A CLASS OF $\mathcal{I}$-CONSERVATIVE MATRICES

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Received 7 April 2005 and in revised form 20 September 2005

By using the concept of $\mathcal{I}$-convergence defined by Kostyrko et al. in 2001, the $\mathcal{I}$-limit superior of real sequences was introduced and the inequality $\mathcal{I}-\limsup(Ax) \le \mathcal{I}-\limsup(x)$ for all $x \in \ell_\infty$ was studied by Demirci in 2001. In this paper, we have characterized a class of $\mathcal{I}$-conservative matrices by studying some new inequalities related to the $\mathcal{I}$-limit superior.

1. Introduction

Let ℓ_∞ and c be the Banach spaces of bounded and convergent sequence $x = (x_k)$ with the usual supremum norm. Let σ be a one-to-one mapping of $\mathbb{N}$, the set of positive integers, into itself and $T : \ell_\infty \to \ell_\infty$ a linear operator defined by $Tx = (Tx_k) = (x_{\sigma(k)})$. An element $\phi \in \ell'_\infty$, the conjugate space of ℓ_∞, is called an invariant mean or a σ-mean if and only if (i) $\phi(x) \ge 0$ when the sequence $x = (x_k)$ has $x_k \ge 0$ for all k, (ii) $\phi(e) = 1$ where $e = (1,1,1,\ldots)$, and (iii) $\phi(Tx) = \phi(x)$ for all $x \in \ell_\infty$. Let M be the set of all σ-means on ℓ_∞. A sublinear functional P on ℓ_∞ is said to generate σ-means if $\phi \in \ell'_\infty$ and $\phi \le P \Rightarrow \phi$ is a σ-mean, and to dominate σ-means if $\phi \le P$ for all $\phi \in M$, where $\phi \le P$ means that $\phi(x) \le P(x)$ for all $x \in \ell_\infty$.

It is shown [8] that the sublinear functional

$$V(x) = \sup_n \limsup_p t_{pn}(x) \tag{1.1}$$

both generates and dominates σ-means, where

$$t_{pn}(x) = \frac{1}{p+1}\left(x_n + x_{\sigma(n)} + \cdots + x_{\sigma^p(n)}\right), \quad t_{-1,n}(x) = 0. \tag{1.2}$$

A bounded sequence x is called σ-convergent to s if $V(x) = -V(-x) = s$. In this case, we write $\sigma - \lim x = s$. Let V_σ be the set of all σ-convergent sequences. We assume throughout this paper that $\sigma^p(n) \ne n$ for all $n \ge 0$ and $p \ge 1$, where $\sigma^p(n)$ is the pth iterate of

International Journal of Mathematics and Mathematical Sciences 2005:21 (2005) 3443–3452
DOI: 10.1155/IJMMS.2005.3443

σ at n. Thus, a σ-mean extends the limit functional onto c in the sense that $\phi(x) = \lim x$ for all $x \in c$ [9]. Consequently, $c \subset V_\sigma$.

By (iii), it is clear that $(Tx - x) \in Z$ for $x \in \ell_\infty$, where Z is the set of all σ-convergent sequences with σ-limit zero.

For $x \in \ell_\infty$, we write

$$l(x) = \liminf x, \qquad L(x) = \limsup x, \qquad W(x) = \inf_{z \in Z} L(x+z). \tag{1.3}$$

It is known that $V(x) = W(x)$ on ℓ_∞ [8].

Let $A = (a_{nk})$ be an infinite matrix of real numbers and $x = (x_k)$ a real sequence such that $Ax = (A_n(x)) = (\sum_k a_{nk} x_k)$ exists for each n. Then, the sequence $Ax = (A_n(x))$ is called an A-transform of x. For two sequence spaces E and F, we say that the matrix A maps E into F if Ax exits and belongs to F for each $x \in E$. By (E,F), we denote the set of all matrices which map E into F.

