CHARACTERIZATION OF SEVERAL CLASSES OF SECOND-ORDER ODEs BY USING DIFFERENTIAL INVARIANTS

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Several classes of second-order ordinary differential equations are characterized intrinsically by means of differential invariants. The method is proved to be computationally feasible.

1. Introduction

The classical theory of differential invariants for second-order ODEs (e.g., see [24, 25, 36, 37, 39, 52, 53]) has experienced a notable development in the last decade. It is due to the interest of its geometric applications (see [2, 14, 17, 18, 20, 21, 22, 23, 26, 27, 28, 34, 35, 42, 45, 49]) and, very specially, due to the new computational aspects of that theory in applying it to a wide class problems, such as symmetries, conservation laws, order reduction; for example, see [3, 4, 5, 6, 9, 10, 15, 16, 29, 30, 31, 32, 38, 43, 44, 48, 50]. The goal of this paper is to characterize several classes of second-order ODEs by using differential invariants with respect to the subgroups of horizontal and vertical transformations of the plane, which are defined and studied in Sections 2.1, 2.2, and 2.3 below. Basically, the method supplies a criterion to know whether a given second-order ordinary differential equation can be put, after a change of variables belonging to one of the two groups under consideration, into a specific normal form. In each case, the criterion reduces to check the vanishing of several algebraic expressions written in terms of differential invariants and, hence, we can recognize whether a concrete equation belongs to a specific normal form by simply running an algorithm in polynomial time. The problem of characterization of ordinary differential equations has been tackled by some authors in the last decades (see, e.g., [5, 17, 18, 20, 22, 23, 26, 27, 34, 35]). All these works are based on the Cartan's equivalence method, as well as Tresse's original papers, and their solutions rely also on differential invariants, which derive from their own theory. Nevertheless, in this paper we propose a rather different approximation to this problem: we calculate explicitly the basis of differential invariants (of each order), and starting from them we obtain necessary and sufficient conditions for the reduction. The proposed method is applied to five classes of second-order ODEs: autonomous differential equations, equations of the second homogeneous type (as defined in [33]), special equations, Painlevé

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transcendents, and linear equations. The first four classes are studied with respect to the group of horizontal diffeomorphisms, whereas the fifth one is studied with respect to the group of vertical diffeomorphisms. The horizontal group can be understood as the group of changes of the independent x-variable, and the vertical group as the changes of the dependent y-variable. Essentially, we have chosen such groups and normal forms in order to illustrate the method of invariants since they are very well-known instances, but the procedure works successfully in many other cases as well. The main results of this paper are Propositions 3.2, 3.5, 3.7, 3.8, and 3.9, which correspond to the horizontal group and Theorems 4.2, 4.4, which correspond to the vertical group. The results are explicit as, in each case, the method provides not only a criterion for recognizing the type of a given equation intrinsically, but also the change of variables reducing the equation to normal form. The results in Propositions 3.2, 3.5, 3.7, 3.8, and 3.9 are new, to our knowledge. The characterization of linearizable second-order equations has already been reached by using different approaches; for example, see [22, 32, 34, 44]. Theorems 4.2 and 4.4 below are a reformulation of linearization criteria in terms of differential invariants. Although these theorems are equivalent to the criteria obtained in the aforementioned works, the approach that we present here seems to be different. The algorithm to calculate the differential invariants introduced in [38] allows us to interpret the conditions for reduction in terms of the function that defines the ordinary differential equation. Nevertheless, sometimes, when the function that defines a given ODE is rather long or complicated, these explicit conditions for the reduction cannot be checked "by hand." Fortunately, such explicit conditions can be implemented easily in any programmable computer algebra system (CAS) such as Mathematica, Maple, MACSYMA, REDUCE, AXIOM, MuPAD, and so forth. On the other hand, in the literature of the topic, there are several packages using the symmetry groups of differential equations in order to carry out the reduction process; for example, see [7, 8, 9, 10, 11, 12, 15, 16, 31, 50].

