result, we have $y \in Fy$ in $E$. □

Next, we present an application of Theorem 2.2. We discuss the differential equation

$$\begin{gathered} y'(t) = f(t, y(t)) \quad \text{a.e. } t \in [0, T), \\ y(0) = y_0 \in \mathbb{R}, \end{gathered} \tag{2.12}$$

where $0 < T \leq \infty$ is fixed. First we introduce some notation. If $u \in C[0, T)$, then for every $n \in \mathbb{N}$, we define the seminorms $\rho_n(u)$ by

$$\rho_n(u) = \sup_{t\in[0,t_n]} |u(t)|, \tag{2.13}$$

where $t_n \uparrow T$. Note that $C[0, T)$ is a locally convex linear topological space. The topology on $C[0, T)$, induced by the seminorms $\{\rho_n\}_{n\in\mathbb{N}}$, is the topology of uniform convergence on every compact interval of $[0, T)$.

Recall that a function $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ is an $L^1$-Carathéodory function if

(a) the map $t \mapsto g(t, y)$ is measurable for all $y \in \mathbb{R}$,

(b) the map $y \mapsto g(t, y)$ is continuous for a.e. $t \in [a,b]$.

Now, $g : [a,b] \times \mathbb{R} \to \mathbb{R}$ is said to be an $L^p$-Carathéodory function ($1 \leq p \leq \infty$) if $g$ is a Carathéodory function and

(c) for any $r > 0$, there exists $\mu_r \in L^p[a,b]$ such that $|y| \leq r$ implies that $|g(t, y)| \leq \mu_r(t)$ for a.e. $t \in [a,b]$.

Finally, a function $g : [0, T) \times \mathbb{R} \to \mathbb{R}$ is an $L^p_{\text{loc}}$-Carathéodory function if (a), (b), and (c) above hold when $g$ is restricted to $[0, t_n] \times \mathbb{R}$ for any $n \in \mathbb{N}$.

THEOREM 2.3. *Suppose that the following conditions are satisfied:*

$$\text{for each } n \in \mathbb{N}, \quad f : [0,t_n] \times \mathbb{R} \longrightarrow \mathbb{R} \text{ is a Carathéodory function}, \tag{2.14}$$

$$\begin{aligned}&\text{there exists an } L^1_{\text{loc}}[0,T) - \text{Carathéodory function } g : [0,T) \times [0,\infty) \longrightarrow [0,\infty)\\ &\text{such that } |f(t,x)| \le g(t,|x|) \quad \text{for a.e. } t \in [0,T) \text{ and all } x \in \mathbb{R},\end{aligned} \tag{2.15}$$

*for each $n \in \mathbb{N}$, the problem*

$$\begin{gathered} v'(t) = g(t,v(t)), \quad \text{a.e. } t \in [0,t_n], \\ v(0) = |y_0| \end{gathered} \tag{2.16}$$

*has a maximal solution $r_n(t)$ on $[0,t_n]$ (here $r_n \in C[0,t_n]$).*

*Then (2.12) has at least one solution $y \in C[0,T)$.*

*Remark 2.4.* One could also obtain a multivalued version of Theorem 2.3 (with (2.12) replaced by a differential inclusion) by using the ideas in the proof below with the ideas in [6].

*Proof.* Here $E = C[0,T)$, $\mathbf{E}^k$ consists of the class of functions in $E$ which coincide on the interval $[0,t_k]$, $E_k = C[0,t_k]$ with of course $\pi_{n,m} = j_n\mu_{n,m}j_m^{-1} : E_m \to E_n$ for $m \ge n$ defined by $\pi_{n,m}(x) = x|_{[0,t_n]}$. We will apply Theorem 2.2 with

$$X = \{u \in C[0,T) : |u|_n \le w_n \text{ for each } n \in \mathbb{N}\}; \tag{2.17}$$