A matrix $A \in (c,c)$ is said to be conservative. It is known [1, page 21] that A is conservative if and only if $\|A\| = \sup_n \sum_k |a_{nk}| < \infty$, $a_k = \lim_n a_{nk}$ for each k, and $a = \lim_n \sum_k a_{nk}$. If A is conservative, the number $\chi = \chi(A) = a - \sum_k a_k$ called the characteristic of A is of importance in summability [1, page 46].

Let E be a subset of $\mathbb{N}$. Natural density δ of E is defined by

$$\delta(E) = \lim_n \frac{1}{n} \left| \{k \le n : k \in E\} \right|, \tag{1.4}$$

where the vertical bars indicate the number of elements in the enclosed set. The number sequence $x = (x_k)$ is said to be statistically convergent to the number l if for every ε, $\delta\{k : |x_k - l| \ge \varepsilon\} = 0$ [4]. In this case, we write $st - \lim x = l$.

A matrix $A \in (c,c)_{\text{reg}}$ is said to be regular and it is known [1, page 21] that A is regular if and only if $\|A\| < \infty$, $\lim_n a_{nk} = 0$ for each k, and $\lim_n \sum_k a_{nk} = 1$. For a given nonnegative regular matrix A, the number

$$\delta_A(E) = \lim_n \sum_{k \in E} a_{nk} \tag{1.5}$$

is said to be the A-density of $E \subseteq \mathbb{N}$ [5]. A sequence $x = (x_k)$ is said to be A-statistical convergent to a number s if for every $\varepsilon > 0$, the set $\{k : |x_k - s| \ge \varepsilon\}$ has A-density zero [5]. In this case, we write $st_A - \lim x = s$. By st_A, we denote the set of all A-statistically convergent sequences.

Let $\mathcal{B} = (\mathcal{B}_i) = (b_{nk}(i))$ be a sequence of infinite matrices. Then, a bounded sequence x is said to be $\mathcal{B}$ summable to the value l if

$$\lim_n \mathcal{B}x = \lim_n \sum_k b_{nk}(i) x_k = l \quad \text{uniformly in } i. \tag{1.6}$$

The matrix $\mathcal{B}$ is regular [11] if and only if $\|\mathcal{B}\| < \infty$, $\lim_n b_{nk}(i) = 0$ for all k, uniformly in i, and $\lim_n \sum_k b_{nk}(i) = 1$ uniformly in i, where $\|\mathcal{B}\| = \sup_{n,i} \sum_k |b_{nk}(i)|$. For a given nonnegative regular matrix sequence $\mathcal{B}$, Kolk [6] introduced the $\mathcal{B}$-density of a subset of $\mathbb{N}$ as follows.

The number

$$\delta_{\mathcal{B}}(E) = \lim_{n} \sum_{k\in E} b_{nk}(i) = d \quad \text{uniformly in } i \tag{1.7}$$

is said to be $\mathcal{B}$-density of E if it exists. In the cases $\mathcal{B} = (A)$ and $\mathcal{B} = (C,1)$, the Cesàro matrix, the $\mathcal{B}$-density reduces to the A-density and natural density, respectively. A sequence $x = (x_k)$ is said to be $\mathcal{B}$-statistically convergent [6] to a number s if for every $\varepsilon > 0$, the set $\{k : |x_k - s| \geq \varepsilon\}$ has $\mathcal{B}$-density zero. The set of all $\mathcal{B}$-statistically convergent sequences is denoted by $st_{\mathcal{B}}$.

Let $X \neq \emptyset$. A class $S \subset 2^X$ of subsets of X is said to be an ideal in X if S satisfies the conditions (i) $\emptyset \in S$, (ii) $Y \cup Z \in S$ whenever $Y, Z \in S$, (iii) $Y \in S$ and $Z \subseteq Y$ implies that $Z \in S$. An ideal is called nontrivial if $X \notin S$. A nontrivial ideal is called admissible if $\{x\} \in S$ for each $x \in X$ [7].