2. Theoretical background

2.1. The notion of an invariant. In order to be able to use the notion of a differential invariant as a function on a jet bundle that is invariant under the induced action of a certain group of transformations (see [1, 14, 36, 37, 44]), a second-order ODE is defined to be a section σ of $p_{21} : J^2(p) \to J^1(p)$, where $p : \mathbb{R}^2 \to \mathbb{R}$ is the projection $p(x,y) = x$. If (x, y, y', y'') is the natural coordinate system on $J^2(p)$, then σ is equivalent to giving the function $\sigma^* y'' = F(x, y, y') \in C^\infty(J^1(p))$. Let $\operatorname{Aut} p$ be the group of automorphisms of p, that is,

$$\operatorname{Aut} p = \{\Phi \in \operatorname{Diff} \mathbb{R}^2 : p \circ \Phi = \phi \circ p,\ \phi \in \operatorname{Diff} \mathbb{R}\}. \tag{2.1}$$

This group acts on the space of sections of p_{21} in a natural way; precisely $\sigma \mapsto \Phi^{(2)} \circ \sigma \circ (\Phi^{(1)})^{-1}$, where $\Phi^{(k)}$ denotes the k-jet prolongation of Φ to $J^k(p)$; that is,

$$\Phi^{(k)}(j_x^k s) = j_{\phi(x)}^k (\Phi \circ s \circ \phi^{-1}), \tag{2.2}$$

for every section $s : \mathbb{R} \to \mathbb{R}^2$, $s(x) = (x, f(x))$, of p. The notion of invariance that we consider is the one relative to the prolongation of this group action to $J^r(p_{21})$. A function

$I : J^r(p_{21}) \to \mathbb{R}$ is said to be a relative differential invariant with respect to a subgroup $\mathcal{G} \subseteq \operatorname{Aut} p$ (or a relative $\mathcal{G}$-invariant) if for all $j^r_{z'}\sigma \in J^r(p_{21})$, $z' = j^1_x s \in J^1(p)$, and every $\Phi \in \mathcal{G}$, there exists an invertible function $\eta \in C^\infty(\mathbb{R}^2)^*$—called the weight of I—such that,

$$I\Big(\big(\Phi^{(2)}\big)^{(r)}\big(j^r_{z'}\sigma\big)\Big) = I\Big(j^r_{\Phi^{(1)}(z')}\Big(\Phi^{(2)} \circ \sigma \circ \big(\Phi^{(1)}\big)^{-1}\Big)\Big) = \eta I\big(j^r_{z'}\sigma\big). \tag{2.3}$$

If $\eta = 1$, then I is called an absolute invariant or simply a differential invariant. This notion of invariance is not very efficient in order to calculate explicitly the differential invariants. Consequently, we introduced the infinitesimal version of invariance which allows us to use algebraic and analytic tools that provide a more operative algorithm, which can be implemented easily by computer. Every differential invariant with respect to $\mathcal{G}$ is also an invariant with respect to the Lie algebra $\mathfrak{g}$ of p-projectable vector fields whose local flow belongs to $\mathcal{G}$; that is, $(X^{(2)})^{(r)}I = 0$ for all $X \in \mathfrak{g}$, as the flow of $(X^{(2)})^{(r)}$ is $(\Phi_t^{(2)})^{(r)}$, Φ_t being the flow of X. In fact, both notions are equivalent; for example, see [41]. Below, we deal with the cases

$$\begin{gathered}\mathcal{G} = \operatorname{Aut}^h p = \{\Phi \in \operatorname{Aut} p : p \circ \Phi = p\},\\ \mathcal{G} = \operatorname{Aut}^v p = \{\bar{\phi} \in \operatorname{Aut} p : \bar{\phi}(x,y) = (\phi(x), y),\ \phi \in \operatorname{Diff}\mathbb{R}\},\end{gathered} \tag{2.4}$$

which we refer to as the "horizontal" and "vertical" subgroups, respectively. Recall that each element of the horizontal subgroup stands for a change of coordinates of the independent variable, whereas each element of the vertical subgroup stands for a change of coordinates of the dependent variable.

