ON THE FINE SPECTRUM OF THE GENERALIZED DIFFERENCE OPERATOR $B(r,s)$ OVER THE SEQUENCE SPACES c_0 AND c

BILÂL ALTAY AND FEYZİ BAŞAR

Received 27 April 2005 and in revised form 15 July 2005

We determine the fine spectrum of the generalized difference operator $B(r,s)$ defined by a band matrix over the sequence spaces c_0 and c, and derive a Mercerian theorem. This generalizes our earlier work (2004) for the difference operator Δ, and includes as other special cases the right shift and the Zweier matrices.

1. Preliminaries, background, and notation

Let X and Y be the Banach spaces and let $T : X \to Y$ also be a bounded linear operator. By $R(T)$, we denote the range of T, that is,

$$R(T) = \{y \in Y : y = Tx,\ x \in X\}. \tag{1.1}$$

By $B(X)$, we also denote the set of all bounded linear operators on X into itself. If X is any Banach space and $T \in B(X)$, then the adjoint T^* of T is a bounded linear operator on the dual X^* of X defined by $(T^* f)(x) = f(Tx)$ for all $f \in X^*$ and $x \in X$.

Let $X \neq \{\theta\}$ be a nontrivial complex normed space and $T : \mathcal{D}(T) \to X$ a linear operator defined on a subspace $\mathcal{D}(T) \subseteq X$. We do not assume that $D(T)$ is dense in X, or that T has closed graph $\{(x,Tx) : x \in D(T)\} \subseteq X \times X$. We mean by the expression "T is *invertible*" that there exists a bounded linear operator $S : R(T) \to X$ for which $ST = I$ on $D(T)$ and $\overline{R(T)} = X$; such that $S = T^{-1}$ is necessarily uniquely determined, and linear; the boundedness of S means that T must be *bounded below*, in the sense that there is $k > 0$ for which $\|Tx\| \geq k\|x\|$ for all $x \in D(T)$. Associated with each complex number, α is the perturbed operator

$$T_\alpha = T - \alpha I, \tag{1.2}$$

defined on the same domain $D(T)$ as T. The *spectrum* $\sigma(T,X)$ consists of those $\alpha \in \mathbb{C}$ for which T_α is not invertible, and the *resolvent* is the mapping from the complement $\sigma(T,X)$ of the spectrum into the algebra of bounded linear operators on X defined by $\alpha \mapsto T_\alpha^{-1}$.

The name resolvent is appropriate since T_α^{-1} helps to solve the equation $T_\alpha x = y$. Thus, $x = T_\alpha^{-1} y$ provided that T_α^{-1} exists. More important, the investigation of properties of T_α^{-1}

International Journal of Mathematics and Mathematical Sciences 2005:18 (2005) 3005–3013
DOI: 10.1155/IJMMS.2005.3005

will be basic for an understanding of the operator T itself. Naturally, many properties of T_α and T_α^{-1} depend on α, and the spectral theory is concerned with those properties. For instance, we will be interested in the set of all α's in the complex plane such that T_α^{-1} exists. Boundedness of T_α^{-1} is another property that will be essential. We will also ask for what α's the domain of T_α^{-1} is dense in X, to name just a few aspects. A *regular value* α of T is a complex number such that T_α^{-1} exists and is bounded and whose domain is dense in X. For our investigation of T, T_α, and T_α^{-1}, we need some basic concepts in the spectral theory which are given as follows (see [8, pages 370–371]).

The *resolvent set* $\rho(T,X)$ of T is the set of all regular values α of T. Furthermore, the spectrum $\sigma(T,X)$ is partitioned into the following three disjoint sets.

The *point (discrete) spectrum* $\sigma_p(T,X)$ is the set such that T_α^{-1} does not exist. A $\alpha \in \sigma_p(T,X)$ is called an *eigenvalue* of T.

The *continuous spectrum* $\sigma_c(T,X)$ is the set such that T_α^{-1} exists and is unbounded and the domain of T_α^{-1} is dense in X.

The *residual spectrum* $\sigma_r(T,X)$ is the set such that T_α^{-1} exists (and may be bounded or not) but the domain of T_α^{-1} is not dense in X.

To avoid trivial misunderstandings, let us say that some of the sets defined above may be empty. This is an existence problem which we will have to discuss. Indeed, it is well-known that $\sigma_c(T,X) = \sigma_r(T,X) = \emptyset$ and the spectrum $\sigma(T,X)$ consists of only the set $\sigma_p(T,X)$ in the finite-dimensional case.

