ON PROPERTIES OF FUZZY LEFT h-IDEALS IN HEMIRINGS WITH t-NORMS

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The notion of T-fuzzy left h-ideals in a hemiring is introduced and basic properties are investigated. Using a collection of left h-ideals of a hemiring S, T-fuzzy left h-ideals of S are established. The notion of a finite-valued T-fuzzy left h-ideal is introduced, and its characterization is given. T-fuzzy relations on a hemiring S are discussed.

1. Introduction

In [23], Zadeh introduced the notion of fuzzy sets and fuzzy set operations. Since then, fuzzy set theory developed by Zadeh and others has evoked great interest among researchers working in different branches of mathematics. Semirings play an important role in studying matrices and determinants. Many aspects of the theory of matrices and determinants over semirings have been studied by Beasley and Pullman [4, 5], Ghosh [10], and others. Although ideals in semirings are useful for many purposes, they do not in general coincide with the usual ring ideals if S is a ring, and for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Indeed, many results in rings apparently have no analogues in semirings using only ideals. Henriksen [12] defined a more restricted class of ideals in semirings, which is called k-ideals, with the property that if the semiring S is a ring, then a complex in S is a k-ideal if and only if it is a ring ideal. A still more restricted class of ideals in hemirings has been given by Iizuka [13]. However, a definition of ideals in any additively commutative semiring S can be given which coincides with Iizuka's definition provided that S is a hemiring, and it is called h-ideal. La Torre [19] investigated h-ideals and k-ideals in hemirings in an effort to obtain analogues of familiar ring theorems. Several authors have discussed a fuzzy theory in semirings (see [2, 3, 7, 8, 9, 11, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25]).

In this paper, we introduce the concept of T-fuzzy left h-ideals in a hemiring and investigated some basic properties. Using a collection of left h-ideals of a hemiring S, T-fuzzy left h-ideals of S are established. Moreover, we introduce the notion of a finite-valued T-fuzzy left h-ideal and give its characterization. At last, we discuss the T-fuzzy relation on a hemiring S.

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2. Preliminaries

A semiring S is a system consisting of a nonempty set S together with two binary operations on S called addition and multiplication such that

(I) S together with addition is a semigroup;

(II) S together with multiplication is a semigroup;

(III) $a(b+c) = ab+ac$ and $(a+b) = ac+bc$ for all $a,b,c \in S$.

A semiring S is said to be additively commutative if $a+b = b+a$ for all $a,b \in S$. A zero element of a semiring S is an element 0 such that $0 \cdot x = x \cdot 0 = 0$ and $0+x = x+0 = x$ for all $x \in S$. By a hemiring, we mean an additively commutative semiring with zero.

A subset A of a semiring S is called a left ideal of S if (I1) A is closed under addition; (I2) $SA \subseteq A$. A left ideal A of S is called a left k-ideal of S if $y,z \in A$, $x \in S$ and $x+y = z$ implies that $x \in A$. Right (k)-ideals are defined similarly. A fuzzy set μ in a semiring S is called a fuzzy left ideal of S if it satisfies (FI1) $\mu(x+y) \geq \min\{\mu(x),\mu(y)\}$ for all $x,y \in S$; (FI2) $\mu(xy) \geq \mu(y)$ for all $x,y \in S$. Note that if μ is a fuzzy left ideal of a hemiring S, then $\mu(0) \geq \mu(x)$ for all $x \in S$. A fuzzy left ideal μ of a semiring S is called a fuzzy left k-ideal of S (see [11]) if for every $x,y,z \in S$, $x+y = z$ implies that $\mu(x) \geq \min\{\mu(y),\mu(z)\}$. Fuzzy right ($k$)-ideals are defined similarly.

Proposition 2.1 (see [19]). *A left h-ideal of a hemiring S is defined to be a left ideal A of S such that*

(I3) *$x+a+z = b+z$ implies that $x \in A$ for any $x,z \in S$ and $a,b \in A$.*

Right h-ideals are defined similarly.

Definition 2.2 (see [17]). A fuzzy left h-ideal of a hemiring S is defined to be a fuzzy left ideal μ of S such that

(FI3) $x+a+z = b+z$ implies that $\mu(x) \geq \min\{\mu(a),\mu(b)\}$ for all $a,b,x,z \in S$.

Fuzzy right h-ideals are defined similarly.

Definition 2.3. A homomorphism of a hemiring S into a hemiring S' such that $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$ for all $x,y \in S$ is defined.

Definition 2.4 (see [1]). By a t-norm T, a function $T : [0,1]\times[0,1] \to [0,1]$ satisfying the following conditions is meant:

(T1) $T(x,1) = x$;

(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$;

(T3) $T(x,y) = T(y,x)$;

(T4) $T(x,T(y,z)) = T(T(x,y),z)$;

for all $x,y,z \in [0,1]$.

For a t-norm T on $[0,1]$, it is denoted by $\Delta_T = \{\alpha \in [0,1] \mid T(\alpha,\alpha) = \alpha\}$. It is clear that every t-norm T has the following property: $T(\alpha,\beta) \leq \min\{\alpha,\beta\}$.

Definition 2.5. Let T be a t-norm. Then the fuzzy set μ in S is satisfying the imaginable property if $\operatorname{Im}\mu \subseteq \Delta_T$.

Definition 2.6. Let S be a hemiring and let μ be a fuzzy subset of S. For any $\alpha \in [0,1]$, then the set $U(\mu;\alpha) = \{x \in S \mid \mu(x) \geq \alpha\}$ is called a level subset of S with respect to μ.

Lemma 2.7 (see [17]). *A fuzzy set μ in S is a fuzzy left h-ideal of S if and only if the level subset $U(\mu;\alpha)$, $\alpha \in [0,1]$ of μ is a left h-ideal of S whenever $U(\mu;\alpha) \neq \emptyset$.*

3. T-fuzzy left h-ideals in hemirings

Definition 3.1. A fuzzy subset μ is called a fuzzy left ideal of a hemiring S with respect to a t-norm T (briefly, T-fuzzy left ideal of S) if it satisfies the following:

(TFI1) $\mu(x+y) \geq T(\mu(x),\mu(y))$;

(TFI2) $\mu(xy) \geq \mu(y)$;

for all $x, y \in S$.

T-fuzzy right ideals are defined similarly.

Definition 3.2. A T-fuzzy left h-ideal of a hemiring S is defined to be a T-fuzzy left ideal μ of S such that

(TFI3) $x+a+z = b+z$ implies that $\mu(x) \geq T(\mu(a),\mu(b))$ for all $a,b,x,z \in S$.

T-fuzzy right h-ideals are defined similarly.

Definition 3.3. A T-fuzzy left h-ideal of a hemiring S is said to be imaginable if it satisfies the imaginable property.

Example 3.4. Let S be the set of natural numbers including 0, and S is a hemiring with usual addition and multiplication. Define a fuzzy set $\mu : S \to [0,1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x \text{ is even or } 0, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

and let $T_m : [0,1] \times [0,1] \to [0,1]$ be a function defined by $T_m(\alpha,\beta) = \max\{\alpha+\beta-1, 0\}$ for all $\alpha,\beta \in [0,1]$. Then, T_m is a t-norm. By routine calculations, we know that μ is an imaginable T-fuzzy left h-ideal of S.

