ANNIHILATORS OF NILPOTENT ELEMENTS

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Let x be a nilpotent element of an infinite ring R (not necessarily with 1). We prove that $A(x)$—the two-sided annihilator of x—has a large intersection with any infinite ideal I of R in the sense that $\operatorname{card}(A(x) \cap I) = \operatorname{card} I$. In particular, $\operatorname{card} A(x) = \operatorname{card} R$; and this is applied to prove that if N is the set of nilpotent elements of R and $R \neq N$, then $\operatorname{card}(R \backslash N) \geq \operatorname{card} N$.

For an element x of a ring R, let $A_\ell(x)$, $A_r(x)$, and $A(x)$ denote, respectively, the left, right and two-sided annihilator of x in R. For a set X, we denote $\operatorname{card} X$ by $|X|$; and say that a subset Y of X is *large* in X if $|Y| = |X|$. We prove that if x is any nilpotent element and I is any infinite ideal of R, then $A(x) \cap I$ is large in I, and in particular $|A_\ell(x)| = |A_r(x)| = |A(x)| = |R|$. The last result is applied to obtain a generalization of a result of Putcha and Yaqub [2] which shows that an infinite nonnil ring has infinitely many nonnilpotent elements. A short proof of their result is given in [1]. We prove a much stronger result showing that the set of nonnilpotent elements of a nonnil ring is at least as large as is its set of nilpotent elements. The following lemma is simple but crucial.

Lemma 1. *Let R be an infinite ring, $(S,+)$ an infinite subgroup of $(R,+)$, and x an element of R. Then either $|Sx| = |S|$ or $|A_\ell(x) \cap S| = |S|$, and similarly $|xS| = |S|$ or $|A_r(x) \cap S| = |S|$.*

Proof. Consider the map $y \mapsto yx$ from $(S,+)$ onto $(Sx,+)$. The kernel is $A_\ell(x) \cap S$, so $|S| = |Sx||A_\ell(x) \cap S|$ and the result follows since S is infinite. □

A subset of a ring R is said to be *root closed* if whenever it contains a power of an element, it also contains the element itself.

Theorem 2. *Let R be an infinite ring and α an infinite cardinal. Then, the following hold.*

(i) *For any left (right) ideal I of R, the set $\{x \in R \mid |A_r(x) \cap I| = \alpha\}$ (resp., $\{x \in R \mid |A_\ell(x) \cap I| = \alpha\}$) is root closed. In particular, if I is infinite, $\{x \in R \mid |A_r(x) \cap I| = |I|\}$ (resp., $\{x \in R \mid |A_\ell(x) \cap I| = |I|\}$) is root closed, so it contains the set N of nilpotent elements of R.*

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(ii) *For any ideal I of R, $\{x \in R \mid |A(x) \cap I| = \alpha\}$ is root closed. In particular, if I is infinite, $\{x \in R \mid |A(x) \cap I| = |I|\}$ is root closed, so it contains N.*

Proof. (i) Let $|A_r(x^n) \cap I| = \alpha$ for some $n \geq 2$ and consider $x^{n-1}(A_r(x^n) \cap I)$. By Lemma 1, either $|x^{n-1}(A_r(x^n) \cap I)| = \alpha$ or $|A_r(x^{n-1}) \cap I| = |A_r(x^{n-1}) \cap (A_r(x^n) \cap I)| = \alpha$. Now $x^{n-1}(A_r(x^n) \cap I) \subseteq A_r(x) \cap I \subseteq A_r(x^{n-1}) \cap I \subseteq A_r(x^n) \cap I$, so $|A_r(x^{n-1}) \cap I| = \alpha$ even when $|x^{n-1}(A_r(x^n) \cap I)| = \alpha$. It follows by induction that $|A_r(x) \cap I| = \alpha$.

