MAPPINGS PRESERVING REGULAR HEXAHEDRONS

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We will prove that if a one-to-one mapping $f:\mathbb{R}^3\to\mathbb{R}^3$ preserves regular hexahedrons, then f is a linear isometry up to translation.

1. Introduction

Let X and Y be normed spaces. A mapping $f:X\to Y$ is called an isometry if f satisfies the equality

$$\|f(x)-f(y)\| = \|x-y\| \tag{1.1}$$

for all $x,y\in X$. A distance $r>0$ is said to be preserved (conservative) by a mapping $f:X\to Y$ if $\|f(x)-f(y)\|=r$ for all $x,y\in X$ with $\|x-y\|=r$.

If f is an isometry, then every distance $r>0$ is conservative by f, and conversely. We can now raise a question whether each mapping that preserves certain distances is an isometry. Indeed, Aleksandrov [1] had raised a question whether a mapping $f:X\to X$ preserving a distance $r>0$ is an isometry, which is now known to us as the Aleksandrov problem.

Beckman and Quarles [2] solved the Aleksandrov problem for finite-dimensional real Euclidean spaces $X=\mathbb{R}^n$ (see also [3, 4, 5, 6, 7, 12, 13, 14, 15, 16, 17, 18, 19]).

Theorem 1.1 (Beckman and Quarles). *If a mapping $f:\mathbb{R}^n\to\mathbb{R}^n$ $(2\le n<\infty)$ preserves a distance $r>0$, then f is a linear isometry up to translation.*

It seems to be interesting to investigate whether the "distance $r>0$" in the above theorem can be replaced by some properties characterized by "geometrical figures" without loss of its validity.

In [8], Jung proved that if a one-to-one mapping $f:\mathbb{R}^n\to\mathbb{R}^n$ $(n\ge 2)$ maps every regular triangle (quadrilateral or hexagon) of side length $a>0$ onto a figure of the same type with side length $b>0$, then there exists a linear isometry $I:\mathbb{R}^n\to\mathbb{R}^n$ up to translation such that $f(x)=(b/a)I(x)$.

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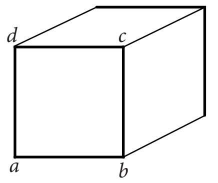

Figure 1.1. Cube A.

Furthermore, the authors [10] proved that if a one-to-one mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ maps every unit circle onto a unit circle, then f is a linear isometry up to translation (see also [9, 11]).

In this connection, we will extend the results of [8] to the more general three-dimensional objects, that is, we prove in this paper that if a one-to-one mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ maps every regular hexahedron onto a regular hexahedron, then f is a linear isometry up to translation. (An isometry $I : \mathbb{R}^3 \to \mathbb{R}^3$ is called a linear isometry up to translation if there exists a point $v \in \mathbb{R}^3$ such that $I(x) - v$ is a linear mapping.)

2. Main theorem

From now on, by a cube we mean a regular hexahedron with side length one. We first make our terms precise as follows. In Figure 1.1, we will call the points a, b, c, d "vertices" and the lines $\overline{ab}$, $\overline{bc}$, $\overline{cd}$, $\overline{da}$ "edges" and the plane bounded by the four edges $\overline{ab}$, $\overline{bc}$, $\overline{cd}$, $\overline{da}$ "face $abcd$" or simply a "face." Further by a cube or hexahedron we will mean the six faces only and not the three-dimensional open set bounded by those six faces. Let us denote the three-dimensional open set bounded by cube A as "Inside of A" or simply as Inside(A).

Suppose that $p \in A$ where p is a point and A is a cube. Firstly let us review the solid angles in three dimensions. If p is a vertex, say $p = a$, then the solid angle that Inside(A) subtends with respect to p is $\pi/2$. If p is a point which belongs to an edge and is not a vertex, then the solid angle that Inside(A) subtends with respect to p is π. If $p \in A$ is neither a vertex nor an edge point, then the solid angle that Inside(A) subtends with respect to p is 2π. Let us denote the solid angle that Inside(A) subtends with respect to $p \in A$ by $\Omega(A, p)$. Therefore for $p \in A$, if $\Omega(A, p) = \pi/2$ or $\Omega(A, p) = \pi$, p is a vertex of A or p is an edge point of A (and not a vertex), respectively. If $\Omega(A, p) = 2\pi$, then p is neither a vertex nor an edge point of a cube A. Now we prove the following lemma.

Lemma 2.1. *Let a one-to-one mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ map every regular hexahedron onto a regular hexahedron. For any A and B cubes, if* Inside$(A) \cap$ Inside$(B) = \varnothing$, *then* Inside$\{f(A)\} \cap$ Inside$\{f(B)\} = \varnothing$.

Proof. First, we show that if $q \notin$ Inside(A), then $f(q) \notin$ Inside$\{f(A)\}$. In other words, we show that if $f(q) \in$ Inside$\{f(A)\}$, then $q \in$ Inside(A). Assume that $q \in A$. Then $f(q) \in f(A)$ and so $f(q) \notin$ Inside$\{f(A)\}$. Suppose that $q \notin$ Inside(A) and $q \notin A$. Then choose another cube B such that $q \in B$ and $B \cap A = \varnothing$. Then $f(B) \cap f(A) = \varnothing$ and therefore $f(q) \notin$ Inside$\{f(A)\}$.

