COMMON FIXED POINTS OF SINGLE-VALUED AND MULTIVALUED MAPS

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We define a new property which contains the property (EA) for a hybrid pair of single- and multivalued maps and give some new common fixed point theorems under hybrid contractive conditions. Our results extend previous ones. As an application, we give a partial answer to the problem raised by Singh and Mishra.

1. Introduction and preliminaries

Let (X,d) be a metric space. Then, for $x \in X$, $A \subset X$, $d(x,A) = \inf\{d(x,y),\ y \in A\}$. We denote $CB(X)$ as the class of all nonempty bounded closed subsets of X. Let H be the Hausdorff metric with respect to d, that is,

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, \tag{1.1}$$

for every $A,B \in CB(X)$. A self-map T defined on X satisfies Rhoades' contractive definition in following sense: (see [19]) for all $x,y \in X$, $x \neq y$,

$$d(Tx,Ty) < \max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}. \tag{1.2}$$

The fixed points theorems for Rhoades-type contraction mapping were investigated by many authors [1, 5, 8, 10, 13, 16, 22] and the more results on this fields can be found in [2, 4, 9, 11, 15, 23]. Hybrid fixed point theory for nonlinear single-valued and multivalued maps is a new development in the domain of contraction-type multivalued theory (see [3, 7, 10, 12, 14, 17, 18, 20] and references therein). In 1998, Jungck and Rhoades [12] introduced the notion of weak compatibility to the setting of single-valued and multivalued maps. In [21], Singh and Mishra introduced the notion of (IT)-commutativity for hybrid pair of single-valued and multivalued maps which need not be weakly compatible. Recently, Aamri and El Moutawakil [1] defined a property (EA) for self-maps which contained the class of noncompatible maps. More recently, Kamran [13] extended the property (EA) for a hybrid pair of single- and multivalued maps and generalized the notion of (IT)-commutativity for such pair.

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The aim of this paper is to define a new property which contains the property (EA) for a hybrid pair of single- and multivalued maps and give some new common fixed point theorems under hybrid contractive conditions. As an application, we give an affirmative (half-) answer (Theorem 2.8) to the open problem in [21].

Now we state some known definitions and facts.

Definition 1.1 [12]. Maps $f : X \to X$ and $T : X \to CB(X)$ are weakly compatible if they commute at their coincidence points, that is, if $fTx = Tfx$ whenever $fx \in Tx$.

Definition 1.2 [21]. Maps $f : X \to X$ and $T : X \to CB(X)$ are said to be (IT)-commuting at $x \in X$ if $fTx \subset Tfx$ whenever $fx \in Tx$.

Definition 1.3 [1]. Maps $f,g : X \to X$ are said to satisfy the property (EA) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in X$.

Definition 1.4 [13]. Maps $f : X \to X$ and $T : X \to CB(X)$ are said to satisfy the property (EA) if there exist a sequence $\{x_n\}$ in X, some t in X, and A in $CB(X)$ such that

$$\lim_{n\to\infty} fx_n = t \in A = \lim_{n\to\infty} Tx_n. \tag{1.3}$$

Definition 1.5 [13]. Let $T : X \to CB(X)$. The map $f : X \to X$ is said to be T-weakly commuting at $x \in X$ if $ffx \in Tfx$.

For the rest of the introduction, we state the following theorem as the prototype in this paper.

Theorem 1.6 (see [13]). *Let f be a self-map of the metric space (X,d) and let F be a map from X into $CB(X)$ such that*

(1) *(f,F) satisfies the property (EA);*

(2) *for all $x \neq y$ in X,*

$$H(Fx,Fy) < \max\left\{d(fx,fy), \frac{d(fx,Fx)+d(fy,Fy)}{2}, \frac{d(fx,Fy)+d(fy,Fx)}{2}\right\}. \tag{1.4}$$

If fX is closed subset of X, then

(a) *f and F have a coincidence point;*

(b) *f and F have a common fixed point provided that f is F-weakly commuting at v and $ffv = fv$ for $v \in C(f,F)$, where $C(f,F) = \{x : x$ is a coincidence point of f and $F\}$.*

2. Main results

We begin with the following definition.

