A NOTE ON THE STRONG LAW OF LARGE NUMBERS FOR ASSOCIATED SEQUENCES

A. NEZAKATI

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We prove that the sequence $\{b_n^{-1}\sum_{i=1}^{n}(X_i - EX_i)\}_{n\geq 1}$ converges a.e. to zero if $\{X_n,\ n \geq 1\}$ is an *associated* sequence of random variables with $\sum_{n=1}^{\infty} b_{k_n}^{-2}\operatorname{Var}(\sum_{i=k_{n-1}+1}^{k_n} X_i) < \infty$ where $\{b_n,\ n \geq 1\}$ is a positive nondecreasing sequence and $\{k_n,\ n \geq 1\}$ is a strictly increasing sequence, both tending to infinity as n tends to infinity and $0 < a = \inf_{n\geq 1} b_{k_n} b_{k_{n+1}}^{-1} \leq \sup_{n\geq 1} b_{k_n} b_{k_{n+1}}^{-1} = c < 1$.

1. Introduction

Let (Ω, F, P) be a probability space and $\{X_n,\ n \geq 1\}$ a sequence of random variables defined on (Ω, F, P). We start with definitions. A finite sequence $\{X_1,\dots,X_n\}$ is said to be *associated* if for any two componentwise nondecreasing functions f and g on R^n,

$$\operatorname{Cov}\left(f(X_1,\dots,X_n), g(X_1,\dots,X_n)\right) \geq 0, \tag{1.1}$$

assuming of course that the covariance exists. The infinite sequence $\{X_n,\ n \geq 1\}$ is said to be *associated* if every finite subfamily is associated. The concept of association was introduced by Esary et al. [1]. There are some results on the strong law of large numbers for associated sequences. Rao [4] developed the Hajek-Renyi inequality for associated sequences and proved the following theorem. Let $\{X_n,\ n \geq 1\}$ be an associated sequence of random variables with

$$\sum_{j=1}^{\infty}\frac{\operatorname{Var}(X_j)}{b_j^2} + \sum_{1\leq j\neq k}^{\infty}\frac{\operatorname{Cov}(X_j, X_k)}{b_j b_k} < \infty, \tag{1.2}$$

where $\{b_n,\ n \geq 1\}$ is a positive nondecreasing sequence of real numbers. Then $b_n^{-1}\sum_{j=1}^{n}(X_j - EX_j)$ converges to zero almost everywhere as $n \to \infty$. In this note we will prove the strong law of large numbers for associated sequences with new conditions.

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2. Result

Theorem 2.1. *Let $\{X_n,\ n \geq 1\}$ be an associated sequence of random variables. If*

$$\sum_{n=1}^{\infty} b_{k_n}^{-2} \operatorname{Var}\left(S_{k_n} - S_{k_{n-1}}\right) < \infty, \tag{2.1}$$

where $S_n = \sum_{i=1}^{n} X_i$ and $\{b_n,\ n \geq 1\}$ is a positive nondecreasing sequence and $\{k_n,\ n \geq 1\}$ is a strictly increasing sequence, both tending to infinity as n tends to infinity and

$$0 < a = \inf_{n \geq 1} b_{k_n} b_{k_{n+1}}^{-1} \leq \sup_{n \geq 1} b_{k_n} b_{k_{n+1}}^{-1} = c < 1. \tag{2.2}$$

Then

$$\lim_{n \to \infty} \frac{1}{b_n} \sum_{k=1}^{n} \left(X_k - EX_k\right) = 0 \quad a.e. \tag{2.3}$$

Proof. We set $k_0 = 0$, $b_0 = 0$, and $T_n = b_{k_n}^{-1} \sum_{j=k_{n-1}+1}^{k_n} Y_j$, where $Y_j = X_j - EX_j$. For any positive integer n, there exists a positive integer m such that $k_{m-1} < n \leq k_m$. Note that $m \to \infty$ as $n \to \infty$. Without loss of generality, we assume that $n > k_1$ and, therefore, $k_{m-1} \geq 1$ and $b_n \geq b_{k_{m-1}} > 0$. We can show that

$$\frac{1}{b_n} \sum_{j=1}^{n} Y_j = \frac{b_{k_{m-1}}}{b_n} \sum_{j=1}^{m-1} \frac{b_{k_j}}{b_{k_{m-1}}} T_j + \frac{1}{b_n} \sum_{j=k_{m-1}+1}^{n} Y_j. \tag{2.4}$$

Since $b_{k_{m-1}} \geq a b_{k_m}$, we conclude that

$$\left| \frac{1}{b_n} \sum_{j=1}^{n} Y_j \right| \leq \left| \sum_{j=1}^{m-1} \frac{b_{k_j}}{b_{k_{m-1}}} T_j \right| + \frac{1}{a b_{k_m}} \max_{k_{m-1} < l \leq k_m} \left| \sum_{j=k_{m-1}+1}^{l} Y_j \right|. \tag{2.5}$$

In order to prove (2.3) it suffices to demonstrate that each of the two terms in the right-hand side of (2.5) converges to zero almost everywhere as $n \to \infty$. The first term on the right-hand side does so due to the Toeplitz lemma (see Loève [2]) provided that

$$\lim_{j \to \infty} T_j = 0 \quad \text{a.e.}, \quad \sup_{m \geq 2} \sum_{j=1}^{m-1} \frac{b_{k_j}}{b_{k_{m-1}}} < \infty, \quad \lim_{n \to \infty} \frac{b_{k_j}}{b_{k_{m-1}}} = 0 \quad \text{for every } j. \tag{2.6}$$