Proposition 3.5. *Let T be a t-norm. Then every imaginable T-fuzzy left h-ideal μ of a hemiring S is a fuzzy left h-ideal of S.*

Proof. Assume that μ is an imaginable T-fuzzy left h-ideal of S, then we have (TFI1) $\mu(x+y) \geq T(\mu(x),\mu(y))$ and (TFI2) $\mu(xy) \geq \mu(y)$ for all $x, y \in S$.

Since μ is imaginable, we have

$$\begin{aligned} \min\{\mu(x),\mu(y)\} &= T(\min\{\mu(x),\mu(y)\},\min\{\mu(x),\mu(y)\}) \\ &\leq T(\mu(x),\mu(y)) \leq \min\{\mu(x),\mu(y)\}, \end{aligned} \tag{3.2}$$

and so $T(\mu(x),\mu(y)) = \min\{\mu(x),\mu(y)\}$.

It follows that

$$\mu(x+y) \geq T(\mu(x),\mu(y)) = \min\{\mu(x),\mu(y)\} \quad \forall x, y \in S. \tag{3.3}$$

Hence, μ is a fuzzy left ideal of S.

Let $x,z,a,b \in S$ be such that $x+a+z=b+z$, then $\mu(x) \geq T(\mu(a),\mu(b))$ since μ is a T-fuzzy left h-ideal of S. It follows that

$$\begin{aligned}\min\{\mu(a),\mu(b)\} &= T(\min\{\mu(a),\mu(b)\},\min\{\mu(a),\mu(b)\}) \\ &\leq T(\mu(a),\mu(b)) \leq \min\{\mu(a),\mu(b)\}\end{aligned} \tag{3.4}$$

since μ is imaginable.

Hence, $\mu(x) \geq T(\mu(a),\mu(b)) = \min\{\mu(a),\mu(b)\}$.

Thus, μ is a fuzzy left h-ideal of S. □

Using Lemma 2.7 and Proposition 3.5, we have the following corollary.

Corollary 3.6. *If μ is an imaginable T-fuzzy left h-ideal of S, then each nonempty level subset $U(\mu;\alpha)$ of μ is a left h-ideal of S.*

The following example shows that there exists a t-norm T such that a fuzzy left h-ideal of S may not be an imaginable T-fuzzy left h-ideal of S.

Example 3.7. Let S be a hemiring in Example 3.4. Define a fuzzy subset $\mu : S \to [0,1]$ such that

$$\mu(x) = \begin{cases} \dfrac{1}{2} & \text{if } x \text{ is even or } 0, \\ \dfrac{1}{4} & \text{otherwise,} \end{cases} \tag{3.5}$$

is a fuzzy left h-ideal of S.

Let $\gamma \in (0,1)$ and define the binary operation T_γ on (0,1) as follows:

$$T_\gamma(\alpha,\beta) = \begin{cases} \alpha \cap \beta & \text{if } \max\{\alpha,\beta\} = 1, \\ 0 & \max\{\alpha,\beta\} < 1,\ \alpha+\beta \leq 1+\gamma, \\ \gamma & \text{otherwise,} \end{cases} \tag{3.6}$$

for all $\alpha,\beta \in [0,1]$.

Then T_γ is a t-norm. It is easy to check that μ is a T-fuzzy left h-ideal of S, but

$$T_\gamma(\mu(0),\mu(0)) = T_\gamma\left(\frac{1}{2},\frac{1}{2}\right) = 0 \neq \mu(0). \tag{3.7}$$

Hence, μ is not an imaginable T-fuzzy left h-ideal of S.

Now, we consider the following theorem.

Theorem 3.8. *Let T be a t-norm and let μ be an imaginable fuzzy set in a hemiring S, then μ is an imaginable T-fuzzy left h-ideal of S if and only if each nonempty level subset $U(\mu;\alpha)$ of μ is a left h-ideal of S.*

Proof. The necessary condition can be given by Corollary 3.6. Conversely, assume that each nonempty level subset $U(\mu;\alpha)$ of μ is a left h-ideal of S. Then, μ is a fuzzy left h-ideal of S by Lemma 2.7, and so $\mu(x+y) \geq \min\{\mu(x),\mu(y)\} \geq T(\mu(x),\mu(y))$ for all $x, y \in S$. Hence, μ is a T-fuzzy left ideal of S.

Now, let $x,z,a,b \in S$ be such that $x+a+z = b+z$. Then, $\mu(x) \geq \min\{\mu(a),\mu(b)\} \geq T(\mu(a),\mu(b))$. Therefore, μ is an imaginable T-fuzzy left h-ideal of S. □

If μ is a fuzzy set in a hemiring S and θ is a mapping from S into itself, we define a mapping $\mu[\theta] : S \to [0,1]$ by $\mu[\theta](x) = \mu(\theta(x))$ for all $x \in S$.

Proposition 3.9. *If μ is a T-fuzzy left h-ideal of a hemiring S and θ is an endomorphism of S, then $\mu[\theta]$ is a T-fuzzy left h-ideal of S.*

Proof. For any $x, y \in S$, we have

$$\begin{aligned} \mu[\theta](x+y) &= \mu(\theta(x+y)) = \mu(\theta(x)+\theta(y)) \\ &\geq T(\mu(\theta(x)),\mu(\theta(y))) \\ &= T(\mu[\theta](x),\mu[\theta](y)), \\ \mu[\theta](xy) &= \mu(\theta(xy)) = \mu(\theta(x)\theta(y)) \\ &\geq \mu(\theta(y)) = \mu[\theta](y). \end{aligned} \tag{3.8}$$

Hence, $\mu[\theta]$ is a T-fuzzy left ideal of S.

Let $x,z,a,b \in S$ be such that $x+a+z = b+z$. Then $\theta(x+a+z) = \theta(b+z)$, and so $\theta(x)+\theta(a)+\theta(z) = \theta(b)+\theta(z)$.

It follows that

$$\mu[\theta](x) = \mu(\theta(x)) \geq T(\mu(\theta(a)),\mu(\theta(b))) = T(\mu[\theta](a),\mu[\theta](b)). \tag{3.9}$$

Therefore, $\mu[\theta]$ is a T-fuzzy left h-ideal of S. □

Let f be a mapping defined on a hemiring S. If ν is a fuzzy set in $f(S)$, then the fuzzy set $\mu = \nu \circ f$ (i.e., the fuzzy set defined by $\mu(x) = \nu(f(x))$ for all $x \in S$) is called the preimage of ν under f.

