MORE ON REVERSE TRIANGLE INEQUALITY IN INNER PRODUCT SPACES

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Refining some results of Dragomir, several new reverses of the generalized triangle inequality in inner product spaces are given. Among several results, we establish some reverses for the Schwarz inequality. In particular, it is proved that if a is a unit vector in a real or complex inner product space $(H;\langle\cdot,\cdot\rangle)$, $r,s>0$, $p\in(0,s]$, $D=\{x\in H,\|rx-sa\|\le p\}$, $x_1,x_2\in D-\{0\}$, and $\alpha_{r,s}=\min\{(r^2\|x_k\|^2-p^2+s^2)/2rs\|x_k\|:1\le k\le 2\}$, then $(\|x_1\|\|x_2\|-\operatorname{Re}\langle x_1,x_2\rangle)/(\|x_1\|+\|x_2\|)^2\le\alpha_{r,s}$.

1. Introduction

It is interesting to know under which conditions the triangle inequality went the other way in a normed space X; in other words, we would like to know if there is a positive constant c with the property that $c\sum_{k=1}^{n}\|x_k\|\le\|\sum_{k=1}^{n}x_k\|$ for any finite set $x_1,\dots,x_n\in X$. Nakai and Tada [7] proved that the normed spaces with this property are precisely those of finite dimensional.

The first authors investigating reverse of the triangle inequality in inner product spaces were Diaz and Metcalf [2] by establishing the following result as an extension of an inequality given by Petrovich [8] for complex numbers.

Theorem 1.1 (Diaz-Metcalf theorem). *Let a be a unit vector in an inner product space $(H;\langle\cdot,\cdot\rangle)$. Suppose the vectors $x_k\in H$, $k\in\{1,\dots,n\}$ satisfy*

$$0\le r\le\frac{\operatorname{Re}\langle x_k,a\rangle}{||x_k||},\quad k\in\{1,\dots,n\}. \tag{1.1}$$

Then

$$r\sum_{k=1}^{n}||x_k||\le\left\|\sum_{k=1}^{n}x_k\right\|, \tag{1.2}$$

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where equality holds if and only if

$$\sum_{k=1}^{n} x_k = r \sum_{k=1}^{n} ||x_k|| a. \tag{1.3}$$

Inequalities related to the triangle inequality are of special interest (cf. [6, Chapter XVII]). They may be applied to get interesting inequalities in complex numbers or to study vector-valued integral inequalities [4, 5].

Using several ideas and following the terminology of [4, 5], we modify or refine some results of Dragomir and ours [1] and get some new reverses of triangle inequality. Among several results, we show that if a is a unit vector in a real or complex inner product space $(H;\langle\cdot,\cdot\rangle)$, $x_k \in H - \{0\}$, $1 \le k \le n$, $\alpha = \min\{\|x_k\| : 1 \le k \le n\}$, $p \in (0, \sqrt{\alpha^2+1})$, $\max\{\|x_k - a\| : 1 \le k \le n\} \le p$, and $\beta = \min\{(\|x_k\|^2 - p^2 + 1)/2\|x_k\| : 1 \le k \le n\}$, then

$$\sum_{k=1}^{n} ||x_k|| - \left\| \sum_{k=1}^{n} x_k \right\| \le \frac{1-\beta}{\beta} \operatorname{Re} \left\langle \sum_{k=1}^{n} x_k, a \right\rangle. \tag{1.4}$$

We also examine some reverses for the celebrated Schwarz inequality. In particular, it is proved that if a is a unit vector in a real or complex inner product space $(H;\langle\cdot,\cdot\rangle)$, $r,s > 0$, $p \in (0,s]$, $D = \{x \in H, \|rx - sa\| \le p\}$, $x_1, x_2 \in D - \{0\}$, and $\alpha_{r,s} = \min\{(r^2\|x_k\|^2 - p^2 + s^2)/2rs\|x_k\| : 1 \le k \le 2\}$, then

$$\frac{||x_1||\,||x_2|| - \operatorname{Re}\langle x_1, x_2\rangle}{\left(||x_1|| + ||x_2||\right)^2} \le \alpha_{r,s}. \tag{1.5}$$

Throughout the paper, $(H;\langle\cdot,\cdot\rangle)$ denotes a real or complex inner product space. We use repeatedly the Cauchy-Schwarz inequality without mentioning it. The reader is referred to [3, 9] for the terminology on inner product spaces.