Definition 2.1. (1) Let $f,g,F,G : X \to X$. The maps pair (f,F) and (g,G) are said to satisfy the *common property (EA)* if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X and some t in X such that

$$\lim_{n\to\infty} Gy_n = \lim_{n\to\infty} Fx_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t \in X. \tag{2.1}$$

(2) Let $f,g : X \to X$ and $F,G : X \to CB(X)$. The maps pair (f,F) and (g,G) are said to satisfy the *common property (EA)* if there exist two sequences $\{x_n\}$, $\{y_n\}$ in X, some t in X, and A, B in $CB(X)$ such that

$$\lim_{n\to\infty} Fx_n = A, \quad \lim_{n\to\infty} Gy_n = B, \quad \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = t \in A\cap B. \tag{2.2}$$

Example 2.2. Let $X = [1,+\infty)$ with the usual metric. Define $f,g : X \to X$ and $F,G : X \to CB(X)$ by $f(x) = 2+x/3$, $g(x) = 2+x/2$, and $F(x) = [1,2+x]$, $G(x) = [3,3+x/2]$ for all $x \in X$. Consider the sequences $\{x_n\} = \{3+1/n\}$, $\{y_n\} = \{2+1/n\}$. Clearly, $\lim_{n\to\infty} Fx_n = [1,5] = A$, $\lim_{n\to\infty} Gy_n = [3,4] = B$, $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = 3 \in A\cap B$. Therefore, (f,F) and (g,G) are said to satisfy the common property (EA).

Theorem 2.3. *Let f, g be two self-maps of the metric space (X,d) and let F, G be two maps from X into $CB(X)$ such that*

(1) *(f,F) and (g,G) satisfy the common property (EA);*

(2) *for all $x \neq y$ in X,*

$$H(Fx,Gy) < \max\left\{d(fx,gy), \frac{d(fx,Fx)+d(gy,Gy)}{2}, \frac{d(fx,Gy)+d(gy,Fx)}{2}\right\}. \tag{2.3}$$

If fX and gX are closed subsets of X, then

(a) *f and F have a coincidence point;*

(b) *g and G have a coincidence point;*

(c) *f and F have a common fixed point provided that f is F-weakly commuting at v and $ffv = fv$ for $v \in C(f,F)$;*

(d) *g and G have a common fixed point provided that g is G-weakly commuting at v and $ggv = gv$ for $v \in C(g,G)$;*

(e) *f, g, F, and G have a common fixed point provided that both (c) and (d) are true.*

Proof. Since (f,F) and (g,G) satisfy the common property (EA), there exist two sequences $\{x_n\}$, $\{y_n\}$ in X and $u \in X$, $A,B \in CB(X)$ such that

$$\begin{gathered} \lim_{n\to\infty} Fx_n = A, \quad \lim_{n\to\infty} Gy_n = B, \\ \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = u \in A\cap B. \end{gathered} \tag{2.4}$$

By virtue of fX and gX being closed, we have $u = fv$ and $u = gw$ for some $v,w \in X$. We claim that $fv \in Fv$ and $gw \in Gw$. Indeed, condition (2) implies that

$$H(Fx_n,Gw) < \max\left\{d(fx_n,gw), \frac{d(fx_n,Fx_n)+d(gw,Gw)}{2}, \frac{d(fx_n,Gw)+d(gw,Fx_n)}{2}\right\}. \tag{2.5}$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} H(A,Gw) &< \max\left\{d(fv,gw), \frac{d(fv,A)+d(gw,Gw)}{2}, \frac{d(fv,Gw)+d(gw,A)}{2}\right\} \\ &= \frac{d(gw,Gw)}{2}. \end{aligned} \tag{2.6}$$

Since $gw = fv \in A$, it follows from the definition of Hausdorff metric that

$$d(gw,Gw) \leq H(A,Gw) \leq \frac{d(gw,Gw)}{2}, \tag{2.7}$$

which implies that $gw \in Gw$.