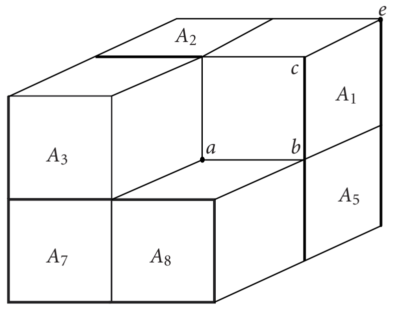

Figure 2.1

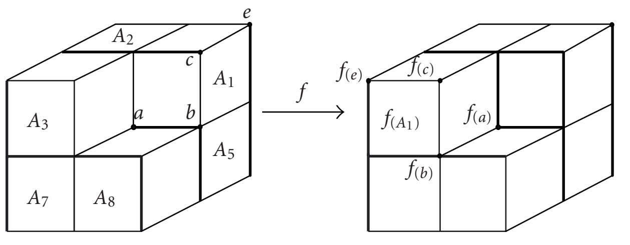

Figure 2.2

Now, let Inside$\{f(A)\} \cap$ Inside$\{f(B)\} \neq \emptyset$. Then Inside$\{f(A)\} \cap f(B) \neq \emptyset$, which means that for some $b \in B$, $f(b) \in$ Inside$\{f(A)\}$. Therefore $b \in$ Inside(A) and Inside$(A) \cap B \neq \emptyset$ by which we conclude that Inside$(A) \cap$ Inside$(B) \neq \emptyset$. □

We show now that if any one-to-one mapping preserves regular hexahedrons, then it is actually an isometry. More precisely, we have the following.

Theorem 2.2. *If a one-to-one mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ maps every regular hexahedron onto a regular hexahedron, then f is a linear isometry up to translation.*

Proof. We show that f preserves the distance $\sqrt{3}$. Let a be a vertex of a cube $A = A_1$. We can then construct 7 more cubes A_i $(i = 2,\ldots,8)$ so that a is the common vertex of 8 cubes A_i $(i = 1,\ldots,8)$ and Inside$(A_i) \cap$ Inside$(A_j) = \emptyset$ for $i \neq j$ (see Figure 2.1). Then $f(a)$ belongs to $f(A_i)$ for $i = 1,\ldots,8$ and by Lemma 2.1 Inside$\{f(A_i)\} \cap$ Inside$\{f(A_j)\} = \emptyset$ for $i \neq j$. Now the solid angle that Inside$\{f(A_i)\}$ subtends with respect to $f(a)$ is at least $\pi/2$ for any i, that is, $\Omega(f(A_i), f(a)) \geq \pi/2$. Since the maximum solid angle with respect to the point $f(a)$ is 4π, $\Omega(f(A_i), f(a)) = \pi/2$ and $f(a)$ is a vertex of $f(A_i)$ for every i. As a conclusion, if a is a vertex of a cube A, then $f(a)$ is a vertex of a cube $f(A)$.

Now given any two points a and e which are separated by the distance $\sqrt{3}$ from each other, form cube A_1 such that they are two vertices of A_1. We form 7 more cubes $A_2,\ldots,A_8$ so that the following conditions are met (see Figure 2.2). Firstly, Inside$(A_i) \cap$ Inside$(A_j) = \emptyset$ for $i \neq j$. a is the common vertex of A_i, $i = 1,\ldots,8$. Each cube A_i has exactly 3 vertices (like the vertex b) each of which is the common vertex of exactly four cubes. They are all separated from a by the distance 1. Each cube A_i has exactly 3 vertices (like the vertex c) each of which is the common vertex of exactly two cubes. They are all separated from a by the distance $\sqrt{2}$. Each cube A_i has exactly one vertex (like the vertex e) which belongs to only one cube A_i and is separated from a by the distance $\sqrt{3}$.

If we use Lemma 2.1, we can obtain $\text{Inside}\{f(A_i)\} \cap \text{Inside}\{f(A_j)\} = \emptyset$ for $i \neq j$. $f(a)$ is the common vertex of $f(A_i)$, $i = 1,\ldots,8$. Each cube $f(A_i)$ has exactly 3 vertices (like the vertex $f(b)$) each of which is the common vertex of exactly four cubes. They are all separated from $f(a)$ by the distance 1. Each cube $f(A_i)$ has exactly 3 vertices (like the vertex $f(c)$) each of which is the common vertex of exactly two cubes. They are all separated from $f(a)$ by the distance $\sqrt{2}$. Each cube $f(A_i)$ has exactly one vertex (like the vertex $f(e)$) which belongs to only one cube $f(A_i)$. It is separated from $f(a)$ by the distance $\sqrt{3}$. Therefore, we conclude that the distance between $f(a)$ and $f(e)$ is $\sqrt{3}$.

Consequently, in view of the theorem of Beckman and Quarles, we conclude that f is a linear isometry up to translation. □

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