2. Reverse of triangle inequality

We start this section by pointing out the following theorem of [1] which is a modification of [5, Theorem 3].

Theorem 2.1. *Let $a_1,\ldots,a_m$ be orthonormal vectors in the complex inner product space $(H;\langle\cdot,\cdot\rangle)$. Suppose that for $1 \le t \le m$, $r_t, \rho_t \in R$, and that the vectors $x_k \in H$, $k \in \{1,\ldots,n\}$ satisfy*

$$0 \le r_t^2 ||x_k|| \le \operatorname{Re}\langle x_k, r_t a_t\rangle, \quad 0 \le \rho_t^2 ||x_k|| \le \operatorname{Im}\langle x_k, \rho_t a_t\rangle, \quad 1 \le t \le m. \tag{2.1}$$

Then

$$\left(\sum_{t=1}^{m} (r_t^2 + \rho_t^2)\right)^{1/2} \sum_{k=1}^{n} ||x_k|| \le \left\| \sum_{k=1}^{n} x_k \right\| \tag{2.2}$$

and the equality holds in (2.2) if and only if

$$\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} ||x_k|| \sum_{t=1}^{m} (r_t + i\rho_t) a_t. \tag{2.3}$$

The following theorem is a strengthen of [5, Corollary 1] and a generalization of [1, Theorem 2].

THEOREM 2.2. *Let a be a unit vector in the complex inner product space $(H; \langle \cdot, \cdot \rangle)$. Suppose that the vectors $x_k \in H - \{0\}$, $k \in \{1, \ldots, n\}$ satisfy*

$$\max\{||rx_k - sa|| : 1 \le k \le n\} \le p, \qquad \max\{||r' x_k - is' a|| : 1 \le k \le n\} \le q, \tag{2.4}$$

where $r, r', s, s' > 0$ and

$$p \le \left((r\alpha)^2 + s^2\right)^{1/2}, \qquad q \le \left((r'\alpha)^2 + s'^2\right)^{1/2}, \qquad \alpha = \min\{||x_k|| : 1 \le k \le n\}. \tag{2.5}$$

Let

$$\alpha_{r,s} = \min\left\{\frac{r^2||x_k||^2 - p^2 + s^2}{2rs||x_k||} : 1 \le k \le n\right\}, \qquad \beta_{r',s'} = \min\left\{\frac{r'^2||x_k||^2 - q^2 + s'^2}{2r's'||x_k||} : 1 \le k \le n\right\}. \tag{2.6}$$

Then

$$\left(\alpha_{r,s}^2 + \beta_{r',s'}^2\right)^{1/2} \sum_{k=1}^{n} ||x_k|| \le \left|\left| \sum_{k=1}^{n} x_k \right|\right| \tag{2.7}$$

and the equality holds if and only if

$$\sum_{k=1}^{n} x_k = (\alpha_{r,s} + i\beta_{r',s'}) \sum_{k=1}^{n} ||x_k|| a. \tag{2.8}$$

Proof. From the first inequality above, we infer that

$$\langle rx_k - sa, rx_k - sa\rangle \le p^2, \qquad r^2||x_k||^2 + s^2 - p^2 \le 2\,\mathrm{Re}\,\langle rx_k, sa\rangle. \tag{2.9}$$

Then

$$\frac{r^2||x_k||^2 + s^2 - p^2}{2rs||x_k||}||x_k|| \le \mathrm{Re}\,\langle x_k, a\rangle. \tag{2.10}$$