2.2. The horizontal group. In [41], the number of functionally independent r-order differential invariants for the group $\operatorname{Aut}^h p$ is proved to be equal to $(1/6)r(r+1)(r+5)+1$ on the dense open subset defined by $y' \neq 0$. A basic question is how to obtain invariants of order $r+1$ starting from invariants of order $\leq r$. The standard procedure goes back to Lie's ideas (see [36, 37, 39]). If we denote by D_x, D_y, and $D_{y'}$ the total derivatives on jet bundles with respect to the variables x, y, and y', respectively (cf. [44]), then, for every $r \in \mathbb{N}$, the operators

$$Y_1 = D_y, \qquad Y_2 = y' D_{y'}, \qquad Y_3 = \frac{1}{y'}\left(D_x + y'' D_{y'}\right) \tag{2.5}$$

transform rth order differential invariants into $(r+1)$th order differential invariants for the horizontal subgroup. For the proof, we refer the reader to [41]. Then, the algebra of $\operatorname{Aut}^h p$-invariants are generated by algebraic operations and derivations with respect to the operators D_x, D_y, and $D_{y'}$ above. Precisely, a system of functionally independent generators of the ring of rth order is given by

$$\begin{gathered} y,\\ I_{abc} = \big(Y_1^a \circ Y_2^b \circ Y_3^c\big)(I), \quad 0 \leq a+b+c \leq r-1,\\ J_{\beta\gamma} = \big(Y_2^\beta \circ Y_3^\gamma\big)(J), \quad 0 \leq \beta+\gamma \leq r-1,\end{gathered} \tag{2.6}$$

where I and J are given below. One can efficiently implement an algorithm by using any computer algebra system (see [38] and the appendix of this paper) to obtain the invariants in the formula (2.6) recursively. As an example, for $r=3$, we have

$$
\begin{gathered}
y, \\
I = \frac{y''_{010}}{y'^2}, \\
I_{001} = \frac{y'\left(y''_{110} + y'' y''_{011}\right) - 2y'' y''_{010}}{y'^4}, \\
I_{010} = -\frac{2y''_{010} + y' y''_{011}}{y'^2}, \\
I_{100} = \frac{y''_{020}}{y'^2}, \\
I_{020} = \frac{4y''_{010} - 3y' y''_{011} + y'^2 y''_{012}}{y'^2}, \\
I_{200} = \frac{y''_{030}}{y'^2}, \\
I_{002} = \frac{8y''^2 y''_{010} - y'\left(2y'' y''_{010} y''_{001} - 2y''_{100} y''_{010} - 5y''^2 y''_{011} - 5y'' y''_{110}\right)}{y'^6} \\
+ \frac{y''_{210} + 2y'' y''_{111} + y'' y''_{001} y''_{011} + y''_{100} y''_{011} + y''^2 y''_{012}}{y'^4}, \\
I_{101} = \frac{y'\left(y''_{120} + y''_{010} y''_{011} + y'' y''_{021}\right) - 2\left(y''^2_{010} + y'' y''_{020}\right)}{y'^4}, \\
I_{011} = \frac{8y'' y''_{010} - y'\left(3y''_{110} + 2y''_{010} y''_{001} + 5y'' y''_{011}\right)}{y'^4} + \frac{y''_{111} + y''_{001} y''_{011} + y'' y''_{012}}{y'^2}, \\
J = \frac{y'' - y' y''_{001}}{y'^2}, \\
J_{10} = -\frac{2y'' + 2y' y''_{001} - y'^2 y''_{002}}{y'^2}, \\
J_{01} = -\frac{2y''^2 + y'\left(2y'' y''_{001} + y''_{100}\right) - y'^2\left(y'' y''_{002} + y''_{101}\right)}{y'^4}, \\
J_{20} = \frac{4y'' - 4y' y''_{001} + 2y'^2 y''_{002} - y'^3 y''_{003}}{y'^2}, \\
J_{11} = \frac{8y''^2 - y'\left(3y''_{100} - 10y'' y''_{001}\right) + y'^2\left(3y''_{101} + 2y''^2_{001} + 4y'' y''_{002}\right)}{y'^4} \\
- \frac{\left(y''_{102} + y'' y''_{003} + y''_{001} y''_{002}\right)}{y'},
\end{gathered}
$$

$$
\begin{aligned}
J_{02} = {} & \frac{8y''^3 - y'\left(10y''^2 y''_{001} + 7y'' y''_{001}\right)}{y'^6} + \frac{y''_{200} + 2y'' y''^2_{001} + 4y''^2 y''_{002} + 5y'' y''_{101} + 2y''_{100} y''_{001}}{y'^4} \\
& - \frac{2y'' y''_{102} + y''_{100} y''_{002} + y''^2 y''_{003} + y'' y''_{001} y''_{002} + y''_{201}}{y'^3}.
\end{aligned}
\tag{2.7}
$$