(ii) Let $|A(x^n) \cap I| = \alpha$ for some $n \geq 2$. Since $A_\ell(x^n) \cap I$ is a left ideal and $|A_r(x^n) \cap (A_\ell(x^n) \cap I)| = |A(x^n) \cap I| = \alpha$, it follows by (i) that $|A_r(x) \cap (A_\ell(x^n) \cap I)| = \alpha = |A_\ell(x^n) \cap (A_r(x) \cap I)|$; and since $A_r(x) \cap I$ is a right ideal, we get, again by (i), that $|A_\ell(x) \cap (A_r(x) \cap I)| = \alpha$, namely $|A(x) \cap I| = \alpha$. □

Applying the previous theorem for $I = R$, we obtain the following corollary.

Corollary 3. *Let x be a nilpotent element of an infinite ring R, then $|A_\ell(x)| = |A_r(x)| = |A(x)| = |R|$.*

The previous corollary will be applied in the proof of the above-mentioned generalization of a result of Putcha and Yaqub [2]. We also need the following result.

Lemma 4. *Let b be a nonnilpotent element of an infinite ring R. If $R\backslash N$ is infinite, then $|A_\ell(b)| \leq |R\backslash N|$ and $|A_r(b)| \leq |R\backslash N|$.*

Proof. Let $x \in A_\ell(b) \cap N$, then $xb = 0$ and $x^n = 0$ for some $n \geq 1$. Let $m \geq n$, then $(b+x)^m = b^m + b^{m-1}x + \cdots + bx^{m-1}$. Since $(b^{m-1}x + \cdots + bx^{m-1})^2 = 0$ and $b \notin N$, $b^{2m} \neq 0$ and $(b+x)^m \neq 0$, so $b + x \notin N$. Hence, the map $x \mapsto b + x$ is $1-1$ from $A_\ell(b) \cap N$ into $R\backslash N$ and therefore $|A_\ell(b) \cap N| \leq |R\backslash N|$. Since $R\backslash N$ is infinite, we get that $|A_\ell(b)| = |A_\ell(b)\backslash N| + |A_\ell(b) \cap N| \leq |R\backslash N| + |R\backslash N| = |R\backslash N|$. □

In a ring with 1, the map $x \mapsto 1 + x$ from N into $R\backslash N$ is $1-1$, so $|R\backslash N| \geq |N|$. The next theorem shows that the same result holds in any nonnil ring. In particular, we get the result of Putcha and Yaqub [2] stating that R is finite when $R\backslash N$ is finite and not empty.

Theorem 5. *Let R be a nonnil ring, then $|R\backslash N| \geq |N|$.*

Proof. We start with R infinite. Suppose $|R\backslash N| < |N|$, then $|N| = |R|$ and $|R\backslash N| < |R|$. By the previous lemma, if $b \in R\backslash N$, $|A_\ell(b)| \leq |R\backslash N|$, so $|A_\ell(b)| < |R|$ and by Lemma 1, $|Rb| = |R|$. Now $|R| = |Rb| \leq |Nb| + |(R\backslash N)b|$ and $|(R\backslash N)b| \leq |R\backslash N| < |R|$, so $|Nb| = |R|$. Therefore, $|\{b + xb | x \in N\}| = |R|$, so since $|R\backslash N| < |R|$, there exists $x \in N$ such that $b + xb \notin R\backslash N$, namely $b + xb \in N$. Since $x \in N$, $1 + x$ is formally invertible, so $A_r(b + xb) = A_r(b)$. By Corollary 3, $|A_r(b + xb)| = |R|$ and by Lemma 4, $|A_r(b)| \leq |R\backslash N| < |R|$, a contradiction.

Now let R be finite and let J be its radical. Since J is nilpotent, if $a \in R$, $a + J$ is nilpotent in R/J if and only if a is nilpotent, and if $a \notin N$, $(a + J) \cap N = \varnothing$. Since R/J is a finite semisimple ring, it has 1, so at least half of its elements are nonnilpotent, hence at least half of the distinct cosets $a + J$, $a \in R$, do not intersect N, and therefore at least half of the elements of R are not nilpotent, so $|R\backslash N| \geq |N|$. □

References

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