Similarly,

$$\frac{r'^2||x_k||^2 - q^2 + s'^2}{2r's'||x_k||}||x_k|| \le \mathrm{Im}\,\langle x_k, a\rangle, \tag{2.11}$$

consequently,

$$\begin{aligned} \alpha_{r,s}||x_k|| &\le \operatorname{Re}\langle x_k, a\rangle, \\ \beta_{r',s'}||x_k|| &\le \operatorname{Im}\langle x_k, a\rangle. \end{aligned} \tag{2.12}$$

Applying Theorem 2.1 for $m = 1$, $r_1 = \alpha_{r,s}$, and $\rho_1 = \beta_{r',s'}$, we deduce the desired inequality. □

The next result is an extension of [1, Corollary 3].

Corollary 2.3. *Let a be a unit vector in the complex inner product space $(H;\langle\cdot,\cdot\rangle)$. Suppose that $x_k \in H$, $k \in \{1,\dots,n\}$, $\max\{\|rx_k - sa\| : 1 \le k \le n\} \le r$, $\max\{\|rx_k - isa\| : 1 \le k \le n\} \le s$, where $r > 0$, $s > 0$, and $\alpha = \min\{\|x_k\| : 1 \le k \le n\}$. Then*

$$\frac{r\alpha}{s\sqrt{2}} \sum_{k=1}^{n} ||x_k|| \le \left\| \sum_{k=1}^{n} x_k \right\|. \tag{2.13}$$

The equality holds if and only if

$$\sum_{k=1}^{n} x_k = r\alpha \frac{(1+i)}{2s} \sum_{k=1}^{n} ||x_k|| a. \tag{2.14}$$

Proof. Apply Theorem 2.2 with $r = r'$, $s = s'$, $p = r$, $q = s$. Note that $\alpha_{r,s} = r\alpha/2s = \beta_{r',s'}$. □

Theorem 2.4. *Let a be a unit vector in $(H;\langle\cdot,\cdot\rangle)$. Suppose that the vectors $x_k \in H$, $k \in \{1,\dots,n\}$ satisfy*

$$\max\{||rx_k - sa|| : 1 \le k \le n\} \le p < \left((r\alpha)^2 + s^2\right)^{1/2}, \tag{2.15}$$

where $r > 0$, $s > 0$ and

$$\alpha = \min_{1\le k\le n} ||x_k||. \tag{2.16}$$

Let

$$\alpha_{r,s} = \min\left\{ \frac{r^2||x_k||^2 - p^2 + s^2}{2rs||x_k||} : 1 \le k \le n \right\}. \tag{2.17}$$

Then

$$\alpha_{r,s} \sum_{k=1}^{n} ||x_k|| \le \left\| \sum_{k=1}^{n} x_k \right\|. \tag{2.18}$$

Moreover, the equality holds if and only if

$$\sum_{k=1}^{n} x_k = \alpha_{r,s} \sum_{k=1}^{n} ||x_k|| a. \tag{2.19}$$

Proof. Proof is similar to that of Theorem 2.2 in which we use Theorem 2.1 with $m = 1$, $\rho_1 = 0$. □

Theorem 2.5. *Let a be a unit vector in $(H;\langle\cdot,\cdot\rangle)$. Suppose that $r,s > 0$, and vectors $x_k \in H - \{0\}$, $k \in \{1,\dots,n\}$ satisfy*

$$\sum_{k=1}^{n} x_k = 0. \tag{2.20}$$

Then

$$\sqrt{r^2\alpha^2 + s^2} \le \max\{||rx_k - sa|| : 1 \le n\}, \tag{2.21}$$

where

$$\alpha = \min_{1\le k\le n} ||x_k||. \tag{2.22}$$

Proof. Let $p = \max\{\|rx_k - sa\| : 1 \le k \le n\}$. If $p < \sqrt{r^2\alpha^2 + s^2}$, then, using Theorem 2.4, we get

$$\alpha_{r,s} \sum_{k=1}^{n} ||x_k|| \le \left\| \sum_{k=1}^{n} x_k \right\| = 0. \tag{2.23}$$