Proposition 3.10. *An onto homomorphic preimage of a T-fuzzy left h-ideal of a hemiring S is T-fuzzy left h-ideal.*

Proof. Let $f : S \to S'$ be an onto homomorphism of hemirings, and let ν be a T-fuzzy left h-ideal of S', and μ the preimage of ν under f. Then we have

$$\begin{aligned} \mu(x+y) &= \nu(f(x+y)) = \nu(f(x)+f(y)) \\ &\geq T(\nu(f(x)),\nu(f(y))) = T(\mu(x),\mu(y)), \\ \mu(xy) &= \nu(f(xy)) = \nu(f(x)f(y)) \geq \nu(f(y)) = \mu(y). \end{aligned} \tag{3.10}$$

Hence, μ is a T-fuzzy left ideal of S. Let $x,z,a,b \in S$ be such that $x+a+z = b+z$, then $f(x+a+z) = f(b+z)$, and so $f(x)+f(a)+f(z) = f(b)+f(z)$.

It follows that

$$\mu(x)=\nu(f(x))\geq T(\nu(f(a)),\nu(f(b)))=T(\mu(a),\mu(b)). \tag{3.11}$$

Therefore, μ is a T-fuzzy left h-ideal of S. □

Let μ be a fuzzy set in a hemiring S and f a mapping defined on S. Then the fuzzy set μ^f in $f(S)$ defined by $\mu^f(y)=\sup_{x\in f^{-1}(y)}\mu(x)$ for all $y\in f(S)$ is called the image of μ under f. A fuzzy set μ in S is said to have the sup property if for every subset $H\subseteq S$, there exists $h_0\in H$ such that $\mu(h_0)=\sup_{h\in H}\mu(h)$.

Proposition 3.11. *An onto homomorphic image of a fuzzy left h-ideal with the* sup *property is a fuzzy left h-ideal.*

Proof. Let $f:S\to S'$ be an onto homomorphism of hemirings and let μ be a fuzzy ideal of S with the sup property. Given $x',y'\in S'$, we let $x_0\in f^{-1}(x')$ and $y_0\in f^{-1}(y')$ be such that

$$\mu(x_0)=\sup_{h\in f^{-1}(x')}\mu(h),\qquad \mu(y_0)=\sup_{h\in f^{-1}(y')}\mu(h), \tag{3.12}$$

respectively. Then we can deduce that

$$\begin{aligned}
\mu^f(x'+y')&=\sup_{z\in f^{-1}(x'+y')}\mu(z)\geq\min\{\mu(x_0),\mu(y_0)\}\\
&=\min\left\{\sup_{h\in f^{-1}(x')}\mu(h),\sup_{h\in f^{-1}(y')}\mu(h)\right\}\\
&=\min\{\mu^f(x'),\mu^f(y')\},\\
\mu^f(x'y')&=\sup_{z\in f^{-1}(x'y')}\mu(z)\geq\mu(y_0)\\
&=\sup_{h\in f^{-1}(y')}\mu(h)=\mu^f(y').
\end{aligned} \tag{3.13}$$

Hence, μ^f is a fuzzy left ideal of S'.

Now, given $x',z',a',b'\in S'$, let $x_0\in f^{-1}(x')$, $z_0\in f^{-1}(z')$, $a_0\in f^{-1}(a')$, and $b_0\in f^{-1}(b')$, respectively, be such that

$$\begin{aligned}
\mu(x_0)&=\sup_{h\in f^{-1}(x')}\mu(h),\\
\mu(z_0)&=\sup_{h\in f^{-1}(z')}\mu(h),\\
\mu(a_0)&=\sup_{h\in f^{-1}(a')}\mu(h),\\
\mu(b_0)&=\sup_{h\in f^{-1}(b')}\mu(h).
\end{aligned} \tag{3.14}$$

Then,

$$\begin{aligned}\mu^f(x') &= \sup_{z\in f^{-1}(x')} \mu(z) \geq \min\{\mu(a_0),\mu(b_0)\} \\ &= \min\left\{\sup_{h\in f^{-1}(a')} \mu(h), \sup_{h\in f^{-1}(b')} \mu(h)\right\} \\ &= \min\{\mu^f(a'),\mu^f(b')\}.\end{aligned} \tag{3.15}$$

Therefore, μ^f is a fuzzy h-ideal of S'. □

The above proposition can be further strengthened. We first give the following definition.

Definition 3.12. A t-norm T on $[0,1]$ is called a continuous t-norm if T is a continuous function from $[0,1]\times[0,1]\to[0,1]$ with respect to the usual topology.

We observe that the function "min" is always a continuous t-norm.

PROPOSITION 3.13. *Let T be a continuous t-norm and let f be a homomorphism on a hemiring S. If μ is a T-fuzzy left h-ideal of S, then μ^f is a T-fuzzy left h-ideal of $f(S)$.*

Proof. Let $A_1 = f^{-1}(y_1)$, $A_2 = f^{-1}(y_2)$ and $A_{12} = f^{-1}(y_1+y_2)$, where $y_1, y_2 \in f(S)$. Consider the set

$$A_1 + A_2 = \{x\in S \mid x = a_1 + a_2 \text{ for some } a_1\in A_1,\ a_2\in A_2\}. \tag{3.16}$$

If $x\in A_1+A_2$, then $x = x_1+x_2$ for some $x_1\in A_1$ and $x_2\in A_2$ so that we have $f(x) = f(x_1+x_2) = f(x_1)+f(x_2) = y_1+y_2$, that is, $x\in f^{-1}(y_1+y_2) = A_{12}$. Thus, $A_1+A_2\subseteq A_{12}$. It follows that

$$\begin{aligned}\mu^f(y_1+y_2) &= \sup\{\mu(x)\mid x\in f^{-1}(y_1+y_2)\} \\ &= \sup\{\mu(x)\mid x\in A_{12}\} \geq \sup\{\mu(x)\mid x\in A_1+A_2\} \\ &\geq \sup\{\mu(x_1+x_2)\mid x_1\in A_1,\ x_2\in A_2\} \\ &\geq \sup\{T(\mu(x_1),\mu(x_2))\mid x_1\in A_1,\ x_2\in A_2\}.\end{aligned} \tag{3.17}$$

Since T is continuous, for every $\epsilon>0$, we see that if $\sup\{\mu(x_1)\mid x\in A_1\} - x_1^* \leq \delta$ and $\sup\{\mu(x_2)\mid x\in A_2\} - x_2^*\leq\delta$, then

$$T(\sup\{\mu(x_1)\mid x_1\in A_1\}, \sup\{\mu(x_2)\mid x_2\in A_2\}) - T(x_1^*,x_2^*)\leq\epsilon. \tag{3.18}$$

Choose $a_1\in A_1$ and $a_2\in A_2$, such that $\sup\{\mu(x_1)\mid x_1\in A_1\} - \mu(a_1)\leq\delta$ and $\sup\{\mu(x_2)\mid x_2\in A_2\} - \mu(a_2)\leq\delta$. Then, we have

$$T(\sup\{\mu(x_1)\mid x_1\in A_1\}, \sup\{\mu(x_2)\mid x_2\in A_2\}) - T(\mu(a_1),\mu(a_2))\leq\epsilon. \tag{3.19}$$

Consequently, we have

$$\begin{aligned}\mu^f(y_1+y_2) &\geq \sup\{T(\mu(x_1),\mu(x_2)) \mid x_1\in A_1, x_2\in A_2\}\\ &\geq T(\sup\{\mu(x_1)\mid x_1\in A_1\},\ \sup\{\mu(x_2)\mid x_2\in A_2\})\\ &= T(\mu^f(y_1),\mu^f(y_2)).\end{aligned}\tag{3.20}$$

Similarly, we can show that $\mu^f(y_1y_2)\geq \mu^f(y_2)$.