From Goldberg [6, pages 58–71], if X is a Banach space and $T \in B(X)$, then there are three possibilities for $R(T)$ and for T^{-1}:

(I) $R(T) = X$,
(II) $R(T) \neq \overline{R(T)} = X$,
(III) $\overline{R(T)} \neq X$,

and

(1) T^{-1} exists and is continuous,
(2) T^{-1} exists but is discontinuous,
(3) T^{-1} does not exist.

Applying Golberg's classification to T_α, we have three possibilities for T_α and for T_α^{-1}:

(I) T_α is surjective,
(II) $R(T_\alpha) \neq \overline{R(T_\alpha)} = X$,
(III) $\overline{R(T_\alpha)} \neq X$,

and

(1) T_α is injective and T_α^{-1} is continuous,
(2) T_α is injective and T_α^{-1} is discontinuous,
(3) T_α is not injective.

If these possibilities are combined in all possible ways, nine different states are created. These are labeled by I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, and III_3. If α is a complex number such that $T_\alpha \in I_1$ or $T_\alpha \in II_1$, then α is in the resolvent set $\rho(T,X)$ of T. The further classification gives rise to the *fine spectrum* of T. If an operator is in state II_2 for example, then $R(T) \neq \overline{R(T)} = X$ and T^{-1} exists but is discontinuous and we write $\alpha \in II_2\sigma(T,X)$.

By a *sequence space*, we understand a linear subspace of the space $w = \mathbb{C}^{\mathbb{N}}$ of all complex sequences which contain ϕ, the set of all finitely nonzero sequences, where $\mathbb{N} = \{0,1,2,\dots\}$.

We write ℓ_∞, c, c_0, and bv for the spaces of all bounded, convergent, null, and bounded variation sequences, respectively. Also by ℓ_p, we denote the space of all p-absolutely summable sequences, where $1 \le p < \infty$.

Let $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $n,k \in \mathbb{N}$, and write

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk}x_k, \quad (n \in \mathbb{N},\ x \in D_{00}(A)), \tag{1.3}$$

where $D_{00}(A)$ denotes the subspace of w consisting of $x \in w$ for which the sum exists as a finite sum. More generally, if μ is a normed sequence space, we can write $D_\mu(A)$ for $x \in w$ for which the sum in (1.3) converges in the norm of μ. We will write

$$(\lambda : \mu) = \{A : \lambda \subseteq D_\mu(A)\} \tag{1.4}$$

for the space of those matrices which send the whole of the sequence space λ into μ in this sense. Our main focus in this note is on the band matrix $A = B(r,s)$, where

$$B(r,s) = \begin{bmatrix} r & 0 & 0 & \cdots \\ s & r & 0 & \cdots \\ 0 & s & r & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (s \neq 0). \tag{1.5}$$

We begin by determining when a matrix A induces a bounded operator from c to c.

Lemma 1.1 (cf. [14, Theorem 1.3.6, page 6]). *The matrix $A = (a_{nk})$ gives rise to a bounded linear operator $T \in B(c)$ from c to itself if and only if*

(1) *the rows of A in ℓ_1 and their ℓ_1 norms are bounded,*
(2) *the columns of A are in c,*
(3) *the sequence of row sums of A is in c.*

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

Corollary 1.2. *$B(r,s) : c \to c$ is a bounded linear operator and $\|B(r,s)\|_{(c:c)} = |r| + |s|$.*

Lemma 1.3 (cf. [14, Example 8.4.5A, page 129]). *The matrix $A = (a_{nk})$ gives raise to a bounded linear operator $T \in B(c_0)$ from c_0 to itself if and only if*

(1) *the rows of A in ℓ_1 and their ℓ_1 norms are bounded,*
(2) *the columns of A are in c_0.*

The operator norm of T is the supremum of the ℓ_1 norms of the rows.