Hence $\alpha_{r,s} = 0$. On the other hand, $(p^2 - s^2)/r^2 < \alpha^2$, so

$$\alpha_{r,s} = \min\left\{ \frac{r^2||x_k||^2 - p^2 + s^2}{2rs||x_k||} : 1 \le k \le n \right\} > 0 \tag{2.24}$$

holds a contradiction. □

Theorem 2.6. *Let $a_1,\dots,a_m$ be orthonormal vectors in the complex inner product space $(H;\langle\cdot,\cdot\rangle)$, $M_t \ge m_t > 0$, $L_t \ge \ell_t > 0$, $1 \le t \le m$, and $x_k \in H - \{0\}$, $k \in \{1,\dots,n\}$ such that*

$$\operatorname{Re}\langle M_t a_t - x_k, x_k - m_t a_t\rangle \ge 0, \qquad \operatorname{Re}\langle L_t i a_t - x_k, x_k - \ell_t i a_t\rangle \ge 0, \tag{2.25}$$

or equivalently

$$\left\| x_k - \frac{m_t + M_t}{2} a_t \right\| \le \frac{M_t - m_t}{2}, \qquad \left\| x_k - \frac{L_t + \ell_t}{2} i a_t \right\| \le \frac{L_t - \ell_t}{2}, \tag{2.26}$$

for all $1 \le k \le n$ and $1 \le t \le m$. Let

$$\begin{aligned} \alpha_{m_t,M_t} &= \min\left\{ \frac{||x_k||^2 + m_t M_t}{(m_t + M_t)||x_k||} : 1 \le k \le n \right\}, \quad 1 \le t \le m, \\ \alpha_{\ell_t,L_t} &= \min\left\{ \frac{||x_k||^2 + \ell_t L_t}{(m_t + M_t)||x_k||} : 1 \le k \le n \right\}, \quad 1 \le t \le m, \end{aligned} \tag{2.27}$$

then

$$\left(\sum_{t=1}^{m}\alpha_{m_t,M_t}^2+\alpha_{\ell_t,L_t}^2\right)^{1/2}\sum_{k=1}^{n}||x_k||\le\left\|\sum_{k=1}^{n}x_k\right\|. \tag{2.28}$$

The equality holds if and only if

$$\sum_{k=1}^{n}x_k=\left(\sum_{k=1}^{n}||x_k||\right)\sum_{t=1}^{m}\left(\alpha_{m_t,M_t}+i\alpha_{\ell_t,L_t}\right)a_t. \tag{2.29}$$

Proof. Given $1\le t\le m$ and all $1\le k\le n$, it follows from $\|x_k-((m_t+M_t)/2)a_t\|\le(M_t-m_t)/2$ that

$$||x_k||^2+m_tM_t\le(m_t+M_t)\operatorname{Re}\langle x_k,a_t\rangle. \tag{2.30}$$

Then

$$\frac{||x_k||^2+m_tM_t}{(m_t+M_t)||x_k||}||x_k||\le\operatorname{Re}\langle x_k,a_t\rangle, \tag{2.31}$$

and so

$$\alpha_{m_t,M_t}||x_k||\le\operatorname{Re}\langle x_k,a_t\rangle. \tag{2.32}$$

Similarly, from the second inequality, we deduce that

$$\alpha_{\ell_t,L_t}||x_k||\le\operatorname{Im}\langle x_k,a_t\rangle. \tag{2.33}$$

Applying Theorem 2.5 for $r_t=\alpha_{m_t,M_t}$ and $\rho_t=\alpha_{\ell_t,L_t}$, we obtain the required inequality. □

We will need [4, Theorem 7]. We mention it for the sake of completeness.