Hence, μ^f is a T-fuzzy left ideal of $f(S)$. Now, let $x_1,z_1,a_1,b_1\in f(S)$ be such that $x_1+a_1+z_1=b_1+z_1$. Similarly, we can show that $\mu^f(x_1)\geq T(\mu(a_1),\mu(b_1))$. Therefore, μ^f is a T-fuzzy left h-ideal of $f(S)$. □

4. Chain conditions

Let μ and ν be fuzzy sets in a hemiring S. Then the $T-h$-product of μ and ν is defined by

$$\mu\circ_h\nu(x)=\begin{cases}\sup\left(T(\mu(a_i),\nu(b_i)\mid i=1,2)\right) & \text{if } x \text{ can be expressed as}\\ & \quad x+a_1b_1+z=a_2b_2+z,\\ 0 & \text{otherwise.}\end{cases}\tag{4.1}$$

Lemma 4.1 (see [1]). *Let T be a t-norm. Then for all α,β,γ, and $\delta\in[0,1]$,*

$$T(T(\alpha,\beta),T(\gamma,\delta))=T(T(\alpha,\gamma),T(\beta,\delta)).\tag{4.2}$$

Proposition 4.2. *Let μ and ν be fuzzy sets in a hemiring S. If they are T-fuzzy left h-ideals of S, then so is $\mu\cap\nu$, where $\mu\cap\nu$ is defined by $(\mu\cap\nu)(x)=T(\mu(x),\nu(x))$ for all $x\in S$. Moreover, if μ and ν are a T-fuzzy right h-ideal and a T-fuzzy left h-ideal, respectively, then $\mu\circ_h\nu\subseteq\mu\cap\nu$.*

Proof. For any $x,y\in S$, we have

$$\begin{aligned}(\mu\cap\nu)(x+y) &= T(\mu(x+y),\nu(x+y))\\ &\geq T(T(\mu(x),\mu(y)),T(\nu(x),\nu(y)))\\ &= T(T(\mu(x),\nu(x)),T(\mu(y),\nu(y)))\\ &= T((\mu\cap\nu)(x),(\mu\cap\nu)(y)),\\ (\mu\cap\nu)(xy) &= T(\mu(xy),\nu(xy))\\ &\geq T(\mu(y),\nu(y))=(\mu\cap\nu)(y).\end{aligned}\tag{4.3}$$

Hence, $\mu\cap\nu$ is a T-fuzzy left ideal of S.

Let $a,b,x,z \in S$ be such that $x+a+z+b+z$. Then,

$$\begin{aligned}(\mu\cap\nu)(x) &= T(\mu(x),\nu(x))\\ &\geq T(T(\mu(a),\mu(b)),T(\nu(a),\nu(b)))\\ &= T(T(\mu(a),\nu(a)),T(\mu(b),\nu(b)))\\ &= T((\mu\cap\nu)(a),(\mu\cap\nu)(b)).\end{aligned} \tag{4.4}$$

Therefore, $\mu\cap\nu$ is a T-fuzzy left h-ideal of S.

Now, let μ and ν be a T-fuzzy right h-ideal and a T-fuzzy left h-ideal of S, respectively. Let $x,z \in S$. The proof is obvious if $(\mu\circ_h\nu)(x) = 0$. Otherwise, for every $a_i,b_i \in S$, $i = 1,2$, satisfying $x+a_1b_1+z = a_2b_2+z$, we have $\mu(x) \geq T(\mu(a_1b_1),\mu(a_2b_2)) \geq T(\mu(a_1),\mu(a_2))$ as μ is a T-fuzzy right h-ideal of S, and $\nu(x) \geq T(\nu(a_1b_1),\nu(a_2b_2)) \geq T(\nu(b_1),\nu(b_2))$ as ν is a T-fuzzy left h-ideal of S. Thus,

$$\begin{aligned}(\mu\circ_h\nu)(x) &= \sup_{x+a_1b_1+z=a_2b_2+z}\left(T(\mu(a_i),\nu(b_i) \mid i=1,2)\right)\\ &\leq T(\mu(x),\nu(x)) = (\mu\cap\nu)(x).\end{aligned} \tag{4.5}$$

Consequently, $\mu\circ_h\nu \subseteq \mu\cap\nu$. □

We know that the intersection of all left h-ideals of a hemiring S is also a left h-ideal of S. Let Λ be a totally ordered set and let $\{A_j \mid j\in\Lambda\}$ be a collection of left h-ideals of S such that for all $i,j\in\Lambda$, $j<i$ if and only if $A_i \subset A_j$. Then, $\bigcap_{j\in\Lambda}A_j$ is a left h-ideal of S. For any subset A of S, denote by $\langle A\rangle_h$ the intersection of all left h-ideals containing A. It is obvious that $\langle A\rangle_h$ is the small left h-ideal of S containing A. We call it the left h-ideal generated by A.

Theorem 4.3. *Let $\{A_j \mid j\in\Lambda\subseteq[0,1]\}$ be a collection of left h-ideals of a hemiring S such that*

(i) $S = \bigcup_{j\in\Lambda}A_j$;

(ii) *$j<i$ if and only if $A_i\subset A_j$ for all $i,j\in\Lambda$.*

Let T be a t-norm. Define an imaginable fuzzy subset μ in S by $\mu(x) = \sup\{j\in\Lambda \mid x\in A_j\}$ for all $x\in S$. Then μ is an imaginable T-fuzzy left h-ideal of S.

Proof. For any $i\in[0,1]$, we consider the following two cases:

$$i = \sup\{j\in\Lambda \mid j<i\}, \qquad i \neq \sup\{j\in\Lambda \mid j<i\}. \tag{4.6}$$

For the first case, we know that $x\in U(\mu;i)$ if and only if $x\in A_j$ for all $j<i$ if and only if $x\in\bigcap_{j<i}A_j$. Hence, $U(\mu;i) = \bigcap_{j<i}A_j$, which is a left h-ideal of S. The second case implies that there exists $\epsilon>0$ such that $(i-\epsilon,i)\cap\Lambda=\varnothing$. We claim that $U(\mu;i) = \bigcup_{j\geq i}A_j$. If $x\in\bigcup_{j\geq i}A_j$, then $x\in A_j$ for some $j\geq i$. It follows that $\mu(x)\geq j\geq i$. Hence $x\in U(\mu;i)$, showing that if $x\in A_j$, then $j\leq i-\epsilon$. Thus $\mu(x)\leq i-\epsilon$, and so $x\notin U(\mu;i)$. Therefore, $U(\mu;i)\subseteq\bigcup_{j\geq i}A_j$, and so $U(\mu;i) = \bigcup_{j\geq i}A_j$. Hence μ is an imaginable T-fuzzy left h-ideal of S by Theorem 3.8. □

THEOREM 4.4. *Let $\{A_n \mid n \in \mathbb{N}\}$ be a family of left h-ideals of a hemiring S which is nested, that is, $S = A_1 \supset A_2 \supset \cdots$. Let μ be a fuzzy set in S defined by*

$$\mu(x) = \begin{cases} \dfrac{n}{n+1} & \text{if } x \in A_n \backslash A_{n+1},\ n = 1,2,3,\dots, \\ 1 & \text{if } x \in \bigcap_{n=1}^{\infty} A_n, \end{cases} \tag{4.7}$$

for all $x \in S$. Then μ is a T-fuzzy left h-ideal of S.