Corollary 1.4. *$B(r,s) : c_0 \to c_0$ is a bounded linear operator and $\|B(r,s)\|_{(c_0:c_0)} = \|B(r,s)\|_{(c:c)}$.*

We summarize the knowledge in the existing literature concerned with the spectrum of the linear operators defined by some particular limitation matrices over some sequence spaces. Wenger [13] examined the fine spectrum of the integer power of the Cesàro operator in c and Rhoades [12] generalized this result to the weighted mean methods. The fine spectrum of the Cesàro operator on the sequence space ℓ_p has been studied by González [7], where $1 < p < \infty$. The spectrum of the Cesàro operator on the sequence spaces c_0 and

bv have also been investigated by Reade [11], Akhmedov and Başar [1], and Okutoyi [10], respectively. The fine spectrum of the Rhaly operators on the sequence spaces c_0 and c has been examined by Yıldırım [15]. Furthermore, Coşkun [4] has studied the spectrum and fine spectrum for p-Cesàro operator acting on the space c_0. More recently, de Malafosse [5] and Altay and Başar [2] have, respectively, studied the spectrum and the fine spectrum of the difference operator on the sequence spaces s_r and c_0, c; where s_r denotes the Banach space of all sequences $x = (x_k)$ normed by

$$\|x\|_{s_r} = \sup_{k\in\mathbb{N}} \frac{|x_k|}{r^k} \quad (r > 0). \tag{1.6}$$

In this work, our purpose is to determine the fine spectrum of the generalized difference operator $B(r,s)$ on the sequence spaces c_0 and c, and to give a Mercerian theorem. The main results of the present work are more general than those of Altay and Başar [2].

2. The spectrum of the operator $B(r,s)$ on the sequence spaces c_0 and c

In this section, we examine the spectrum, the point spectrum, the continuous spectrum, the residual spectrum, and the fine spectrum of the operator $B(r,s)$ on the sequence spaces c_0 and c. Finally, we also give a Mercerian theorem.

Theorem 2.1. $\sigma(B(r,s),c_0) = \{\alpha \in \mathbb{C} : |\alpha - r| \le |s|\}$.

Proof. Firstly, we prove that $(B(r,s)-\alpha I)^{-1}$ exists and is in $(c_0 : c_0)$ for $|\alpha - r| > |s|$, and secondly the operator $B(r,s)-\alpha I$ is not invertible for $|\alpha - r| \le |s|$.

Let $\alpha \notin \sigma(B(r,s),c_0)$. Since $B(r,s)-\alpha I$ is triangle, $(B(r,s)-\alpha I)^{-1}$ exists and solving $(B(r,s)-\alpha I)x = y$ for x in terms of y gives the matrix $(B(r,s)-\alpha I)^{-1}$. The nth row turns out to be

$$\frac{(-s)^{n-k}}{(r-\alpha)^{n-k+1}} \tag{2.1}$$

in the kth place for $k \le n$ and zero otherwise. Thus, we observe that

$$\left\|(B(r,s)-\alpha I)^{-1}\right\|_{(c_0:c_0)} = \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \frac{(-s)^{n-k}}{(r-\alpha)^{n-k+1}} \right| = \left| \frac{1}{r-\alpha} \right| \sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left| \frac{s}{r-\alpha} \right|^{k} < \infty, \tag{2.2}$$

that is, $(B(r,s)-\alpha I)^{-1} \in (c_0 : c_0)$.

Let $\alpha \in \sigma(B(r,s),c_0)$ and $\alpha \ne r$. Since $B(r,s)-\alpha I$ is triangle, $(B(r,s)-\alpha I)^{-1}$ exists but one can see by (2.2) that

$$\left\|(B(r,s)-\alpha I)^{-1}\right\|_{(c_0:c_0)} = \infty \tag{2.3}$$

whenever $\alpha \in \sigma(B(r,s),c_0)$, that is, $(B(r,s)-\alpha I)^{-1}$ is not in $B(c_0)$.

If $\alpha = r$, then the operator $B(r,s) - \alpha I = B(0,s)$ is represented by the matrix

$$B(0,s) = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ s & 0 & 0 & \cdots \\ 0 & s & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.4}$$

Since $\overline{R(B(0,s))} \neq c_0$, $B(0,s)$ is not invertible. This completes the proof. □

Theorem 2.2. $\sigma_p(B(r,s),c_0) = \emptyset$.

Proof. Suppose that $B(r,s)x = \alpha x$ for $x \neq \theta = (0,0,0,\ldots)$ in c_0. Then, by solving the system of linear equations

$$\begin{aligned} rx_0 &= \alpha x_0, \\ sx_0 + rx_1 &= \alpha x_1, \\ sx_1 + rx_2 &= \alpha x_2, \\ &\vdots \\ sx_k + rx_{k+1} &= \alpha x_{k+1}, \\ &\vdots \end{aligned} \tag{2.5}$$

we find that if x_{n_0} is the first nonzero entry of the sequence $x = (x_n)$, then $\alpha = r$ and

$$x_{n_0+k} = 0 \tag{2.6}$$

for all $k \in \mathbb{N}$. This contradicts the fact that $x_{n_0} \neq 0$, which completes the proof. □

If $T : c_0 \to c_0$ is a bounded linear operator with the matrix A, then it is known that the adjoint operator $T^* : c_0^* \to c_0^*$ is defined by the transpose A^t of the matrix A. It should be noted that the dual space c_0^* of c_0 is isometrically isomorphic to the Banach space ℓ_1 of absolutely summable sequences normed by $\|x\| = \sum_{k=0}^{\infty} |x_k|$.