Let $\mathcal{I}$ be a nontrivial ideal in $\mathbb{N}$. A sequence $x = (x_k)$ is said to be $\mathcal{I}$-convergent to a number l if for every $\varepsilon > 0$, $\{k : |x_k - l| > \varepsilon\} \in \mathcal{I}$ [7]. In this case, we write $\mathcal{I} - \lim x = l$. It is clear that a $\mathcal{I}$-convergent sequence need not be bounded. Let $F_{\mathcal{I}}(b)$ be the set of all $\mathcal{I}$-convergent and bounded sequences.

Note that in the cases $\mathcal{I}_\delta = \{E \subseteq \mathbb{N} : \delta(E) = 0\}$, $\mathcal{I}_{\delta_A} = \{E \subseteq \mathbb{N} : \delta_A(E) = 0\}$, and $\mathcal{I}_{\delta_{\mathcal{B}}} = \{E \subseteq \mathbb{N} : \delta_{\mathcal{B}}(E) = 0\}$, the $\mathcal{I}$-convergence is reduced to the statistically convergence, A-statistically convergence, and $\mathcal{B}$-statistically convergence, respectively.

An admissible ideal $\mathcal{I}$ in $\mathbb{N}$ is said to satisfy the additive property if for every countable system $\{Y_1, Y_2, \dots\}$ of mutually disjoint sets in $\mathcal{I}$, there exist sets $Z_j \subseteq \mathbb{N}$ $(j = 1, 2, \dots)$ such that the symmetric differences $Y_j \Delta Z_j$ $(j = 1, 2, \dots)$ are finite and $\bigcup_j Z_j \in \mathcal{I}$ [7].

Demirci [3] has introduced the concepts $\mathcal{I}$-limit superior and inferior. For a real number sequence x, let B_x and A_x denote the sets $\{b \in \mathbb{R} : \{k : x_k > b\} \notin \mathcal{I}\}$ and $\{a \in \mathbb{R} : \{k : x_k < a\} \notin \mathcal{I}\}$, respectively, and also let $\mathcal{I}$ be admissible. Then,

$$\begin{aligned} \mathcal{I} - \limsup x &= \begin{cases} \sup B_x & \text{if } B_x \neq \emptyset, \\ -\infty & \text{if } B_x = \emptyset, \end{cases} \\ \mathcal{I} - \liminf x &= \begin{cases} \inf A_x & \text{if } A_x \neq \emptyset, \\ \infty & \text{if } A_x = \emptyset. \end{cases} \end{aligned} \tag{1.8}$$

It is shown [3] that $\mathcal{I} - \limsup x = \beta$ if and only if for every $\varepsilon > 0$, $\{k : x_k < \beta - \varepsilon\} \notin \mathcal{I}$ and $\{k : x_k > \beta + \varepsilon\} \in \mathcal{I}$. Also, $\mathcal{I} - \liminf x = \alpha$ if and only if for every $\varepsilon > 0$, $\{k : x_k < \alpha + \varepsilon\} \notin \mathcal{I}$ and $\{k : x_k < \alpha - \varepsilon\} \in \mathcal{I}$. Recall that a sequence $x = (x_k)$ is said to be $\mathcal{I}$-bounded if there exists an $N > 0$ such that $\{k : |x_k| > N\} \in \mathcal{I}$. It is proved in [3] that a $\mathcal{I}$-bounded sequence x is $\mathcal{I}$-convergent if and only if $\mathcal{I} - \limsup x = \mathcal{I} - \liminf x$.

For all $x \in \ell_\infty$, the inequality

$$\mathcal{I} - \limsup A(x) \leq \mathcal{I} - \limsup(x) \tag{1.9}$$

has been studied in [3].

In this paper, we have characterized a class of matrices $A \in (c, F_{\mathcal{I}}(b))$ by studying some new inequalities related to the $\mathcal{I}$-limit superior and limit inferior.

2. The main results

Firstly, we will begin with the following lemma.