Proof. Let $x, y \in S$. Suppose that $x \in A_k \backslash A_{k+1}$ and $y \in A_r \backslash A_{r+1}$ for $k = 1,2,\dots; r = 1,2,\dots$. Without loss of generality, we may assume that $k \le r$. Then $x + y \in A_k$, and so

$$\mu(x+y) \ge \frac{k}{k+1} = \min\{\mu(x),\mu(y)\} \ge T(\mu(x),\mu(y)). \tag{4.8}$$

If $x, y \in \bigcap_{n=1}^{\infty} A_n$, then $x + y \in \bigcap_{n=1}^{\infty} A_n$, and thus

$$\mu(x+y) = 1 = T(\mu(x),\mu(y)). \tag{4.9}$$

If $x \in \bigcap_{n=1}^{\infty} A_n$ and $y \notin \bigcap_{n=1}^{\infty} A_n$, then there exists $i \in \mathbb{N}$ such that $y \in A_i \backslash A_{i+1}$. It follows that $x + y \in A_i$ so that

$$\mu(x+y) \ge \frac{i}{i+1} = \min\{\mu(x),\mu(y)\} \ge T(\mu(x),\mu(y)). \tag{4.10}$$

Similarly, we know that $\mu(x+y) \ge T(\mu(x),\mu(y))$ whenever $x \notin \bigcap_{n=1}^{\infty} A_n$ and $y \in \bigcap_{n=1}^{\infty} A_n$. Now, if $y \in A_k \backslash A_{k+1}$ for some $k = 1,2,3,\dots$, then $xy \in A_k$ for all $x \in S$. Hence, $\mu(xy) \ge k/(k+1) = \mu(y)$. If $y \in \bigcap_{n=1}^{\infty} A_n$, then $xy \in \bigcap_{n=1}^{\infty} A_n$ for all $x \in S$. Thus $\mu(xy) = 1 = \mu(y)$. Hence μ is a T-fuzzy left ideal of S. Now, let $a, b, x, z \in S$ be such that $x + a + z = b + z$. If $a, b \in A_r \backslash A_{r+1}$ for some $r = 1,2,3,\dots$, then $x \in A_r$ as A_r is a left h-ideal of S. Thus

$$\mu(x) \ge \frac{r}{r+1} = \min\{\mu(a),\mu(b)\} \ge T(\mu(a),\mu(b)). \tag{4.11}$$

If $a, b \in \bigcap_{n=1}^{\infty} A_n$, then $x \in \bigcap_{n=1}^{\infty} A_n$, and so $\mu(x) = 1 = T(\mu(a),\mu(b))$. Assume that $a \in A_r \backslash A_{r+1}$ for some $r = 1,2,3,\dots$, and $b \in \bigcap_{n=1}^{\infty} A_n$ (or, $a \in \bigcap_{n=1}^{\infty} A_n$ and $b \in A_r \backslash A_{r+1}$ for some $r = 1,2,3,\dots$). Then $x \in A_r$, and so $\mu(x) = r/(r+1) = \min\{\mu(a),\mu(b)\} \ge T(\mu(a),\mu(b))$. Consequently, μ is a T-fuzzy left h-ideal of S. □

Let $\mu : S \to [0,1]$ be a fuzzy set. The smallest T-fuzzy left h-ideal containing μ is called the T-fuzzy left h-ideal generated by μ, and μ is said to be n-valued if $\mu(S)$ is a finite set of n elements. When no specific n is intended, we call μ a finite-valued fuzzy set.

Theorem 4.5. *A T-fuzzy left h-ideal ν of a hemiring S is finite valued if and only if it is generated by a finite-valued fuzzy set μ in S.*

Proof. If $\nu : S \to [0,1]$ is a finite-valued T-fuzzy left h-ideal of S, then one may choose $\mu = \nu$. Consequently, assume that $\mu : S \to [0,1]$ is an n-valued fuzzy set with n distinct values $t_1, t_2, \ldots, t_n$, where $t_1 > t_2 > \cdots > t_n$. Let G^i be the inverse image of t_i under μ, that is, $G^i = \mu^{-1}(t_i)$. Obviously, $\bigcup_{i=1}^{j} G^i \subseteq \bigcup_{i=1}^{r} G^i$ when $j < r$. Denote by A^j the left h-ideal of S generated by the set $\bigcup_{i=1}^{j} G^i$. Then we have the following chain of left h-ideals:

$$A^1 \subseteq A^2 \subseteq \cdots \subseteq A^n = S. \tag{4.12}$$

Define a fuzzy set $\nu : S \to [0,1]$ by

$$\mu(x) = \begin{cases} t_1 & \text{if } x \in A^1, \\ t_j & \text{if } x \in A^j \backslash A^{j-1};\ j = 2,3,\ldots,n. \end{cases} \tag{4.13}$$

We claim that ν is a T-fuzzy left h-ideal of S generated by μ. Let $x, y \in S$ and let i and j be the smallest integers such that $x \in A^i$ and $y \in A^j$. We may assume that $i > j$ without loss of generality. Then $x + y \in A^i$ and $yx \in A^j$, and so $\nu(yx) \geq t_j = \nu(y)$ and $\nu(x+y) \geq t_i = \min\{t_i, t_j\} = \min\{\nu(x), \nu(y)\} \geq T(\nu(x), \nu(y))$. Hence, ν is a T-fuzzy left ideal of S. Now, let $a, b, x, z \in S$ be such that $x + a + z = b + z$. If $a \in A^i$ and $b \in A^j$ for some $i < j$, then $a, b \in A^j$, and so $x \in A^j$ as A^j is a left h-ideal of S. Thus, $\nu(x) \geq t_j = \min\{\nu(a), \nu(b)\} \geq T(\nu(a), \nu(b))$. If $a, b \in A^j \backslash A^{j-1}$ for $j = 2,3,\ldots,n$, then $x \in A^j$. Hence, $\nu(x) \geq t_j = T(\nu(a), \nu(b))$. Therefore, ν is a T-fuzzy left h-ideal of S. If $x \in S$ and $\mu(x) = t_j$, then $x \in G^j$ and so $x \in A^j$. But we get $\nu(x) \geq t_j = \mu(x)$. Consequently, $\mu \subseteq \nu$. Let γ be any T-fuzzy left h-ideal of S containing μ. Then, $\bigcup_{i=1}^{j} G^i = U(\mu; t_j) \subseteq U(\gamma; t_j)$, and thus $A^j \subseteq U(\gamma; t_j)$. Hence, $\nu \subseteq \gamma$ and ν is generated by μ. Note that $|\operatorname{Im}\mu| = n = |\operatorname{Im}\nu|$, thus completing the proof. □

A semiring S is said to be left h-Noetherian (see [20]) if it satisfies the ascending chain condition on left h-ideals of S.