Theorem 2.3. $\sigma_p(B(r,s)^*, c_0^*) = \{\alpha \in \mathbb{C} : |\alpha - r| < |s|\}$.

Proof. Suppose that $B(r,s)^* x = \alpha x$ for $x \neq \theta$ in $c_0^* \cong \ell_1$. Then, by solving the system of linear equations

$$\begin{aligned} rx_0 + sx_1 &= \alpha x_0, \\ rx_1 + sx_2 &= \alpha x_1, \\ rx_2 + sx_3 &= \alpha x_2, \\ &\vdots \\ rx_k + sx_{k+1} &= \alpha x_k, \\ &\vdots \end{aligned} \tag{2.7}$$

we observe that

$$x_n = \left(\frac{\alpha - r}{s}\right)^n x_0. \tag{2.8}$$

This shows that $x \in \ell_1$ if and only if $|\alpha - r| < |s|$, as asserted. □

Now, we may give the following lemma required in the proof of theorems given in the present section.

Lemma 2.4 [6, page 59]. *T has a dense range if and only if T^* is one to one.*

Theorem 2.5. $\sigma_r(B(r,s),c_0) = \{\alpha \in \mathbb{C} : |\alpha - r| < |s|\}$.

Proof. We show that the operator $B(r,s) - \alpha I$ has an inverse and $\overline{R(B(r,s) - \alpha I)} \neq c_0$ for α satisfying $|\alpha - r| < |s|$. For $\alpha \neq r$, the operator $B(r,s) - \alpha I$ is triangle, hence has an inverse. For $\alpha = r$, the operator $B(r,s) - \alpha I$ is one to one, hence has an inverse. But $B(r,s)^* - \alpha I$ is not one to one by Theorem 2.3. Now, Lemma 2.4 yields the fact that $\overline{R(B(r,s) - \alpha I)} \neq c_0$ and this step concludes the proof. □

Theorem 2.6. *If $\alpha = r$, then $\alpha \in III_1\sigma(B(r,s),c_0)$.*

Proof. Since the operator $B(r,s) - \alpha I = B(0,s)$ for $\alpha = r$, $B(0,s) \in III_1$ or $\in III_2$ by Theorem 2.5. To verify the fact that $B(0,s)$ has a bounded inverse, it is enough to show that $B(0,s)$ is bounded below. Indeed, one can easily see for all $x \in c_0$ that

$$||B(0,s)x|| \geq \frac{|s|}{2}\|x\|, \tag{2.9}$$

which means that $B(0,s)$ is bounded below. This completes the proof. □

Theorem 2.7. *If $\alpha \neq r$ and $\alpha \in \sigma_r(B(r,s),c_0)$, then $\alpha \in III_2\sigma(B(r,s),c_0)$.*

Proof. By Theorem 2.5, $B(r,s) - \alpha I \in III_1$ or $\in III_2$. Hence, by (2.2), the inverse of the operator $B(r,s) - \alpha I$ is discontinuous. Therefore, $B(r,s) - \alpha I$ has an unbounded inverse. □

Theorem 2.8. $\sigma_c(B(r,s),c_0) = \{\alpha \in \mathbb{C} : |\alpha - r| = |s|\}$.

Proof. For this, we prove that the operator $B(r,s) - \alpha I$ has an inverse and

$$\overline{R(B(r,s) - \alpha I)} = c_0, \tag{2.10}$$

if $\alpha \in \sigma_c(B(r,s),c_0)$. Since $\alpha \neq r$, $B(r,s) - \alpha I$ is triangle and has an inverse. Therefore, $B(r,s)^* - \alpha I$ is one to one by Theorem 2.3 and (2.10) holds from Lemma 2.4. This is what we wished to prove. □

Theorem 2.9. *If $\alpha \in \sigma_c(B(r,s),c_0)$, then $\alpha \in II_2\sigma(B(r,s),c_0)$.*

Proof. By (2.2), the inverse of the operator $B(r,s) - \alpha I$ is discontinuous. Therefore, $B(r,s) - \alpha I$ has an unbounded inverse.