In these formulas, we denote by y''_{abc}, $0 \le a+b+c \le r$, the coordinates induced on $J^r(p_{21})$; that is,

$$
y''_{abc}\left(j^r_{z'}\sigma\right) = \frac{\partial^{a+b+c} F}{\partial x^a \partial y^b \partial y'^c}(z'),
\tag{2.8}
$$

where $\sigma : J^1(p) \to J^2(p)$ is the section associated with the differential equation $y'' = F(x, y, y')$.

2.3. The vertical group. For the group $\mathrm{Aut}^v p$, the number of functionally independent r-order invariants is given as follows (see [40]):

$$
\begin{gathered}
1, \quad \text{if } r = 0, 1, \\
2, \quad \text{if } r = 2, \\
\frac{1}{6} r(r-2)(r+5) + 1, \quad \text{if } r \ge 3.
\end{gathered}
\tag{2.9}
$$

As in the horizontal case, there exist three operators,

$$
Z_1 = \frac{D_{y'}}{\sqrt{y''_{003}}}, \qquad Z_2 = D_x + y' D_y + y'' D_{y'}, \qquad Z_3 = \frac{2D_y + y''_{001} D_{y'}}{\sqrt{y''_{003}}},
\tag{2.10}
$$

and two functions, K and V, which are vertical invariants of order 2 and 4, respectively, such that a system of functionally independent generators of the ring of rth order differential invariants is given by

$$
\begin{gathered}
x, \\
K_{abc} = \left(Z_1^a \circ Z_2^b \circ Z_3^c\right)(K), \quad 0 \le a+b+c \le r-2, \\
K_{\gamma\beta} = \left(Z_3^\gamma \circ Z_2^\beta \circ Z_1\right)(K), \quad 1 \le \beta + \gamma \le r-3,\ \gamma \ne 0, \\
Z_1^\alpha(V), \quad 0 \le \alpha \le r-4,
\end{gathered}
\tag{2.11}
$$

where $V = y''_{004} / y''^{3/2}_{003}$, and K is defined below. Again, by using the algorithm explained in the appendix, we obtain the following basis for the third order:

$$
\begin{aligned}
&x,\\
K &= \frac{y_{001}''^{2}}{2} - y'' y_{002}'' + 2y_{010}'' - y' y_{011}'' - y_{101}'',\\
K_{001} &= \frac{1}{\sqrt{y_{003}''}}\big(- y'' y_{001}'' y_{003}'' - 2y_{010}'' y_{002}'' + 3y_{001}'' y_{011}'' - 2y'' y_{012}''\\
&\qquad - y' y_{001}'' y_{012}'' + 4y_{020}'' - 2y' y_{021}'' - y_{001}'' y_{102}'' - 2y_{111}''\big),\\
K_{010} &= y' \big(y_{001}'' y_{011}'' + 2y_{020}'' - y_{010}'' y_{002}'' - 2y_{111}'' - 2y'' y_{012}''\big)\\
&\qquad - y'^{2} y_{021}'' - y_{100}'' y_{002}'' + y_{001}'' y_{101}'' + y'' y_{011}'' - 2y'' y_{102}'' + 2y_{110}'' - y_{201}'' - y''^{2} y_{003}'',\\
K_{100} &= \frac{1}{\sqrt{y_{003}''}}\big(- y'' y_{003}'' + y_{011}'' - y' y_{012}'' - y_{102}''\big).
\end{aligned}
\tag{2.12}
$$

Moreover, it is easily checked that $R_1 = y_{003}''$ is a relative $\mathrm{Aut}^v(p)$-invariant of weight ψ_y^{-2} (where $\bar{x} = x$, $\bar{y} = \psi(x,y)$ is the change of variables), so that the numerators of K_{001} and K_{100} are relative invariants as well:

$$
R_2 = \sqrt{y_{003}''}K_{100}, \qquad R_3 = \sqrt{y_{003}''}K_{001}. \tag{2.13}
$$