The third condition is satisfied because by the hypothesis the sequence $\{b_n,\ n \geq 1\}$ monotonically increases without bounds. The second condition holds because

$$\begin{gathered} \frac{b_{k_j}}{b_{k_{m-1}}} = \prod_{i=j}^{m-2} \frac{b_{k_i}}{b_{k_{i+1}}} \leq c^{m-j-1}, \\ \sum_{j=1}^{m-1} \frac{b_{k_j}}{b_{k_{m-1}}} \leq \sum_{j=1}^{m-1} c^{m-j-1} = \frac{1-c^m}{1-c} < \frac{1}{1-c}, \end{gathered} \tag{2.7}$$

since by the hypothesis $b_{k_j} \le c b_{k_{j+1}}$, $c \in (0,1)$. Thus, the first term in the right-hand side of (2.5) converges to zero almost everywhere as $m \to \infty$ if the sequence $\{T_n,\ n \ge 1\}$ also does so. By the hypothesis, Let ϵ be an arbitrary positive number. With the use of the Markov inequality, we obtain

$$\epsilon^2 \sum_{n=2}^{\infty} P(|T_n| > \epsilon) \le \sum_{n=2}^{\infty} E|T_n|^2 \le \sum_{n=2}^{\infty} b_{k_n}^{-2} \operatorname{Var}(S_{k_n} - S_{k_{n-1}}) < \infty. \tag{2.8}$$

The finiteness of the last series in the right-hand side is guaranteed by condition (2.1). In view of the Borel-Cantelli lemma, the sequence $\{T_n,\ n \ge 1\}$ converges to zero a.e. Let us turn to the second term in the right-hand side of (2.5). Applying Chebyschev's inequality, we get that, for any $\epsilon > 0$,

$$\epsilon^2 P\left(\frac{1}{b_{k_m}} \max_{k_{m-1} < l \le k_m} \left| \sum_{j=k_{m-1}+1}^{l} Y_j \right| > \epsilon\right) \le \frac{1}{b_{k_m}^2} E\left(\max_{k_{m-1} < l \le k_m} \left| \sum_{j=k_{m-1}+1}^{l} Y_j \right|^2\right). \tag{2.9}$$

We now apply the Kolmogorov-type inequality, valid for partial sums of associated random variables $\{Y_j, k_{m-1}+1 \le j \le k_m\}$ with mean zero (cf. Newman and Wright [3, Theorem 2]). Hence, from (2.1), we have

$$\begin{aligned} \epsilon^2 \sum_{m=2}^{\infty} P\left(\frac{1}{b_{k_m}} \max_{k_{m-1} < l \le k_m} \left| \sum_{j=k_{m-1}+1}^{l} Y_j \right| > \epsilon\right) &\le \sum_{m=2}^{\infty} \frac{1}{b_{k_m}^2} E\left[\sum_{j=k_{m-1}+1}^{k_m} Y_j\right]^2 \\ &\le \sum_{m=2}^{\infty} \frac{\operatorname{Var}\left(\sum_{j=k_{m-1}+1}^{k_m} Y_j\right)}{b_{k_m}^2} \\ &\le \sum_{m=2}^{\infty} \frac{\operatorname{Var}(S_{k_m} - S_{k_{m-1}})}{b_{k_m}^2} < \infty. \end{aligned} \tag{2.10}$$

By virtue of the Borel-Cantelli lemma, the sequence

$$\left\{\frac{1}{b_{k_m}} \max_{k_{m-1} < l \le k_m} \left| \sum_{j=k_{m-1}+1}^{l} Y_j \right| \right\}_{m \ge 1} \tag{2.11}$$

converges to zero almost everywhere. Thus, the theorem is proved. □

Theorem 2.2. *Let $\{X_n,\ n \ge 1\}$ be an associated sequence of random variables with*

$$\operatorname{Var}(X_j) + \sum_{1 \le k \ne j}^{\infty} \operatorname{Cov}(X_j, X_k) = O(1), \tag{2.12}$$

for all $j \ge 1$. Then

$$\frac{\sum_{j=1}^{n} (X_j - EX_j)}{(n \log n)^{1/2} \log\log n} \longrightarrow 0 \quad \text{a.e. as } n \longrightarrow \infty. \tag{2.13}$$

Proof. Under condition (2.12), there exists the constant of B such that

$$\operatorname{Var}\left(S_{k_n} - S_{k_{n-1}}\right) \leq B\left(k_n - k_{n-1}\right) \leq Bk_n. \tag{2.14}$$

The sequence $b_n = (n\log n)^{1/2}\log\log n$ and $k_n = 2^{n+1}$, $n = 1,2,\ldots$, satisfy the hypotheses of Theorem 2.1, which proves Theorem 2.2. □

Example 2.3. Let $\{X_n,\ n \geq 1\}$ be an associated sequence with $\operatorname{Var}(X_i) = 1$ and $\operatorname{Cov}(X_i,X_j) = \rho^{|i-j|}$, $0 < \rho < 1$ for every i and j. Then

$$\operatorname{Var}\left(X_i\right) + \sum_{1\leq j\neq i}^{\infty} \operatorname{Cov}\left(X_i,X_j\right) \leq 1 + 2\sum_{k=1}^{\infty}\rho^k < \infty. \tag{2.15}$$

Therefore, we can apply Theorem 2.2.

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A. Nezakati: Faculty of Mathematics, Shahrood University of Technology, Shahrood, P.O. Box 36155-316, Iran

E-mail address: nezakati@shahrood.ac.ir