here $|u|_n = \sup_{t\in I_n} |u(t)|$, where $I_n = [0,t_n]$ and $w_n = \sup_{t\in I_n} r_n(t) + 1$. On any interval $I_n = [0,t_n]$ $(n \in \mathbb{N})$, we let $F$ on $C(I_n)$ be defined by

$$Fy(t) = y_0 + \int_0^t f(s,y(s))\,ds. \tag{2.18}$$

Fix $n \in \mathbb{N}$. Notice that

$$\overline{X_n} = \{u \in C[0,t_n] : |u|_n \le w_n\}. \tag{2.19}$$

Clearly, (2.6) holds with $x_0 = 0$ and a standard argument from the literature [8] guarantees that

$$F : \overline{X_n} \longrightarrow E_n \text{ is continuous and compact}, \tag{2.20}$$

so (2.7) and (2.8) hold.

To show that (2.9), fix $n \in \mathbb{N}$ and let $y \in C(I_n)$ be such that $y = \lambda F y$ for $\lambda \in (0,1)$. We claim $|y|_n < w_n$ and if this is true, then $y \notin \partial X_n$ and hence (2.9) is true. Let $t \in I_n$ and we now show that $|y(t)| < w_n$. If $|y(t)| \le |y_0|$, we are finished so it remains to discuss the case when $|y(t)| > |y_0|$. In this case, there exists $a \in [0,t)$ with

$$|y(s)| > |y_0| \quad \text{for } s \in (a,t],\ |y(a)| = |y_0|. \tag{2.21}$$

Also

$$|y(s)|' \le |y'(s)| \le g(s,|y(s)|) \quad \text{a.e. on } (a,t), \tag{2.22}$$

so

$$\begin{gathered} |y(s)|' \le g(s,|y(s)|), \quad \text{a.e. on } (a,t), \\ |y(a)| = |y_0|. \end{gathered} \tag{2.23}$$

Now a standard comparison theorem for ordinary differential equations in the real case [5, Theorem 1.10.2] guarantees that $|y(s)| \le r_n(s)$ for $s \in [a,t]$, so in particular $|y(t)| \le r_n(t) < w_n$, so (2.9) is true.

It remains to show that (2.10). To see this, fix $n \in \{2,3,\dots\}$ and suppose that $y \in \overline{X_n}$ solves

$$\begin{gathered} y'(t) = f(t,y(t)), \quad \text{a.e. on } [0,t_n], \\ y(0) = y_0. \end{gathered} \tag{2.24}$$

Next, fix $k \in \{1,\dots,n-1\}$. We must show that $y \in \overline{X_k}$. Now since $t_n \uparrow T$, notice that $[0,t_k] \subseteq [0,t_n]$ so as a result,

$$\begin{gathered} y'(t) = f(t,y(t)), \quad \text{a.e. on } [0,t_k], \\ y(0) = y_0. \end{gathered} \tag{2.25}$$

Let $t \in [0,t_k]$ and essentially the same argument as above guarantees that $|y(t)| < w_k$ so $|y|_k < w_k$. Thus $y \in \overline{X_k}$ and (2.10) holds.

The result now follows immediately from Theorem 2.2. □

Our final result was motivated by Urysohn-type operators.

Theorem 2.5. *Let $E$ and $E_n$ be as described in the beginning of Section 2 and let $F : X \to 2^E$, where $X \subseteq E$. Suppose that the following conditions are satisfied:*

$$x_0 \in \text{pseudo}-\text{int}(X), \tag{2.26}$$

$$\overline{X_1} \supseteq \overline{X_2} \supseteq \cdots, \tag{2.27}$$

$$\text{for each } n \in \mathbb{N}, \quad F_n : \overline{X_n} \longrightarrow CK(E_n) \text{ is upper semicontinuous}, \tag{2.28}$$

$$\begin{gathered} \text{for each } n \in \mathbb{N}, \quad M \subseteq \overline{X_n} \quad \text{with } M \subseteq \operatorname{co}(\{j_n\mu_n(x_0)\} \cup F_n(M)) \\ \text{with } \overline{M} = \overline{C} \text{ and } C \subseteq M \quad \text{countable, implies that } \overline{M} \text{ is compact}, \end{gathered} \tag{2.29}$$