On the other hand, by condition (2) again, we have

$$H(Fv,Gy_n) < \max\left\{d(fv,gy_n), \frac{d(fv,Fv)+d(gy_n,Gy_n)}{2}, \frac{d(fv,Gy_n)+d(gy_n,Fv)}{2}\right\}. \tag{2.8}$$

Similarly, we obtain

$$d(fv,Fv) \leq H(Fv,B) \leq \frac{d(fv,Fv)}{2}. \tag{2.9}$$

Hence $fv \in Fv$. Thus f and F have a coincidence point v, g and G have a coincidence point w. This ends the proofs of part (a) and part (b).

Furthermore, by virtue of condition (c), we obtain $ffv = fv$ and $ffv \in Ffv$. Thus $u = fu \in Fu$. This proves (c). A similar argument proves (d). Then (e) holds immediately. □

Remark 2.4. In Theorem 2.3, if F, G are two maps from K into $CB(X)$, where K is a closed subset of X. In this case, it is necessary to assume that (X,d) is a metrically convex metric space. In this direction, many excellent works have appeared (see [5, 21]).

Corollary 2.5 (see [13, Theorem 3.10]). *Let f be a self-map of the metric space (X,d) and let F be a map from X into $CB(X)$ such that*

(1) *(f,F) satisfies the property (EA);*

(2) *for all $x \neq y$ in X,*

$$H(Fx,Fy) < \max\left\{d(fx,fy), \frac{d(fx,Fx)+d(fy,Fy)}{2}, \frac{d(fx,Fy)+d(fy,Fx)}{2}\right\}. \tag{2.10}$$

If fX is closed subset of X, then

(a) *f and F have a coincidence point;*

(b) *f and F have a common fixed point provided that f is F-weakly commuting at v and $ffv = fv$ for $v \in C(f,F)$.*

Proof. Let $F = G$ and $f = g$, then the results follow from Theorem 2.3 immediately. □

If $f = g$, we can conclude the following corollary.

Corollary 2.6. *Let f be a self-map of the metric space (X,d) and let F, G be two maps from X into $CB(X)$ such that*
(1) *(f,F) and (f,G) satisfy the common property (EA);*
(2) *for all $x \neq y$ in X,*

$$H(Fx,Gy) < \max\left\{d(fx,fy), \frac{d(fx,Fx)+d(fy,Gy)}{2}, \frac{d(fx,Gy)+d(fy,Fx)}{2}\right\}. \tag{2.11}$$

If fX is closed subset of X, then
(a) *f, G and F have a coincidence point;*
(b) *f, G and F have a common fixed point provided that f is both F-weakly commuting and G-weakly commuting at v and $ffv = fv$ for $v \in C(f,F)$.*

If both F and G are single-valued maps in Theorem 2.3, then we have the following corollary.

Corollary 2.7. *Let f, g, F, and G be four self-maps of the metric space (X,d) such that*
(1) *(f,F) and (g,G) satisfy the common property (EA);*
(2) *for all $x \neq y$ in X,*

$$d(Fx,Gy) < \max\left\{d(fx,gy), \frac{d(fx,Fx)+d(gy,Gy)}{2}, \frac{d(fx,Gy)+d(gy,Fx)}{2}\right\}. \tag{2.12}$$

If fX and gX are closed subsets of X, then
(a) *f and F have a coincidence point;*
(b) *g and G have a coincidence point;*
(c) *f and F have a common fixed point provided that f is F-weakly commuting at v and $ffv = fv$ for $v \in C(f,F)$;*
(d) *g and G have a common fixed point provided that g is G-weakly commuting at v and $ggv = gv$ for $v \in C(g,G)$;*
(e) *f, g, F, and G have a common fixed point provided that both (c) and (d) are true.*

Theorem 2.8. *Let f, g be two self-maps of the complete metric space (X,d), let $\lambda \in (0,1)$ be a constant, and let F, G be two maps from X into $CB(X)$ such that for all $x \neq y$ in X,*

$$H(Fx,Gy) \leq \lambda \max\left\{d(fx,gy), d(fx,Fx), d(gy,Gy), \frac{d(fx,Gy)+d(gy,Fx)}{2}\right\}. \tag{2.13}$$

If fX and gX are closed subsets of X and $FX \subset gX$, $GX \subset fX$, then
(a) *f and F have a coincidence point;*
(b) *g and G have a coincidence point;*
(c) *f and F have a common fixed point provided that f is F-weakly commuting at v and $ffv = fv$ for $v \in C(f,F)$;*