Lemma 2.1. *$A \in (c, F_{\mathcal{I}}(b))$ if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty, \tag{2.1}$$

$$\mathcal{I} - \lim_n a_{nk} = t_k \quad \text{for every } k, \tag{2.2}$$

$$\mathcal{I} - \lim_n \sum_k a_{nk} = t. \tag{2.3}$$

Proof. Assume that $A \in (c, F_{\mathcal{I}}(b))$. Then, (2.1) follows from the fact that $(c, F_{\mathcal{I}}(b)) \subset (\ell_\infty, \ell_\infty)$. For the necessity of the other conditions it is enough to consider the sequences (e_k) and e, respectively, where (e_k) is the sequence whose kth place is 1 and the others are all zero.

Conversely, suppose that the conditions (2.1)–(2.3) hold. Let $x \in c$ and $\lim x = l$. Then, for any given $\varepsilon > 0$, there exists a $k_0 \in \mathbb{N}$ such that $|x_k - l| \leq \varepsilon$ whenever $k \geq k_0$. Now, we can write

$$Ax = \sum_k a_{nk}(x_k - l) + l \sum_k a_{nk}. \tag{2.4}$$

By an easy calculation, one can see that

$$\mathcal{I} - \lim_n \sum_k a_{nk}(x_k - l) = \sum_k t_k(x_k - l). \tag{2.5}$$

So, by applying $\mathcal{I} - \lim_n$ in (2.4), we get that

$$\mathcal{I} - \lim_n Ax = lt + \sum_k t_k(x_k - l). \tag{2.6}$$

This completes the proof. □

In what follows, a matrix $A \in (c, F_{\mathcal{I}}(b))$ is said to be $\mathcal{I}$-conservative. In the case A is $\mathcal{I}$-conservative, the number

$$K_{\mathcal{I}} = K_{\mathcal{I}}(A) = t - \sum_k t_k \tag{2.7}$$

is said to be $\mathcal{I}$-characteristic of A.

To the proof of our main results, we need two lemmas which can be proved by the same technique used in [2, Lemmas 2.3-2.4], respectively.

Lemma 2.2. *Let A be $\mathcal{I}$-conservative and $\lambda > 0$. Then,*

$$\mathcal{I} - \limsup_n \sum_k |a_{nk} - t_k| \leq \lambda \tag{2.8}$$

if and only if

$$\mathcal{I}-\limsup_n \sum_k (a_{nk}-t_k)^+ \le \frac{\lambda+K_{\mathcal{I}}}{2},$$
$$\mathcal{I}-\limsup_n \sum_k (a_{nk}-t_k)^- \le \frac{\lambda-K_{\mathcal{I}}}{2}. \tag{2.9}$$

LEMMA 2.3. *Let* $\|A\|<\infty$ *and* $\mathcal{I}-\lim_n |a_{nk}|=0$. *Then there exists a* $y\in\ell_\infty$ *such that* $\|y\|\le 1$ *and*

$$\mathcal{I}-\limsup \sum_k a_{nk}y_k = \mathcal{I}-\limsup \sum_k |a_{nk}|. \tag{2.10}$$

THEOREM 2.4. *Let A be $\mathcal{I}$-conservative. Then, for some constant* $\lambda\ge |K_{\mathcal{I}}|$ *and for all* $x\in\ell_\infty$,

$$\mathcal{I}-\limsup_n \sum_k (a_{nk}-t_k)x_k \le \frac{\lambda+K_{\mathcal{I}}}{2}L(x)-\frac{\lambda-K_{\mathcal{I}}}{2}l(x) \tag{2.11}$$

if and only if

$$\mathcal{I}-\limsup_n \sum_k |a_{nk}-t_k| \le \lambda. \tag{2.12}$$

Proof. Let (2.11) hold. Define $B=(b_{nk})$ by $b_{nk}=(a_{nk}-t_k)$ for all n, k. Then, since A is $\mathcal{I}$-conservative, the matrix B satisfies the hypothesis of Lemma 2.3. Hence, we have from (2.11) for a $y\in\ell_\infty$ with $\|y\|\le 1$ that