$$\text{for each } n \in \mathbb{N}, \quad y \notin (1-\lambda)j_n\mu_n(x_0) + \lambda F_n y \text{ in } E_n \quad \forall \lambda \in (0,1),\ y \in \partial X_n, \tag{2.30}$$

$$\text{for each } n \in \mathbb{N}, \text{ the map } \mathcal{K}_n : \overline{X_n} \longrightarrow 2^{E_n}, \text{ given by}$$
$$\mathcal{K}_n(y) = \bigcup_{m=n}^{\infty} F_m(y) \text{ (see Remark 2.6), satisfies that} \tag{2.31}$$
$$\text{if } C \subseteq \overline{X_n} \text{ is countable with } C \subseteq \mathcal{K}_n(C), \text{ then } \overline{C} \text{ is compact,}$$

$$\begin{gathered}\text{if there exist a } w \in X \text{ and a sequence } \{y_n\}_{n\in\mathbb{N}} \text{ with } y_n \in \overline{X_n} \text{ and } y_n \in F_n y_n \text{ in } E_n \\ \text{such that for every } k \in \mathbb{N} \text{ there exists a subsequence } S \subseteq \{k+1, k+2, \ldots\} \text{ of } \mathbb{N} \\ \text{with } y_n \longrightarrow w \text{ in } E_k \text{ as } n \longrightarrow \infty \text{ in } S, \text{ then } w \in Fw \text{ in } E.\end{gathered} \tag{2.32}$$

*Then $F$ has a fixed point in $X$.*

*Remark 2.6.* The definition of $\mathcal{K}_n$ is as follows. If $y \in \overline{X_n}$ and $y \notin \overline{X_{n+1}}$, then $\mathcal{K}_n(y) = F_n(y)$, whereas if $y \in \overline{X_{n+1}}$ and $y \notin \overline{X_{n+2}}$, then $\mathcal{K}_n(y) = F_n(y) \cup F_{n+1}(y)$, and so on.

*Proof.* Fix $n \in \mathbb{N}$. Let $\sum_n = \{x \in \overline{X_n} : x \in F_n x \text{ in } E_n\}$. Now, Theorem 1.1 guarantees that there exists $y_n \in \sum_n$ with $y_n \in F_n y_n$ in $E_n$. We look at $\{y_n\}_{n\in\mathbb{N}}$. Note that $y_n \in \overline{X_1}$ for $n \in \mathbb{N}$ from (2.27). In addition with $C = \{y_n\}_1^\infty$, we have from assumption (2.31) that $\overline{C}(\subseteq E_1)$ is compact; note that $y_n \in \mathcal{K}_1(y_n)$ in $E_1$ for each $n \in \mathbb{N}$. Thus there exist a subsequence $N_1^\star$ of $\mathbb{N}$ and a $z_1 \in \overline{X_1}$ with $y_n \to z_1$ in $E_1$ as $n \to \infty$ in $N_1^\star$. Let $N_1 = N_1^\star \setminus \{1\}$. Proceed inductively to obtain subsequences of integers

$$N_1^\star \supseteq N_2^\star \supseteq \ldots, \quad N_k^\star \subseteq \{k, k+1, \ldots\} \tag{2.33}$$

and $z_k \in \overline{X_k}$ with $y_n \to z_k$ in $E_k$ as $n \to \infty$ in $N_k^\star$. Note that $z_{k+1} = z_k$ in $E_k$ for $k \in \mathbb{N}$. Also let $N_k = N_k^\star \setminus \{k\}$.

Fix $k \in \mathbb{N}$. Let $y = z_k$ in $E_k$. Notice that $y$ is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n \in F_n y_n$ in $E_n$ for $n \in N_k$ and $y_n \to y$ in $E_k$ as $n \to \infty$ in $N_k$ (since $y = z_k$ in $E_k$) together with (2.32) imply that $y \in Fy$ in $E$. □

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