Theorem 2.7. *Let a be a unit vector in $(H;\langle\cdot,\cdot\rangle)$, and $x_k\in H-\{0\}$, $k\in\{1,\dots,n\}$. If $r_k\ge0$, $k\in\{1,\dots,n\}$ such that*

$$||x_k||-\operatorname{Re}\langle x_k,a\rangle\le r_k, \tag{2.34}$$

then

$$\sum_{k=1}^{n}||x_k||-\left\|\sum_{k=1}^{n}x_k\right\|\le\sum_{k=1}^{n}r_k. \tag{2.35}$$

The equality holds if and only if

$$\begin{aligned}&\sum_{k=1}^{n}||x_k||\ge\sum_{k=1}^{n}r_k,\\ &\sum_{k=1}^{n}x_k=\left(\sum_{k=1}^{n}||x_k||-\sum_{k=1}^{n}r_k\right)a.\end{aligned} \tag{2.36}$$

Theorem 2.8. *Let a be a unit vector in $(H;\langle\cdot,\cdot\rangle)$, and $x_k \in H - \{0\}$, $k \in \{1,\dots,n\}$. Let*

$$\alpha = \min\{||x_k|| : 1 \le k \le n\}, \qquad p \in (0,\sqrt{\alpha^2+1}), \qquad \max\{||x_k - a|| : 1 \le k \le n\} \le p,$$
$$\beta = \min\left\{\frac{||x_k||^2 - p^2 + 1}{2||x_k||} : 1 \le k \le n\right\}. \tag{2.37}$$

Then

$$\sum_{k=1}^{n} ||x_k|| - \left\|\sum_{k=1}^{n} x_k\right\| \le \frac{1-\beta}{\beta}\operatorname{Re}\left\langle \sum_{k=1}^{n} x_k, a\right\rangle. \tag{2.38}$$

The equality holds if and only if

$$\sum_{k=1}^{n} ||x_k|| \ge \frac{1-\beta}{\beta}\operatorname{Re}\left\langle \sum_{k=1}^{n} x_k, a\right\rangle,$$
$$\sum_{k=1}^{n} x_k = \left(\sum_{k=1}^{n} ||x_k|| - \frac{1-\beta}{\beta}\operatorname{Re}\left\langle \sum_{k=1}^{n} x_k, a\right\rangle\right)a. \tag{2.39}$$

Proof. Since $\max\{\|x_k - a\| : 1 \le k \le n\} \le p$, we have

$$\langle x_k - a, x_k - a\rangle \le p^2, \qquad ||x_k||^2 + 1 - p^2 \le 2\operatorname{Re}\langle x_k, a\rangle,$$
$$\frac{||x_k||^2 - p^2 + 1}{2||x_k||}||x_k|| \le \operatorname{Re}\langle x_k, a\rangle, \qquad \beta||x_k|| \le \operatorname{Re}\langle x_k, a\rangle, \tag{2.40}$$
$$||x_k|| \le \frac{1}{\beta}\operatorname{Re}\langle x_k, a\rangle,$$

for all $k \in \{1,\dots,n\}$. Then

$$||x_k|| - \operatorname{Re}\langle x_k, a\rangle \le \frac{1-\beta}{\beta}\operatorname{Re}\langle x_k, a\rangle, \quad k \in \{1,\dots,n\}. \tag{2.41}$$

Applying Theorem 2.7 for $r_k = ((1-\beta)/\beta)\operatorname{Re}\langle x_k, a\rangle$, $k \in \{1,\dots,n\}$, we deduce the desired inequality. □

As a corollary, we obtain a result similar to [4, Theorem 9].

Corollary 2.9. *Let a be a unit vector in $(H;\langle\cdot,\cdot\rangle)$, and $x_k \in H - \{0\}$, $k \in \{1,\dots,n\}$. Let*

$$\max\{||x_k - a|| : 1 \le k \le n\} \le 1,$$
$$\alpha = \min\{||x_k|| : 1 \le k \le n\}. \tag{2.42}$$

Then

$$\sum_{k=1}^{n} ||x_k|| - \left\|\sum_{k=1}^{n} x_k\right\| \le \frac{2-\alpha}{\alpha}\operatorname{Re}\left\langle \sum_{k=1}^{n} x_k, a\right\rangle. \tag{2.43}$$