By Theorem 2.3, $B(r,s)^* - \alpha I$ is one to one. By Lemma 2.4, $B(r,s) - \alpha I$ has dense range.

To verify that the operator $B(r,s)-\alpha I$ is not surjective, it is sufficient to show that there is no sequence $x=(x_n)$ in c_0 such that $(B(r,s)-\alpha I)x=y$ for some $y\in c_0$. Let us consider the sequence $y=(1,0,0,\dots)\in c_0$. For this sequence, we obtain $x_n=\{s/(\alpha-r)\}^n/(r-\alpha)$. This yields that $x\notin c_0$, that is, $B(r,s)-\alpha I$ is not onto. This completes the proof. □

Theorem 2.10. $\sigma(B(r,s),c)=\{\alpha\in\mathbb{C}:|\alpha-r|\le|s|\}$.

Proof. This is obtained in the similar way that is used in the proof of Theorem 2.1. □

Theorem 2.11. $\sigma_p(B(r,s),c)=\varnothing$.

Proof. The proof may be obtained by proceeding as in proving Theorem 2.2. So, we omit the details. □

If $T:c\to c$ is a bounded matrix operator with the matrix A, then $T^*:c^*\to c^*$ acting on $\mathbb{C}\oplus\ell_1$ has a matrix representation of the form

$$\begin{bmatrix}\chi & 0\\ b & A^t\end{bmatrix}, \tag{2.11}$$

where χ is the limit of the sequence of row sums of A minus the sum of the limit of the columns of A, and b is the column vector whose kth entry is the limit of the kth column of A for each $k\in\mathbb{N}$. For $B(r,s):c\to c$, the matrix $B(r,s)^*\in B(\ell_1)$ is of the form

$$B(r,s)^*=\begin{bmatrix}r+s & 0\\ 0 & B(r,s)^t\end{bmatrix}. \tag{2.12}$$

Theorem 2.12. $\sigma_p(B(r,s)^*,c^*)=\{\alpha\in\mathbb{C}:|\alpha-r|<|s|\}\cup\{r+s\}$.

Proof. Suppose that $B(r,s)^*x=\alpha x$ for $x\neq\theta$ in ℓ_1. Then by solving the system of linear equations

$$\begin{aligned}(r+s)x_0&=\alpha x_0,\\ rx_1+sx_2&=\alpha x_1,\\ rx_2+sx_3&=\alpha x_2,\\ &\vdots\\ rx_k+sx_{k+1}&=\alpha x_k,\\ &\vdots\end{aligned} \tag{2.13}$$

we obtain that

$$x_n=\left(\frac{\alpha-r}{s}\right)^{n-1}x_1\quad(n\ge 2). \tag{2.14}$$

If $x_0\neq 0$, then $\alpha=r+s$. So, $\alpha=r+s$ is an eigenvalue with the corresponding eigenvector $x=(x_0,0,0,\dots)$. If $\alpha\neq r+s$, then $x_0=0$ and one can see by (2.14) that $x\in\ell_1$ if and only if $|\alpha-r|<|s|$. □

Theorem 2.13. $\sigma_r(B(r,s),c) = \sigma_p(B(r,s)^*,c^*)$.

Proof. The proof is obtained by the analogy with the proof of Theorem 2.5. □

Theorem 2.14. $\sigma_c(B(r,s),c) = \{\alpha \in \mathbb{C} : |\alpha - r| = |s|\} \setminus \{r+s\}$.

Proof. This is similar to the proof of Theorem 2.8 with $\alpha \neq r+s$ such that $|\alpha - r| = |s|$. □

Since the fine spectrum of the operator $B(r,s)$ on c can be derived by analogy to that space c_0, we omit the detail and give it without proof. Therefore, we have

$$\begin{gathered} B(r,s) - \alpha I \in I_1, \quad \alpha \notin \sigma(B(r,s),c), \\ \alpha \in III_1\sigma(B(r,s),c), \quad \alpha = r, \\ \alpha \in II_2\sigma(B(r,s),c), \qquad \alpha \in \sigma_c(B(r,s),c), \\ \alpha \in III_2\sigma(B(r,s),c), \qquad \alpha \in \sigma_r(B(r,s),c) \setminus \{r\}. \end{gathered} \tag{2.15}$$

Theorem 2.15. $\sigma(B(r,s),\ell_\infty) = \{\alpha \in \mathbb{C} : |\alpha - r| \leq |s|\}$.