(d) *g and G have a common fixed point provided that g is G-weakly commuting at v and ggv = gv for* $v \in C(g,G)$;

(e) f, g, F, *and G have a common fixed point provided that both (c) and (d) are true.*

Proof. For any given $x_0 \in X$, by virtue of $FX \subset gX$, there is $x_1 \in X$ such that $y_1 = gx_1 \in Fx_0$. Now since Fx_0 and Gx_1 are closed sets and $y_1 \in Fx_0$, we can find $y_2 \in Gx_1$ such that

$$d(y_1,y_2) \le H(Fx_0,Gx_1)+\lambda. \tag{2.14}$$

Since $GX \subset fX$, there exists x_2 such that $fx_2 = y_2 \in Gx_1$, then we choose $y_3 \in Fx_2$ satisfying

$$d(y_2,y_3) \le H(Gx_1,Fx_2)+\lambda^2, \tag{2.15}$$

and $y_3 = gx_3$ for some $x_3 \in X$.

We continue this process to obtain a sequence $\{y_n\}$ in X such that

$$\begin{gathered} y_{2n} = fx_{2n} \in Gx_{2n-1}, \qquad y_{2n+1} = gx_{2n+1} \in Fx_{2n}, \\ d(y_{2n},y_{2n+1}) \le H(Gx_{2n-1},Fx_{2n})+\lambda^{2n}, \\ d(y_{2n-1},y_{2n}) \le H(Fx_{2n-2},Gx_{2n-1})+\lambda^{2n-1}, \quad n=1,2,\dots. \end{gathered} \tag{2.16}$$

Let $a_n = d(y_n,y_{n+1})$, then

$$\begin{aligned} a_{2n} &= d(y_{2n},y_{2n+1}) \le H(Gx_{2n-1},Fx_{2n})+\lambda^{2n} \\ &\le \lambda \max\Big\{ d(fx_{2n},gx_{2n-1}), d(fx_{2n},Fx_{2n}), d(gx_{2n-1},Gx_{2n-1}), \\ &\qquad \frac{d(fx_{2n},Gx_{2n-1})+d(gx_{2n-1},Fx_{2n})}{2}\Big\}+\lambda^{2n}. \end{aligned} \tag{2.17}$$

By $fx_{2n} \in Gx_{2n-1}$, we have

$$d(gx_{2n-1},Gx_{2n-1}) \le d(gx_{2n-1},fx_{2n}), \qquad d(fx_{2n},Fx_{2n}) \le H(Gx_{2n-1},Fx_{2n}). \tag{2.18}$$

Thus, we rewrite (2.17) as

$$a_{2n} \le \lambda \max\left\{ d(fx_{2n},gx_{2n-1}), \frac{d(gx_{2n-1},Fx_{2n})}{2}\right\}+\lambda^{2n}. \tag{2.19}$$

Hence, we obtain

$$a_{2n} \le \lambda \max\left\{ a_{2n-1}, \frac{a_{2n-1}+a_{2n}}{2}\right\}+\lambda^{2n}. \tag{2.20}$$

If $a_{2n-1} \le a_{2n}$ for some n, we have $a_{2n} \le \lambda^{2n}/(1-\lambda)$. Otherwise, we get

$$a_{2n} \le \lambda a_{2n-1}+\lambda^{2n}. \tag{2.21}$$

Therefore, by (2.20), we achieve

$$a_{2n} \leq \max\left\{\lambda a_{2n-1} + \lambda^{2n}, \frac{\lambda^{2n}}{1-\lambda}\right\}. \tag{2.22}$$

On the other hand,

$$\begin{aligned} a_{2n-1} &\leq H(Gx_{2n-1}, Fx_{2n-2}) + \lambda^{2n-1} \\ &\leq \lambda \max\Big\{d(fx_{2n-2}, gx_{2n-1}), d(fx_{2n-2}, Fx_{2n-2}), d(gx_{2n-1}, Gx_{2n-1}), \\ &\qquad \frac{d(fx_{2n-2}, Gx_{2n-1}) + d(gx_{2n-1}, Fx_{2n-2})}{2}\Big\} + \lambda^{2n-1}. \end{aligned} \tag{2.23}$$