Notation 2.1. In what follows, we use the following notation: $f^\sigma = f \circ j^r\sigma$, where σ is a section of p_{21} and $f \in C^\infty(J^r(p_{21}))$. For example, if $y'' = F(x,y,y')$ is the differential equation associated with σ, then from (2.12) we obtain $K^\sigma = (1/2)F_{y'}^2 - FF_{y'y'} + 2F_y - y'F_{yy'} - F_{xy'}$.

3. Four classes of equations for $\mathrm{Aut}^h p$

As mentioned in Section 1, differential invariants are a powerful tool in order to classify ODEs. The aim of this section is to show how to apply this method in four concrete examples: autonomous differential equations, second homogeneous-type equations, special equations, and Painlevé transcendents. Moreover, the criteria obtained for characterization are easily implemented by any computer algebra system as it is shown in Section 5.

3.1. Autonomous, special, and second homogeneous-type second-order ODEs. Let us consider an arbitrary ordinary differential equation of second order,

$$
\sigma \equiv \big(y'' = F(x,y,y')\big), \tag{3.1}
$$

and a horizontal change of variables,

$$
\begin{aligned}
\bar{x} = \phi(x), &\qquad \phi' \neq 0,\\
\bar{y} = y.&
\end{aligned}
\tag{3.2}
$$

Then, taking the formulas

$$\frac{dy}{dx} = \phi' \frac{dy}{d\bar{x}}, \qquad \frac{d^2y}{dx^2} = \phi'' \frac{dy}{d\bar{x}} + \phi'^2 \frac{d^2y}{d\bar{x}^2} \tag{3.3}$$

into account, a straightforward argument yields.

Lemma 3.1. *Equation (3.1) can be reduced to an autonomous differential equation*

$$\frac{d^2y}{d\bar{x}^2} = f\left(y, \frac{dy}{d\bar{x}}\right) \tag{3.4}$$

under the change of variables (3.2) if and only if the following PDE holds:

$$0 = \frac{\phi''}{\phi'^3} y' F_{y'} + \frac{1}{\phi'^2} F_x - 2\frac{\phi''}{\phi'^3} F + \frac{1}{\phi'}\left(\frac{1}{\phi'}\right)'' y'. \tag{3.5}$$

This result allows us to obtain the criterion for the reduction of an arbitrary second-order ODE to a particular type in terms of differential invariants. Consequently, the following results hold.

Proposition 3.2. *Equation (3.1) can be reduced to an autonomous differential equation by means of (3.2) on the open subset $y' \neq 0$ if and only if the following conditions hold:*

(1) *if $I^{\sigma}_{010} \neq 0$, then*

$$0 = I^{\sigma}_{101} I^{\sigma}_{010} - I^{\sigma}_{110} I^{\sigma}_{001} - I^{\sigma} \left(I^{\sigma}_{010}\right)^2, \tag{3.6}$$

$$0 = I^{\sigma}_{010} I^{\sigma}_{011} + J^{\sigma} \left(I^{\sigma}_{010}\right)^2 - I^{\sigma}_{001} I^{\sigma}_{020} + I^{\sigma}_{010} I^{\sigma}_{001}, \tag{3.7}$$

$$0 = I^{\sigma}_{002} I^{\sigma}_{010} - I^{\sigma}_{001} I^{\sigma}_{011}. \tag{3.8}$$

In this case, the change of variables is given by

$$\begin{gathered} \bar{x} = \int \exp\left(-\int \frac{F_{xy}}{y' F_{yy'} - 2F_y} dx\right) dx + \alpha, \quad \alpha \in \mathbb{R}, \\ \bar{y} = y, \end{gathered} \tag{3.9}$$

(2) *if $I^{\sigma}_{010} = 0$, then $F(x, y, y') = \xi(y) y'^2 + \varphi(y')$, where $\varphi, \xi \in C^{\infty}(\mathbb{R})$.*