Theorem 4.6. *If S is an h-Noetherian hemiring, then every T-fuzzy left h-ideal of S is finite valued.*

Proof. Let $\mu : S \to [0,1]$ be a T-fuzzy left h-ideal of S which is not finite valued. Then, there exists an infinite sequence of distinct numbers $\mu(0) = t_1 > t_2 > \cdots > t_n > \cdots$, where $t_i = \mu(x_i)$ for some $x_i \in S$. This sequence induces an infinite sequence of distinct left h-ideals of S:

$$U(\mu; t_1) \subset U(\mu; t_2) \subset \cdots \subset U(\mu; t_n) \subset \cdots. \tag{4.14}$$

This is a contradiction. □

Combining Theorems 4.5 and 4.6, we have the following corollary.

Corollary 4.7. *If S is an h-Noetherian hemiring, then every T-fuzzy left h-ideal of S is generated by a finite fuzzy set in S.*

5. T-product of T-fuzzy left h-ideals

Definition 5.1 (see [6]). A fuzzy relation on any set S is a fuzzy set $\mu : S \times S \to [0,1]$.

Definition 5.2. Let T be a t-norm. If μ is a fuzzy relation on a set S and ν is a fuzzy set S, then μ is a T-fuzzy relation on ν if $\mu(x,y) \leq T(\nu(x),\nu(y))$ for all $x,y \in S$.

Definition 5.3. Let T be a t-norm and let μ and ν be fuzzy sets in a set S. Then direct T-product of μ and ν is defined by

$$(\mu \times \nu)(x,y) = T\big(\mu(x),\nu(y)\big) \quad \forall x,y \in S. \tag{5.1}$$

Lemma 5.4. *Let T be a t-norm and let μ and ν be fuzzy sets in a set S. Then,*
(i) *$\mu \times \nu$ is a T-fuzzy relation on S;*
(ii) *$U(\mu \times \nu;\alpha) = U(\mu;\alpha) \times U(\nu;\alpha)$ for all $\alpha \in [0,1]$.*

Proof. The proof is obvious. □

Definition 5.5. Let T be a t-norm. If ν is a fuzzy set S, the strongest T-fuzzy relation on S that is a T-fuzzy relation on ν is μ_ν, given by

$$\mu_\nu(x,y) = T\big(\nu(x),\nu(y)\big) \quad \forall x,y \in S. \tag{5.2}$$

Lemma 5.6. *For a given fuzzy set ν in a set S, let μ_ν be the strongest T-fuzzy relation S. Then for $\alpha \in [0,1]$, $U(\mu_\nu;\alpha) = U(\nu;\alpha) \times U(\nu;\alpha)$.*

Proof. The proof is obvious. □

Proposition 5.7. *For a given fuzzy set ν in a hemiring S, let μ_ν be the strongest T-fuzzy relation on S. If μ_ν is an imaginable T-fuzzy h-ideal of $S \times S$, then $\nu(a) \leq \nu(0)$ for all $a \in S$.*

Proof. If μ_ν is a T-fuzzy left h-ideal of $S \times S$, then $\mu_\nu(a,a) \leq \mu_\nu(0,0)$ for all $a \in S$. This means that $T(\nu(a),\nu(a)) \leq T(\nu(0),\nu(0))$ for all $a \in S$. Since μ is imaginable, then $\nu(a) \leq \nu(0)$ for all $a \in S$. □

The following proposition is an immediate consequence of Lemma 5.6, and we omit the proof.

Proposition 5.8. *Let μ and ν be T-fuzzy left h-ideals of a hemiring S, then the level left h-ideals of μ_ν are given by $U(\mu_\nu;\alpha) = U(\nu;\alpha) \times U(\nu;\alpha)$ for all $\alpha \in [0,1]$.*

Theorem 5.9. *Let T be a t-norm and let μ and ν be T-fuzzy left h-ideals of a hemiring S. Then, $\mu \times \nu$ is a T-fuzzy left h-ideal of $S \times S$.*

Proof. Let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be any elements of $S \times S$. Then,

$$\begin{aligned}(\mu\times\nu)(x+y) &= (\mu\times\nu)\big((x_1,x_2)+(y_1,y_2)\big)\\ &= (\mu\times\nu)(x_1+y_1,x_2+y_2)\\ &= T(\mu(x_1+y_1),\nu(x_2+y_2))\\ &\geq T\big(T(\mu(x_1),\mu(y_1)),T(\nu(x_2),\nu(y_2))\big)\\ &= T\big(T(\mu(x_1),\nu(x_2)),T(\mu(y_1),\nu(y_2))\big)\\ &= T\big((\mu\times\nu)(x_1,x_2),(\mu\times\nu)(y_1,y_2)\big)\\ &= T\big((\mu\times\nu)(x),(\mu\times\nu)(y)\big),\\ (\mu\times\nu)(xy) &= (\mu\times\nu)\big((x_1,x_2)(y_1,y_2)\big) = (\mu\times\nu)(x_1y_1,x_2y_2)\\ &= T\big(\mu(x_1y_1),\nu(x_2y_2)\big) \geq T\big(\mu(y_1),\nu(y_2)\big)\\ &= (\mu\times\nu)(y_1,y_2) = (\mu\times\nu)(y).\end{aligned} \tag{5.3}$$

Hence, $\mu\times\nu$ is a T-fuzzy left ideal of $S\times S$.

Let $x = (x_1,x_2)$, $z = (z_1,z_2)$, $a = (a_1,a_2)$, and $b = (b_1,b_2)$ be such that $x+a+z = b+z$. Then $(x_1,x_2)+(a_1,a_2)+(z_1,z_2) = (b_1,b_2)+(z_1,z_2)$, and so $x_1+a_1+z_1 = b_1+z_1$ and $x_2+a_2+z_2 = b_2+z_2$. It follows that

$$\begin{aligned}(\mu\times\nu)(x) &= (\mu\times\nu)(x_1,x_2) = T\big(\mu(x_1),\nu(x_2)\big)\\ &\geq T\big(T(\mu(a_1),\mu(b_1)),T(\nu(a_2),\nu(b_2))\big)\\ &= T\big(T(\mu(a_1),\nu(a_2)),T(\mu(b_1),\nu(b_2))\big)\\ &= T\big((\mu\times\nu)(a_1,a_2),(\mu\times\nu)(b_1,b_2)\big)\\ &= T\big((\mu\times\nu)(a),(\mu\times\nu)(b)\big).\end{aligned} \tag{5.4}$$