Proof. It is known by Cartlidge [3] that if a matrix operator A is bounded on c, then $\sigma(A,c) = \sigma(A,\ell_\infty)$. Now, the proof is immediate from Theorem 2.10 with $A = B(r,s)$. □

Subsequent to stating the concept of Mercerian theorem, we conclude this section by giving a Mercerian theorem. Let A be an infinite matrix and the set c_A denotes the convergence field of that matrix A. A theorem which proves that $c_A = c$ is called a Mercerian theorem, after Mercer, who proved a significant theorem of this type [9, page 186]. Now, we may give our final theorem.

Theorem 2.16. *Suppose that α satisfies the inequality $|\alpha(1-r)+r| > |s(1-\alpha)|$. Then the convergence field of $A = \alpha I + (1-\alpha)B(r,s)$ is c.*

Proof. If $\alpha = 1$, there is nothing to prove. Let us suppose that $\alpha \neq 1$. Then, one can observe by Theorem 2.10 and the choice of α that $B(r,s) - [\alpha/(\alpha-1)]I$ has an inverse in $B(c)$. That is to say, that

$$A^{-1} = \frac{1}{1-\alpha}\left(B(r,s) - \frac{\alpha}{\alpha-1}I\right)^{-1} \in B(c). \tag{2.16}$$

Since A is a triangle and is in $B(c)$, A^{-1} is also conservative which implies that $c_A = c$; see [14, page 12]. □

Acknowledgments

We have benefited a lot from the referees' reports. So, we thank to the reviewers for careful reading and making some useful comments which improved the presentation of the note.

References

[1] A. M. Akhmedov and F. Başar, *On the fine spectrum of the Cesàro operator in c_0*, Math. J. Ibaraki Univ. **36** (2004), 25–32.

[2] B. Altay and F. Başar, *On the fine spectrum of the difference operator Δ on c_0 and c*, Inform. Sci. **168** (2004), no. 1-4, 217–224.

[3] J. P. Cartlidge, *Weighted mean matrices as operators on ℓ^p*, Ph.D. dissertation, Indiana University, Indiana, 1978.

[4] C. Coşkun, *The spectra and fine spectra for p-Cesàro operators*, Turkish J. Math. **21** (1997), no. 2, 207–212.

[5] B. de Malafosse, *Properties of some sets of sequences and application to the spaces of bounded difference sequences of order μ*, Hokkaido Math. J. **31** (2002), no. 2, 283–299.

[6] S. Goldberg, *Unbounded Linear Operators*, Theory and applications, Dover, New York, 1985.

[7] M. González, *The fine spectrum of the Cesàro operator in $l_p\,(1 < p < \infty)$*, Arch. Math. (Basel) **44** (1985), no. 4, 355–358.

[8] E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley & Sons, New York, 1978.

[9] I. J. Maddox, *Elements of Functional Analysis*, Cambridge University Press, London, 1970.

[10] J. T. Okutoyi, *On the spectrum of C_1 as an operator on bv*, Comm. Fac. Sci. Univ. Ankara Sér. A_1 Math. Statist. **41** (1992), no. 1-2, 197–207 (1994).

[11] J. B. Reade, *On the spectrum of the Cesàro operator*, Bull. London Math. Soc. **17** (1985), no. 3, 263–267.

[12] B. E. Rhoades, *The fine spectra for weighted mean operators*, Pacific J. Math. **104** (1983), no. 1, 219–230.

[13] R. B. Wenger, *The fine spectra of the Hölder summability operators*, Indian J. Pure Appl. Math. **6** (1975), no. 6, 695–712.

[14] A. Wilansky, *Summability Through Functional Analysis*, North-Holland Mathematics Studies, vol. 85, North-Holland, Amsterdam, 1984.

[15] M. Yıldırım, *On the spectrum and fine spectrum of the compact Rhaly operators*, Indian J. Pure Appl. Math. **27** (1996), no. 8, 779–784.

Bilâl Altay: Matematik Eğitimi Bölümü, Eğitim Fakültesi, İnönü Üniversitesi, 44280 Malatya, Turkey

E-mail address: baltay@inonu.edu.tr

Feyzi Başar: Matematik Eğitimi Bölümü, Eğitim Fakültesi, İnönü Üniversitesi, 44280 Malatya, Turkey

E-mail address: fbasar@inonu.edu.tr