Since $gx_{2n-1} \in Fx_{2n-2}$, we have

$$\begin{aligned} d(gx_{2n-1}, Gx_{2n-1}) &\leq H(Gx_{2n-1}, Fx_{2n-2}), \\ d(fx_{2n-2}, Fx_{2n-2}) &\leq d(gx_{2n-1}, fx_{2n-2}). \end{aligned} \tag{2.24}$$

Thus, we obtain

$$a_{2n-1} \leq \lambda \max\left\{a_{2n-2}, \frac{a_{2n-2} + a_{2n-1}}{2}\right\} + \lambda^{2n-1}. \tag{2.25}$$

Similarly, we get

$$a_{2n-1} \leq \max\left\{\lambda a_{2n-2} + \lambda^{2n-1}, \frac{\lambda^{2n-1}}{1-\lambda}\right\}. \tag{2.26}$$

By (2.22) and (2.26), we obtain

$$a_n \leq \max\left\{\lambda a_{n-1} + \lambda^n, \frac{\lambda^n}{1-\lambda}\right\}, \quad n = 1, 2, \ldots. \tag{2.27}$$

It is easy to see that

$$a_n \leq \max\left\{\lambda^n (a_0 + n), \frac{\lambda^n}{1-\lambda}\right\}, \quad n = 1, 2, \ldots. \tag{2.28}$$

Thus, there exists $n_0 > 0$ such that for $n \geq n_0$,

$$a_n \leq \lambda^n (a_0 + n). \tag{2.29}$$

Hence $\lim_{n\to\infty} a_n = 0$.

In order to prove that $\{y_n\}$ is Cauchy sequence, for any $\varepsilon > 0$, we choose a sufficiently large number N such that

$$\lambda^N (a_0 + N) \leq \frac{\varepsilon(1-\lambda)}{2}, \quad \lambda^N \leq \frac{\varepsilon(1-\lambda)^2}{4}. \tag{2.30}$$

Thus, for any positive integer k, we obtain

$$\begin{aligned} d(y_N, y_{N+k}) &\le \sum_{i=0}^{k-1} a_{N+i} \le \sum_{i=0}^{k-1} \lambda^{N+i}(a_0+N+i) \\ &< \lambda^N(a_0+N)\frac{1}{1-\lambda} + \lambda^N\left(\sum_{i=0}^{k-1} i\lambda^i\right) \\ &< \lambda^N(a_0+N)\frac{1}{1-\lambda} + \lambda^N\frac{2}{(1-\lambda)^2} \le \varepsilon. \end{aligned} \tag{2.31}$$

This implies that $\{y_n\}$ is a Cauchy sequence. Thus there is u satisfying

$$\lim_{n\to\infty} y_n = u = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n+1}. \tag{2.32}$$

Since fX and gX are closed, there exist a, b such that $fa = u = gb$. A similar argument proves that

$$\begin{gathered} \lim_{n\to\infty} Fx_{2n} = \lim_{n\to\infty} Gx_{2n+1}, \\ u \in \lim_{n\to\infty} Fx_{2n} = \lim_{n\to\infty} Gx_{2n+1}. \end{gathered} \tag{2.33}$$

Then (f,F) and (g,G) satisfy the common property (EA). The rest of the proof follows Theorem 2.3 immediately, then the proof of Theorem 2.8 is complete. □

Corollary 2.9. *Let f, g be two self-maps of the complete metric space (X,d), let $\lambda \in (0,1)$ be a constant, and let F, G be two maps from X into $CB(X)$ such that for all $x \ne y$ in X,*

$$\begin{aligned} H(Fx,Gy) \le{}& \alpha d(fx,gy) + \beta \max\{d(fx,Fx), d(gy,Gy)\} \\ &+ \gamma \max\{d(fx,Gy)+d(gy,Fx), d(fx,Fx)+d(gy,Gy)\}, \end{aligned} \tag{2.34}$$

and $\alpha+\beta+2\gamma<1$. If fX and gX are closed subsets of X and $FX \subset gX$, $GX \subset fX$, then