The equality holds if and only if

$$\begin{aligned}\sum_{k=1}^{n}||x_k|| &\geq \frac{2-\alpha}{\alpha}\operatorname{Re}\left\langle\sum_{k=1}^{n}x_k,a\right\rangle,\\ \sum_{k=1}^{n}x_k &= \left(\sum_{k=1}^{n}||x_k|| - \frac{2-\alpha}{\alpha}\operatorname{Re}\left\langle\sum_{k=1}^{n}x_k,a\right\rangle\right)a.\end{aligned} \tag{2.44}$$

Proof. Apply Theorem 2.8 with $\beta = \alpha/2$. □

Theorem 2.10. *Let a be a unit vector in $(H;\langle\cdot,\cdot\rangle)$, $M \geq m > 0$, and $x_k \in H - \{0\}$, $k \in \{1,\ldots,n\}$ such that*

$$\operatorname{Re}\langle Ma - x_k, x_k - ma\rangle \geq 0 \tag{2.45}$$

or equivalently

$$\left\|x_k - \frac{m+M}{2}a\right\| \leq \frac{M-m}{2}. \tag{2.46}$$

Let

$$\alpha_{m,M} = \min\left\{\frac{||x_k||^2 + mM}{(m+M)||x_k||} : 1 \leq k \leq n\right\}. \tag{2.47}$$

Then

$$\sum_{k=1}^{n}||x_k|| - \left\|\sum_{k=1}^{n}x_k\right\| \leq \frac{1-\alpha_{m,M}}{\alpha_{m,M}}\operatorname{Re}\left\langle\sum_{k=1}^{n}x_k,a\right\rangle. \tag{2.48}$$

The equality holds if and only if

$$\begin{aligned}\sum_{k=1}^{n}||x_k|| &\geq \frac{1-\alpha_{m,M}}{\alpha_{m,M}}\operatorname{Re}\left\langle\sum_{k=1}^{n}x_k,a\right\rangle,\\ \sum_{k=1}^{n}x_k &= \left(\sum_{k=1}^{n}||x_k|| - \frac{1-\alpha_{m,M}}{\alpha_{m,M}}\operatorname{Re}\left\langle\sum_{k=1}^{n}x_k,a\right\rangle\right)a.\end{aligned} \tag{2.49}$$

Proof. For each $1 \leq k \leq n$, it follows from the inequality

$$\left\|x_k - \frac{m+M}{2}a\right\| \leq \frac{M-m}{2} \tag{2.50}$$

that

$$\left\langle x_k - \frac{m+M}{2}a, x_k - \frac{m+M}{2}a\right\rangle \leq \left(\frac{M-m}{2}\right)^2. \tag{2.51}$$

Hence

$$||x_k||^2 + mM \leq (m+M)\operatorname{Re}\langle x_k,a\rangle. \tag{2.52}$$

So that

$$\alpha_{m,M}||x_k|| \leq \operatorname{Re}\langle x_k, a\rangle, \tag{2.53}$$

consequently

$$||x_k|| - \operatorname{Re}\langle x_k, a\rangle \leq \frac{1-\alpha_{m,M}}{\alpha_{m,M}} \operatorname{Re}\langle x_k, a\rangle. \tag{2.54}$$

Now apply Theorem 2.7 for $r_k = ((1-\alpha_{m,M})/\alpha_{m,M})\operatorname{Re}\langle x_k, a\rangle$, $k \in \{1,\ldots,n\}$. □

3. Reverses of Schwarz inequality

In this section, we provide some reverses of the Schwarz inequality. The first theorem is an extension of [4, Proposition 5.1].