$$\begin{aligned}\mathcal{I}-\limsup_n \sum_k |b_{nk}| &= \mathcal{I}-\limsup_n \sum_k b_{nk}y_k\\ &\le \frac{\lambda+K_{\mathcal{I}}}{2}L(y)-\frac{\lambda-K_{\mathcal{I}}}{2}l(y)\\ &\le \left(\frac{\lambda+K_{\mathcal{I}}}{2}+\frac{\lambda-K_{\mathcal{I}}}{2}\right)\|y\|=\lambda,\end{aligned} \tag{2.13}$$

which yields (2.12).

Conversely, let (2.12) hold and $x\in\ell_\infty$. Then, for any $\varepsilon>0$, there exits a $k_0\in\mathbb{N}$ such that $l(x)-\varepsilon<x_k<L(x)+\varepsilon$ whenever $k>k_0$. Now, we can write

$$\sum_k (a_{nk}-t_k)x_k = \sum_{k\le k_0}(a_{nk}-t_k)x_k+\sum_{k>k_0}(a_{nk}-t_k)^+x_k-\sum_{k>k_0}(a_{nk}-t_k)^-x_k. \tag{2.14}$$

Since A is $\mathcal{I}$-conservative and by Lemma 2.2, we obtain

$$\begin{aligned}\mathcal{I}-\limsup_n \sum_k (a_{nk}-t_k)x_k &\le (L(x)+\varepsilon)\left(\frac{\lambda+K_{\mathcal{I}}}{2}\right)-(l(x)-\varepsilon)\left(\frac{\lambda-K_{\mathcal{I}}}{2}\right)\\ &= \frac{\lambda+K_{\mathcal{I}}}{2}L(x)-\frac{\lambda-K_{\mathcal{I}}}{2}l(x)+\lambda\varepsilon,\end{aligned} \tag{2.15}$$

which yields (2.11), since ε is arbitrary. □

When $K_{\mathcal{I}} > 0$ and $\lambda = K_{\mathcal{I}}$, we can conclude from Theorem 2.4 the following result.

THEOREM 2.5. *Let A be $\mathcal{I}$-conservative. Then, for all $x \in \ell_\infty$,*

$$\mathcal{I} - \limsup_n \sum_k (a_{nk} - t_k)x_k \le K_{\mathcal{I}} L(x) \tag{2.16}$$

if and only if

$$\mathcal{I} - \lim_n \sum_k |a_{nk} - t_k| \le K_{\mathcal{I}}. \tag{2.17}$$

In the cases $\mathcal{I} = \mathcal{I}_{\delta_{\mathcal{B}}}$ and $\mathcal{I} = \mathcal{I}_{\delta_A}$, we respectively have the following results from Theorem 2.4.

THEOREM 2.6. (a) *Let $A \in (c, st_{\mathcal{B}} \cap \ell_\infty)$. Then, for some constant $\lambda \ge |K_{\mathcal{B}}|$ and for all $x \in \ell_\infty$,*

$$st_{\mathcal{B}} - \limsup_n \sum_k (a_{nk} - t_k)x_k \le \frac{\lambda + K_{\mathcal{B}}}{2} L(x) - \frac{\lambda - K_{\mathcal{B}}}{2} l(x) \tag{2.18}$$

if and only if

$$st_{\mathcal{B}} - \limsup_n \sum_k |a_{nk} - t_k| \le \lambda. \tag{2.19}$$

(b) *Let $A \in (c, st_A \cap \ell_\infty)$. Then, for some constant $\lambda \ge |K_A|$ and for all $x \in \ell_\infty$,*

$$st_A - \limsup_n \sum_k (a_{nk} - t_k)x_k \le \frac{\lambda + K_A}{2} L(x) - \frac{\lambda - K_A}{2} l(x) \tag{2.20}$$

if and only if

$$st_A - \limsup_n \sum_k |a_{nk} - t_k| \le \lambda. \tag{2.21}$$

Also, if $\mathcal{I} = \mathcal{I}_\delta$, Theorem 2.4 appears as in [2, Theorem 2.5].