Therefore, $\mu\times\nu$ is a T-fuzzy left h-ideal of $S\times S$. □

Corollary 5.10. *Let μ and ν be imaginable T-fuzzy left h-ideals of a hemiring S. Then, $\mu\times\nu$ is an imaginable T-fuzzy left h-ideal of $S\times S$.*

Proof. By Theorem 5.9, we have $\mu\times\nu$ is a T-fuzzy left h-ideal of $S\times S$. Let $x = (x_1,x_2)$ be any element of $S\times S$. Then

$$\begin{aligned}T\big((\mu\times\nu)(x),(\mu\times\nu)(x)\big) &= T\big((\mu\times\nu)(x_1,x_2),(\mu\times\nu)(x_1,x_2)\big)\\ &= T\big(T(\mu(x_1),\nu(x_2)),T(\mu(x_1),\nu(x_2))\big)\\ &= T\big(T(\mu(x_1),\mu(x_1)),T(\nu(x_2),\nu(x_2))\big)\\ &= T\big(\mu(x_1),\nu(x_2)\big) = (\mu\times\nu)(x_1,x_2) = (\mu\times\nu)(x).\end{aligned} \tag{5.5}$$

This shows that $\mu\times\nu$ is an imaginable T-fuzzy left h-ideal of $S\times S$. □

As the converse of Corollary 5.10, we have a question as follows. If $\mu\times\nu$ is an imaginable T-fuzzy left h-ideal of $S\times S$, then are both μ and ν imaginable T-fuzzy left h-ideal of S? The following example give a negative answer, that is, if $\mu\times\nu$ is an imaginable T-fuzzy left h-ideal of $S\times S$, then $\mu\times\nu$ need not be imaginable T-fuzzy left h-ideal of S.

Example 5.11. Let S be a hemiring with $|S|\geq 2$ and let $s\in[0,1]$. Define imaginable fuzzy sets μ and ν in S by $\mu(x)=s$ and

$$\nu(x)=\begin{cases} s & \text{if } x=0,\\ 1 & \text{otherwise,}\end{cases} \tag{5.6}$$

for all $x\in S$, respectively.

If $y=0$, then $\nu(y)=s$, and thus

$$(\mu\times\nu)(x,y)=T\big(\mu(x),\nu(x)\big)=T(s,s)=s. \tag{5.7}$$

If $y\neq 0$, then $\nu(y)=1$, and thus

$$(\mu\times\nu)(x,y)=T\big(\mu(x),\nu(y)\big)=T(s,1)=s. \tag{5.8}$$

That is, $\mu\times\nu$ is a constant function, and so $\mu\times\nu$ is an imaginable T-fuzzy left h-ideal of $S\times S$. Now, μ is an imaginable T-fuzzy left h-ideal of S, but ν is not an imaginable T-fuzzy left h-ideal of S since for $x\neq 0$, we have $\nu(0)=s<1=\nu(x)$.

We will generalize the idea to the product of n T-fuzzy left h-ideals, We first need to generalize the domain of T to $\prod_{i=1}^{n}[0,1]$ as follows.

Definition 5.12 (see [1]). The function $T_n:\prod_{i=1}^{n}[0,1]\to[0,1]$ is defined by

$$T_n(\alpha_1,\alpha_2,\dots,\alpha_n)=T\big(\alpha_i,T_{n-1}(\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_n)\big) \tag{5.9}$$

for all $1\leq i\leq n$, where $n\geq 2, T_2=T$, and $T_1=id$(identity).

Lemma 5.13 (see [1]). *For every $\alpha_i,\beta_i\in[0,1]$, where $1\leq i\leq n$ and $n\geq 2$,*

$$T_n\big(T(\alpha_1,\beta_1),T(\alpha_2,\beta_2),\dots,T(\alpha_n,\beta_n)\big)=T\big(T_n(\alpha_1,\alpha_2,\dots,\alpha_n),T_n(\beta_1,\beta_2,\dots,\beta_n)\big). \tag{5.10}$$

Proposition 5.14. *Let T be a t-norm and let $\{S_i\}_{i=1}^{n}$ be the finite collection of hemirings and $S=\prod_{i=1}^{n}S_i$ the T-product of S_i. Let μ_i be a T-fuzzy h-ideal of S_i, where $1\leq i\leq n$. Then, $\mu=\prod_{i=1}^{n}\mu_i$ defined by*

$$\mu(x_1,x_2,\dots,x_n)=\prod_{i=1}^{n}\mu_i(x_1,x_2,\dots,x_n)=T_n\big(\mu_1(x_1),\mu_2(x_2),\dots,\mu_n(x_n)\big) \tag{5.11}$$

is a T-fuzzy left h-ideal of S.

The proof is similar to the proof of Theorem 5.9.

Definition 5.15. Let μ and ν be fuzzy sets in S. Then, the T-product of μ and ν, written as $[\mu \cdot \nu]_T$, is defined by $[\mu \cdot \nu]_T(x) = T(\mu(x), \nu(x))$ for all $x \in S$.

Theorem 5.16. *Let μ and ν be T-fuzzy left h-ideals of a hemiring S. If T^* is a t-norm which dominates T, that is, $T^*(T(\alpha,\beta),T(\gamma,\delta)) \geq T(T^*(\alpha,\gamma),T^*(\beta,\delta))$ for all $\alpha,\beta,\gamma,\delta \in [0,1]$, then T^*-product of μ and ν, $[\mu \cdot \nu]_{T^*}$, is a T-fuzzy left h-ideal of S.*

Proof. Let $x, y \in S$, then we have

$$\begin{aligned}
[\mu \cdot \nu]_{T^*}(x+y) &= T^*(\mu(x+y), \nu(x+y)) \\
&\geq T^*(T(\mu(x),\mu(y)), T(\nu(x),\nu(y))) \\
&\geq T(T^*(\mu(x),\nu(x)), T^*(\mu(y),\nu(y))) \\
&= T([\mu \cdot \nu]_{T^*}(x), [\mu \cdot \nu]_{T^*}(y)), \\
[\mu \cdot \nu]_{T^*}(xy) &= T^*(\mu(xy), \nu(xy)) \\
&\geq T^*(\mu(y), \nu(y)) = [\mu \cdot \nu]_{T^*}(y).
\end{aligned} \tag{5.12}$$

Hence, $[\mu \cdot \nu]_{T^*}$ is a T-fuzzy left ideal of S.

Now, let $x, z, a, b \in S$ be such that $x + a + z = b + z$. Then,

$$\begin{aligned}
[\mu \cdot \nu]_{T^*}(x) &= T^*(\mu(x), \nu(x)) \\
&\geq T^*(T(\mu(a),\mu(b)), T(\nu(a),\nu(b))) \\
&\geq T(T^*(\mu(a),\nu(a)), T(\mu(b),\nu(b))) \\
&= T([\mu \cdot \nu]T^*(a), [\mu \cdot \nu]_{T^*}(b)).
\end{aligned} \tag{5.13}$$

Therefore, $[\mu \cdot \nu]_{T^*}$ is a T-fuzzy left h-ideal of S. □

Let $f : S \to S'$ be an onto homomorphism of hemirings. Let T and T^* be t-norms such that T^* dominates T. If μ and ν are T-fuzzy left h-ideals of S', then the T^*-product of μ and ν, $[\mu \cdot \nu]_{T^*}$, is a T-fuzzy left h-ideal of S'. Since every onto homomorphic inverse image of a T-fuzzy left h-ideal is a T-fuzzy left h-ideal, the inverse images $f^{-1}(\mu), f^{-1}(\nu)$, and $f^{-1}([\mu \cdot \nu]_{T^*})$ are T-fuzzy left h-ideals of S. The next theorem provides the relation between $f^{-1}([\mu \cdot \nu]_{T^*})$ and T^*-product $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}$ of $f^{-1}(\mu)$ and $f^{-1}(\nu)$.