Theorem 3.1. *Let a be a unit vector in $(H;\langle\cdot,\cdot\rangle)$. Suppose that $r,s>0$, $p\in(0,s]$, and*

$$D = \{x \in H, \|rx - sa\| \leq p\}. \tag{3.1}$$

If $0 \neq x_1 \in D$, $0 \neq x_2 \in D$, then

$$\frac{||x_1||\,||x_2|| - \operatorname{Re}\langle x_1, x_2\rangle}{\left(||x_1|| + ||x_2||\right)^2} \leq \frac{1}{2}\left(1 - \left(\frac{r^2||x_1||^2 - p^2 + s^2}{2rs||x_1||}\right)^2\right) \tag{3.2}$$

or

$$\frac{||x_1||\,||x_2|| - \operatorname{Re}\langle x_1, x_2\rangle}{\left(||x_1|| + ||x_2||\right)^2} \leq \frac{1}{2}\left(1 - \left(\frac{r^2||x_2||^2 - p^2 + s^2}{2rs||x_2||}\right)^2\right). \tag{3.3}$$

Proof. Put $\alpha_{r,s} = \min\{(r^2\|x_k\|^2 - p^2 + s^2)/2rs\|x_k\| : 1 \leq k \leq 2\}$. By Theorem 2.4, we obtain

$$\alpha_{r,s}(||x_1|| + ||x_2||) \leq ||x_1 + x_2||. \tag{3.4}$$

Then

$$\alpha_{r,s}^2\left(||x_1||^2 + 2||x_1||\,||x_2|| + ||x_2||^2\right) \leq ||x_1||^2 + 2\operatorname{Re}\langle x_1, x_2\rangle + ||x_2||^2. \tag{3.5}$$

Set $\alpha_{r,s}^2 = 1 - t^2$. Then

$$\frac{||x_1||\,||x_2|| - \operatorname{Re}\langle x_1, x_2\rangle}{\left(||x_1|| + ||x_2||\right)^2} \leq \frac{1}{2}t^2, \tag{3.6}$$

namely,

$$\frac{||x_1||\,||x_2|| - \operatorname{Re}\langle x_1, x_2\rangle}{\left(||x_1|| + ||x_2||\right)^2} \leq \frac{1}{2}(1 - \alpha_{r,s}^2). \tag{3.7}$$

□

Corollary 3.2. *Let a be a unit vector in $(H;\langle\cdot,\cdot\rangle)$. Suppose that $r,s>0$ and*

$$D=\{x\in H,\|rx-sa\|\le s\}. \tag{3.8}$$

If $x,y\in D$ and $0<\|x\|<\|y\|$, then

$$\frac{\|x\|\|y\|-\operatorname{Re}\langle x,y\rangle}{(\|x\|+\|y\|)^2}\le\frac{1}{2}\left(1-\left(\frac{r\|x\|}{2s}\right)^2\right). \tag{3.9}$$

Proof. In the notation of the proof of Theorem 3.1, we get from $p=s$, $x_1=x$, $x_2=y$ that $\alpha_{r,s}=r\|x\|/2s$. Now apply Theorem 3.1. □

Corollary 3.3. *Let a be a unit vector in $(H;\langle\cdot,\cdot\rangle)$, $M\ge m>0$, and $x_k\in H-\{0\}$, $k=1,2$ such that*

$$\operatorname{Re}\langle Ma-x_k,x_k-ma\rangle\ge 0 \tag{3.10}$$

or equivalently

$$\left\|x_k-\frac{m+M}{2}a\right\|\le\frac{M-m}{2}. \tag{3.11}$$

Then

$$\frac{\|x_1\|\|x_2\|-\operatorname{Re}\langle x_1,x_2\rangle}{(\|x_1\|+\|x_2\|)^2}\le\frac{1}{2}\left(1-\left(\frac{\|x_1\|^2+mM}{(m+M)\|x_1\|}\right)^2\right) \tag{3.12}$$

or

$$\frac{\|x_1\|\|x_2\|-\operatorname{Re}\langle x_1,x_2\rangle}{(\|x_1\|+\|x_2\|)^2}\le\frac{1}{2}\left(1-\left(\frac{\|x_2\|^2+mM}{(m+M)\|x_2\|}\right)^2\right). \tag{3.13}$$

Proof. Put $r=1$, $s=(m+M)/2$, $p=(M-m)/2$, $x=x_1$, and $y=x_2$ in Theorem 3.1. □

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