Proof. Set $G = y' F_{yy'} - 2F_y$. Then, using the algorithm implemented to calculate invariants (see [38]), we have $G = y'^2 I^{\sigma}_{010}$, and the formulas (3.6)–(3.8) are equivalent to the following:

$$0 = \left(\frac{F_{xy}}{G}\right)_y, \tag{3.10}$$

$$0 = \left(\frac{F_{xy}}{G}\right)_{y'}, \tag{3.11}$$

$$0 = y' \left(\frac{F_{xy}}{G}\right)_x + y' \left(\frac{F_{xy}}{G}\right)^2 + \left(\frac{F_{xy}}{G}\right)(2F - y' F_{y'}) + F_x. \tag{3.12}$$

In fact, it is readily checked that the following equation holds:

$$\text{r. h. s.}(3.6) = G^2 y'^{-5}\,\text{r. h. s.}(3.10), \qquad \text{r. h. s.}(3.7) = G^2 y'^{-4}\,\text{r. h. s.}(3.11), \tag{3.13}$$

where "r. h. s." means "right-hand side of." The formula (3.8) needs an explanation. We have

$$y'\left(\frac{F_{xy}}{G}\right)_x + y'\left(\frac{F_{xy}}{G}\right)^2 + \frac{F_{xy}}{G}(2F - y'F_{y'}) + F_x = y'^3\frac{\text{r. h. s.}(3.8)}{(I_{010}^{\sigma})^2} - y'F\frac{\text{r. h. s.}(3.7)}{(I_{010}^{\sigma})^2}. \tag{3.14}$$

If $\tau \equiv (y'' = f(y,y'))$ is an autonomous equation, as $f_x = 0$, we have

$$y'^3 (I_{010}^{\tau})^{-2}\,\text{r. h. s.}(3.8) = 0. \tag{3.15}$$

(1) As a simple calculation shows, the conditions (3.6)–(3.8) hold for the equation $d^2y/d\bar{x}^2 = f(y, dy/d\bar{x})$. Since these conditions are written in terms of invariants, they also hold for any differential equation obtained from the autonomous equation by a change of the form (3.2). Conversely, assume the conditions (3.6)–(3.8) hold true (and consequently (3.10)–(3.12) also hold true). Consequently, taking into account the conditions (3.10) and (3.11), it is easy to check that the function F_{xy}/G depends only on the variable x. Consequently, if we use the following change of variables:

$$\begin{gathered}\bar{x} = \phi(x) = \int \exp\left(-\int \frac{F_{xy}}{G}dx\right)dx + \alpha, \quad \alpha \in \mathbb{R}, \\ \bar{y} = y,\end{gathered} \tag{3.16}$$

(3.12) is written as follows:

$$0 = \left(\frac{2\phi''^2 - \phi'\phi'''}{\phi'^2}\right)y' - 2\frac{\phi''}{\phi'}F + \frac{\phi''}{\phi'}y'F_{y'} + F_x. \tag{3.17}$$

Now, multiplying by $1/\phi'^2$, we obtain

$$0 = \frac{1}{\phi'}\left(\frac{1}{\phi'}\right)'' y' - 2\frac{\phi''}{\phi'^3}F + \frac{\phi''}{\phi'^3}y'F_{y'} + \frac{1}{\phi'^2}F_x. \tag{3.18}$$

As a consequence, Lemma 3.1 holds true. (2) Next, assume $G = y'F_{yy'} - 2F_y = 0$. The general solution to this PDE is

$$F(x,y,y') = A(x,y)y'^2 + B(x,y'), \quad A, B \in C^{\infty}(\mathbb{R}^2). \tag{3.19}$$

Using formulas (3.3) and (3.19), a simple computation shows

$$A(x,y)y'^2 + B(x,y') = \frac{\phi''(x)}{\phi'(x)}y' + \phi'(x)^2 f\left(y, \frac{y'}{\phi'(x)}\right). \tag{3.20}$$

Substituting $\phi'(x)y'$ for y', we obtain

$$A(x,y)\phi'(x)^2y'^2 + B(x,\phi'(x)y') = \phi''(x)y' + \phi'(x)^2 f(y,y'). \tag{3.21}$$