THEOREM 2.7. *Let A and λ be as in Theorem 2.4. Then, for all $x \in \ell_\infty$,*

$$\mathcal{I} - \limsup_n \sum_k (a_{nk} - t_k)x_k \le \frac{\lambda + K_{\mathcal{I}}}{2} V(x) + \frac{\lambda - K_{\mathcal{I}}}{2} V(-x) \tag{2.22}$$

if and only if (2.12) holds and

$$\mathcal{I} - \lim_n \sum_k |a_{nk} - a_{n,\sigma(k)} - t_k + t_{\sigma(k)}| = 0. \tag{2.23}$$

Proof. Let (2.22) hold. Then, since $V(x) \le L(x)$ and $V(-x) \le -l(x)$ for all $x \in \ell_\infty$, (2.12) follows from Theorem 2.4.

Define a matrix $C = (c_{nk})$ by $c_{nk} = (b_{nk} - b_{n,\sigma(k)})$ for all n, k, where b_{nk} is defined as in Theorem 2.4. Then, we have the hypothesis of Lemma 2.3. Now, choose the sequence y such that $y_k = 0$ for $k \notin \sigma(\mathbb{N})$. Then, $(y_k - y_{\sigma(k)}) \in Z$ and also, by the same argument used in [10, Theorem 23], one can easily see that

$$\sum_k b_{nk}\big(y_k - y_{\sigma(k)}\big) = \sum_k c_{nk}\, y_{\sigma(k)}. \tag{2.24}$$

Hence, (2.22) implies that

$$\begin{aligned} \mathcal{I} - \limsup_n \sum_k |c_{nk}| &= \mathcal{I} - \limsup_n \sum_k c_{nk}\, y_{\sigma(k)} \\ &= \mathcal{I} - \limsup_n \sum_k b_{nk}\big(y_k - y_{\sigma(k)}\big) \\ &\le \frac{\lambda + K_{\mathcal{I}}}{2} V\big(y_k - y_{\sigma(k)}\big) + \frac{\lambda - K_{\mathcal{I}}}{2} V\big(y_{\sigma(k)} - y_k\big) = 0. \end{aligned} \tag{2.25}$$

This yields (2.23).

Conversely, suppose that (2.12) and (2.23) hold. Then, for any $x \in \ell_\infty$, we have (2.24). Hence, since $(x_k - x_{\sigma(k)}) \in Z$, (2.23) implies that $B \in (Z, F_{\mathcal{I}}(b))$ with $\mathcal{I} - \lim Bz = 0$, $(z \in Z)$. We also see from the assumption that (2.11) holds. Thus, by taking infimum over $z \in Z$ in (2.11), we observe that

$$\begin{aligned} \inf_{z \in Z}\left(\mathcal{I} - \limsup_n \sum_k b_{nk}(x_k + z_k)\right) &\le \frac{\lambda + K_{\mathcal{I}}}{2} L(x+z) - \frac{\lambda - K_{\mathcal{I}}}{2} l(x+z) \\ &= \frac{\lambda + K_{\mathcal{I}}}{2} W(x) + \frac{\lambda - K_{\mathcal{I}}}{2} W(-x). \end{aligned} \tag{2.26}$$

On the other hand, since $\mathcal{I} - \lim Bz = 0$,

$$\begin{aligned} \inf_{z \in Z}\left(\mathcal{I} - \limsup_n \sum_k b_{nk}(x_k + z_k)\right) &\ge \mathcal{I} - \limsup_n \sum_k b_{nk}x_k + \inf_{z \in Z}\left(\mathcal{I} - \limsup_n \sum_k b_{nk}z_k\right) \\ &= \mathcal{I} - \limsup_n \sum_k b_{nk}x_k. \end{aligned} \tag{2.27}$$

Since $W(x) = V(x)$ for all $x \in \ell_\infty$, we conclude that (2.22) holds and the proof is completed. □

When $K_{\mathcal{I}} > 0$ and $\lambda = K_{\mathcal{I}}$, we have the following result.