- (a) *f and F have a coincidence point;*
- (b) *g and G have a coincidence point;*
- (c) *f and F have a common fixed point provided that f is F-weakly commuting at v and $ffv = fv$ for $v \in C(f,F)$;*
- (d) *g and G have a common fixed point provided that g is G-weakly commuting at v and $ggv = gv$ for $v \in C(g,G)$;*
- (e) *f, g, F, and G have a common fixed point provided that both (c) and (d) are true.*

Proof. Let $\lambda = \alpha+\beta+2\gamma$. Following (2.34) and $\max\{d(fx,Fx), d(gy,Gy)\} \ge (d(fx,Fx)+d(gy,Gy))/2$, it is easy to see that

$$H(Fx,Gy) \le \lambda \max\left\{d(fx,gy), d(fx,Fx), d(gy,Gy), \frac{d(fx,Gy)+d(gy,Fx)}{2}\right\}. \tag{2.35}$$

Thus by Theorem 2.8, we arrive to the conclusion in Corollary 2.9. □

The next theorem involves a function φ. Various conditions on φ have been investigated by different authors [4, 6, 15, 16]. Let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ continue and satisfy the following conditions:

(A_1) φ is nondecreasing on $\mathbb{R}^+$,

(A_2) $0 < \varphi(t) < t$, for each $t \in (0,+\infty)$.

Theorem 2.10. *Let f, g be two self-maps of the metric space (X,d) and let $F,G : X \to X$ be two maps from X into $CB(X)$ such that*

(1) *(f,F) and (g,G) satisfy the common property (EA);*

(2) *for all $x \neq y$ in X,*

$$H(Fx,Gy) \leq \varphi\big(\max\{d(fx,gy),d(fx,Fx),d(gy,Gy),d(fx,Gy),d(gy,Fx)\}\big). \quad (2.36)$$

If fX and gX are closed subsets of X, then

(a) *f and F have a coincidence point;*

(b) *g and G have a coincidence point;*

(c) *f and F have a common fixed point provided that f is F-weakly commuting at v and $ffv = fv$ for $v \in C(f,F)$;*

(d) *g and G have a common fixed point provided that g is G-weakly commuting at v and $ggv = gv$ for $v \in C(g,G)$;*

(e) *f, g, F, and G have a common fixed point provided that both (c) and (d) are true.*

Proof. Since (f,F) and (g,G) satisfy the common property (EA), there exist two sequences $\{x_n\}$, $\{y_n\}$ in X and $u \in X$, $A,B \in CB(X)$ such that

$$\begin{aligned} &\lim_{n\to\infty} Fx_n = A, \lim_{n\to\infty} Gy_n = B, \\ &\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gy_n = u \in A \cap B. \end{aligned} \quad (2.37)$$

By virtue of fX and gX being closed, we have $u = fv$ and $u = gw$ for some $v,w \in X$. We claim that $fv \in Fv$ and $gw \in Gw$. Indeed, condition (2) implies that

$$H(Fx_n,Gw) \leq \varphi\big(\max\{d(fx_n,gw),d(fx_n,Fx_n),d(gw,Gw),d(fx_n,Gw),d(gw,Fx_n)\}\big). \quad (2.38)$$

Taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} H(A,Gw) &\leq \varphi\big(\max\{d(fv,gw),d(fv,A),d(gw,Gw),d(fv,Gw),d(gw,A)\}\big) \\ &\leq \varphi(d(gw,Gw)) < d(gw,Gw). \end{aligned} \quad (2.39)$$

Since $gw = fv \in A$, it follows from the definition of Hausdorff metric that

$$d(gw,Gw) \leq H(A,Gw) < d(gw,Gw), \quad (2.40)$$

which implies that $gw \in Gw$.

On the other hand, by condition (2) again, we have

$$H(Fv,Gy_n) \le \varphi(\max\{d(fv,gy_n),d(fv,Fv),d(gy_n,Gy_n),d(fv,Gy_n),d(gy_n,Fv)\}). \tag{2.41}$$

Similarly, we obtain

$$d(fv,Fv) \le H(Fv,B) < d(fv,Fv). \tag{2.42}$$

Hence $fv \in Fv$. Thus f and F have a coincidence point v, g and G have a coincidence point w. This ends the proofs of part (a) and part (b). The rest of proof is similar to the argument of Theorem 2.3. □

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