Theorem 5.17. *Let $f : S \to S'$ be an onto homomorphism of hemirings. Let T^* be a t-norm such that T^* dominates T. Let μ and ν be T-fuzzy left h-ideals of S'. If $[\mu \cdot \nu]_{T^*}$ is the T^*-product of μ and ν, and $[f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}$ is the T^*-product of $f^{-1}(\mu)$ and $f^{-1}(\nu)$, then*

$$f^{-1}([\mu \cdot \nu]_{T^*}) = [f^{-1}(\mu) \cdot f^{-1}(\nu)]_{T^*}. \tag{5.14}$$

Proof. Let $x \in S$, then we have

$$\begin{aligned} f^{-1}([\mu\cdot\nu]_{T^*})(x) &= [\mu\cdot\nu]_{T^*}(f(x)) \\ &= T^*(\mu(f(x)),\nu(f(x))) \\ &= T^*(f^{-1}(\mu)(x), f^{-1}(\nu)(x)) \\ &= \left[f^{-1}(\mu)\cdot f^{-1}(\nu)\right]_{T^*}(x). \end{aligned} \tag{5.15}$$

This completes the proof. □

Theorem 5.18. *Let ν be an imaginable fuzzy set in a hemiring S and let μ_ν be the strongest T-fuzzy relation on S. Then ν is an imaginable T-fuzzy left h-ideal of S if and only if μ_ν is an imaginable T-fuzzy left h-ideal of $S\times S$.*

Proof. Assume that ν is an imaginable T-fuzzy left h-ideal of S. Let $x=(x_1,x_2)$, $y=(y_1,y_2)\in S\times S$. Then we have

$$\begin{aligned} \mu_\nu(x+y) &= \mu_\nu((x_1,x_2)+(y_1,y_2)) \\ &= \mu_\nu(x_1+y_1,x_2+y_2) = T(\nu(x_1+y_1),\nu(x_2+y_2)) \\ &\geq T(T(\nu(x_1),\nu(y_1)),T(\nu(x_2),\nu(y_2))) \\ &= T(T(\nu(x_1),\nu(x_2)),T(\nu(y_1),\nu(y_2))) \\ &= T(\mu_\nu(x_1,x_2),\mu_\nu(y_1,y_2)) = T(\mu_\nu(x),\mu_\nu(y)), \\ T(\mu_\nu(x),\mu_\nu(y)) &= \mu_\nu((x_1,x_2)(y_1,y_2)) = \mu_\nu(x_1y_1,x_2y_2), \\ T(\nu(x_1y_1),\nu(x_2y_2)) &\geq T(\nu(y_1),\nu(y_2)) = \mu_\nu(y_1,y_2) = \mu_\nu(y). \end{aligned} \tag{5.16}$$

Hence μ_ν is a T-fuzzy left ideal of $S\times S$.

Now, let $a=(a_1,a_2)$, $b=(b_1,b_2)$, $x=(x_1,x_2)$, $z=(z_1,z_2)\in S\times S$ be such that $x+a+z=b+z$. That is,

$$(x_1,x_2)+(a_1,a_2)+(z_1,z_2) = (b_1,b_2)+(z_1,z_2). \tag{5.17}$$

Then $x_1+a_1+z_1=b_1+z_1$ and $x_2+a_2+z_2=b_2+z_2$. Thus,

$$\begin{aligned} \mu_\nu(x) = \mu_\nu(x_1,x_2) &= T(\nu(x_1),\nu(x_2)) \\ &\geq T(T(\nu(a_1),\nu(b_1)),T(\nu(a_2),\nu(b_2))) \\ &= T(T(\nu(a_1),\nu(a_2)),T(\nu(b_1),\nu(b_2))) \\ &= T(\mu_\nu(a_1,a_2),\mu_\nu(b_1,b_2)) \\ &= T(\mu_\nu(a),\mu_\nu(b)). \end{aligned} \tag{5.18}$$

Therefore, μ_ν is a T-fuzzy left h-ideal of $S\times S$.

For any $x = (x_1, x_2) \in S \times S$, then

$$\begin{aligned} T(\mu_\nu(x), \mu_\nu(x)) &= T(\mu_\nu(x_1, x_2), \mu_\nu(x_1, x_2)) \\ &= T(T(\nu(x_1), \nu(x_2)), T(\nu(x_1), \nu(x_2))) \\ &= T(T(\nu(x_1), \nu(x_1)), T(\nu(x_2), \nu(x_2))) \\ &= T(\nu(x_1), \nu(x_2)) = \mu_\nu(x_1, x_2) = \mu_\nu(x). \end{aligned} \tag{5.19}$$

Thus, μ_ν is an imaginable T-fuzzy left h-ideal of $S \times S$.

Conversely, suppose that μ_ν is an imaginable T-fuzzy left h-ideal of $S \times S$. Let $x, y \in S$. Then we have

$$\begin{aligned} \nu(x+y) &= T(\nu(x+y), \nu(x+y)) = \mu_\nu(x+y, x+y) \\ &= \mu_\nu((x,x) + (y,y)) \geq T(\mu_\nu(x,x), \mu_\nu(y,y)) \\ &= T(T(\nu(x), \nu(x)), T(\nu(y), \nu(y))) = T(\nu(x), \nu(y)) \end{aligned} \tag{5.20}$$

since ν is imaginable.

Next, we have

$$\nu(xy) = T(\nu(xy), \nu(xy)) = \mu_\nu((x,x)(y,y)) \geq \mu_\nu(y,y) = T(\nu(y), \nu(y)) = \nu(y). \tag{5.21}$$

Hence, ν is an imaginable T-fuzzy left ideal of S.

Let $a, b, x, z \in S$ be such that $x + a + z = b + z$. Then, $(x,x) + (a,a) + (z,z) = (b,b) + (z,z)$. Since μ_ν is an imaginable T-fuzzy left h-ideal of $S \times S$, it follows that

$$\begin{aligned} \nu(x) &= T(\nu(x), \nu(x)) = \mu_\nu(x,x) \geq T(\mu_\nu(a,a), \mu_\nu(b,b)) \\ &= T(T(\nu(a), \nu(a)), T(\nu(b), \nu(b))) = T(\nu(a), \nu(b)). \end{aligned} \tag{5.22}$$

Consequently, ν is an imaginable T-fuzzy left h-ideal of S. □

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