Theorem 2.8. *Let A be $\mathcal{I}$-conservative. Then, for all $x \in \ell_\infty$,*

$$\mathcal{I} - \limsup_n \sum_k (a_{nk} - t_k)x_k \le K_{\mathcal{I}} V(x) \tag{2.28}$$

if and only if (2.17) and (2.23) hold.

The following results can be derived from Theorem 2.7 for the special cases $\mathcal{I} = \mathcal{I}_{\delta_{\mathcal{B}}}$ and $\mathcal{I} = \mathcal{I}_{\delta_A}$.

Theorem 2.9. (a) *Let $A \in (c, st_{\mathcal{B}} \cap \ell_\infty)$. Then, for some constant $\lambda \geq |K_{\mathcal{B}}|$ and for all $x \in \ell_\infty$,*

$$st_{\mathcal{B}} - \limsup_n \sum_k (a_{nk} - t_k) x_k \leq \frac{\lambda + K_{\mathcal{B}}}{2} V(x) + \frac{\lambda - K_{\mathcal{B}}}{2} V(-x) \tag{2.29}$$

if and only if (2.19) holds and

$$st_{\mathcal{B}} - \lim_n \sum_k \left| a_{nk} - a_{n,\sigma(k)} - t_k + t_{\sigma(k)} \right| = 0. \tag{2.30}$$

(b) *Let $A \in (c, st_A \cap \ell_\infty)$. Then, for some constant $\lambda \geq |K_A|$ and for all $x \in \ell_\infty$,*

$$st_A - \limsup_n \sum_k (a_{nk} - t_k) x_k \leq \frac{\lambda + K_A}{2} V(x) + \frac{\lambda - K_A}{2} V(-x) \tag{2.31}$$

if and only if (2.21) holds and

$$st_A - \lim_n \sum_k \left| a_{nk} - a_{n,\sigma(k)} - t_k + t_{\sigma(k)} \right| = 0. \tag{2.32}$$

Further, for $\mathcal{I} = \mathcal{I}_\delta$, Theorem 2.7 is reduced to [2, Theorem 2.7].

Theorem 2.10. *Let A and λ be as in Theorem 2.4. Then, for all $x \in \ell_\infty$,*

$$\mathcal{I} - \limsup_n \sum_k (a_{nk} - t_k) x_k \leq \frac{\lambda + K_{\mathcal{I}}}{2} \gamma(x) + \frac{\lambda - K_{\mathcal{I}}}{2} \gamma(-x) \tag{2.33}$$

if and only if (2.12) holds and

$$\mathcal{I} - \lim_n \sum_{k \in E} \left| a_{nk} - t_k \right| = 0 \tag{2.34}$$

for every $E \in \mathcal{I}$, where $\gamma(x) = \mathcal{I} - \limsup_k x_k$.

Proof. If (2.33) holds, since $\gamma(x) \leq L(x)$ and $\gamma(-x) \leq -l(x)$, (2.12) follows from Theorem 2.4. To show the necessity of (2.34), for any $E \in \mathcal{I}$, let us define a matrix $D = (d_{nk})$ by $d_{nk} = a_{nk} - t_k$, $k \in E$; otherwise, it equals zero for all n. Then, clearly, D satisfies the conditions of Lemma 2.2, and therefore there exists a $y \in \ell_\infty$ such that $\|y\| \leq 1$ and

$$\mathcal{I} - \limsup_n \sum_k d_{nk} y_k = \mathcal{I} - \limsup_n \sum_k \left| d_{nk} \right|. \tag{2.35}$$

Now, for the same E, we choose the sequence y as

$$y_k = \begin{cases} 1, & k \in E, \\ 0, & k \notin E. \end{cases} \tag{2.36}$$

Then, since $\mathcal{I}-\lim y=\gamma(y)=\gamma(-y)=0$, (2.33) implies that

$$\mathcal{I}-\limsup_n \sum_{k\in E}|d_{nk}| \leq \frac{\lambda+K_{\mathcal{I}}}{2}\gamma(y)+\frac{\lambda-K_{\mathcal{I}}}{2}\gamma(-y)=0, \tag{2.37}$$

which yields (2.34).

Conversely, suppose that the conditions of the theorem hold and $x\in\ell_\infty$. Let $E_1=\{k: x_k>\gamma(x)+\varepsilon\}$ and $E_2=\{k: x_k<\gamma(x)-\varepsilon\}$. Then, since $E_1,E_2\in\mathcal{I}$, $E=E_1\cap E_2\in\mathcal{I}$. Now, we can write

$$\sum_k (a_{nk}-t_k)x_k=\sum_{k\in E}(a_{nk}-t_k)x_k+\sum_{k\notin E}(a_{nk}-t_k)^+x_k-\sum_{k\notin E}(a_{nk}-t_k)^-x_k. \tag{2.38}$$

Thus, by (2.34) and Lemma 2.2, (2.33) is obtained since

$$\mathcal{I}-\limsup_n \sum_k (a_{nk}-t_k)x_k \leq \frac{\lambda+K_{\mathcal{I}}}{2}\gamma(x)+\frac{\lambda-K_{\mathcal{I}}}{2}\gamma(-x)+\lambda\varepsilon \tag{2.39}$$

and ε is arbitrary. □

When $K_{\mathcal{I}}>0$ and $\lambda=K_{\mathcal{I}}$, we have the following result.

Theorem 2.11. *Let A be $\mathcal{I}$-conservative. Then, for all $x\in\ell_\infty$,*

$$\mathcal{I}-\limsup_n \sum_k (a_{nk}-t_k)x_k \leq K_{\mathcal{I}}\gamma(x) \tag{2.40}$$

if and only if (2.17) and (2.34) hold.

We can choose $\mathcal{I}=\mathcal{I}_{\delta_{\mathcal{B}}}$ and $\mathcal{I}=\mathcal{I}_{\delta_A}$ in Theorem 2.10 to obtain the following results.

Theorem 2.12. (a) *Let $A\in(c,st_{\mathcal{B}}\cap\ell_\infty)$. Then, for some constant $\lambda\geq|K_{\mathcal{B}}|$ and for all $x\in\ell_\infty$,*

$$st_{\mathcal{B}}-\limsup_n \sum_k (a_{nk}-t_k)x_k \leq \frac{\lambda+K_{\mathcal{B}}}{2}\gamma(x)+\frac{\lambda-K_{\mathcal{B}}}{2}\gamma(-x) \tag{2.41}$$

if and only if (2.19) holds and

$$st_{\mathcal{B}}-\lim_n \sum_{k\in E}|a_{nk}-t_k|=0, \tag{2.42}$$

for every $E\in\mathcal{I}$.

(b) *Let $A\in(c,st_A\cap\ell_\infty)$. Then, for some constant $\lambda\geq|K_A|$ and for all $x\in\ell_\infty$,*

$$st_A-\limsup_n \sum_k (a_{nk}-t_k)x_k \leq \frac{\lambda+K_A}{2}\gamma(x)+\frac{\lambda-K_A}{2}\gamma(-x) \tag{2.43}$$

if and only if (2.21) holds and

$$st_A - \lim_n \sum_{k\in E} |a_{nk} - t_k| = 0, \tag{2.44}$$

for every $E \in \mathcal{I}$.

Moreover, Theorem 2.10 is a dual case of [2, Theorem 2.6] for $\mathcal{I} = \mathcal{I}_\delta$.

Acknowledgment

We wish to thank the referees for valuable suggestions and comments which improved the paper considerably.

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