Letting $y' = 0$, (3.21) yields $B(x,0) = g(x)^2 f(y,0)$, and we distinguish two subcases:

(2.1) If $b(x) = B(x,0) \neq 0$, then $f(y,0) = \lambda \in \mathbb{R}^*$; hence $\phi'(x) = \sqrt{b(x)/\lambda}$. Letting $y' = 0$ in (3.19), we have $F(x,y,0) = b(x)$. The change of variable is thus explicitly given by $\phi(x) = \sqrt{F(x,y,0)/\lambda}$. Substituting ϕ' into (3.21), we deduce

$$f(y,y') = A(x,y)y'^2 + \frac{1}{b(x)}\left(\lambda B\left(x, y'\sqrt{\frac{b(x)}{\lambda}}\right) - y'\frac{b'(x)}{2\sqrt{b(x)/\lambda}}\right). \tag{3.22}$$

Rescaling, we can assume $\lambda = 1$. Hence,

$$f(y,y') = A(x,y)y'^2 + \frac{B(x,y'\sqrt{b(x)})}{b(x)} - \frac{1}{2}y'b(x)^{-3/2}b'(x), \tag{3.23}$$

and consequently,

$$\left(A(x,y)y'^2 + \frac{B(x,y'\sqrt{b(x)})}{b(x)} - \frac{1}{2}y'b(x)^{-3/2}b'(x)\right)_x = 0. \tag{3.24}$$

Expanding the latter equation and substituting $y'/\sqrt{b(x)}$ for y', we obtain

$$\begin{aligned} 0 = A_x(x,y)b(x)y'^2 + b(x)B_x(x,y') + \frac{1}{2}b'(x)y'B_{y'}(x,y') \\ - b'(x)B(x,y') - \frac{1}{2}b''(x)y' + \frac{3}{4}b'(x)^2b(x)^{-1}y'. \end{aligned} \tag{3.25}$$

By taking derivatives with respect to y in (3.25), we conclude $A_{xy} = 0$. Accordingly, $A(x,y) = a_1(x) + a_2(y)$. Moreover, the quasi-linear differential equation (3.25) can be integrated elementarily, and its general solution is

$$\begin{aligned} B(x,y') = \frac{1}{2}\frac{b'(x)}{b(x)}y' + \frac{3}{2}y'\sqrt{b(x)}\int b'(x)^2b(x)^{-5/2}dx \\ + a_1(x)y'^2 + b(x)\varphi(y'b(x)^{-1/2}). \end{aligned} \tag{3.26}$$

Hence, $F(x,y,y') = (a_1(x) + a_2(y))y'^2 + B(x,y')$ and by imposing that this function satisfies (3.5) with $\phi'(x) = \sqrt{b(x)}$, we deduce that the functions a_1, b are constant thus concluding.

(2.2) If $b(x) = B(x,0) = 0$, then $f(y,0) = 0$; hence $f(y,y') = y'\bar{f}(y,y')$, $B(x,y') = y'\bar{B}(x,y')$. Substituting both expressions in (3.21), we have

$$\bar{f}(y,y') = \frac{1}{\phi'(x)^2}\left(A(x,y)\phi'(x)^2y' + \bar{B}(x,\phi'(x)y') - \phi''(x)\right). \tag{3.27}$$

Hence,

$$\left(\frac{A(x,y)\phi'(x)^2y' + \bar{B}(x,\phi'(x)y') - \phi''(x)}{\phi'(x)^2}\right)_x = 0. \tag{3.28}$$

Expanding the equation above and substituting $y'/\phi'(x)$ for y', it yields

$$\begin{aligned}0 &= A_x(x,y)\phi'(x)^2y' + \bar{B}_x(x,y')\phi'(x)^3 + \bar{B}_{y'}(x,y')\phi''(x)\phi'(x)y' \\ &\quad - \bar{B}(x,y')\phi''(x)\phi'(x)^2 - \phi'''(x)\phi'(x)^2 + 2\phi'(x)\phi''(x)^2.\end{aligned} \tag{3.29}$$

As above, we have $A(x,y) = a_1(x) + a_2(y)$ and letting $y' = 0$ in (3.29), we obtain

$$\left(\frac{\phi''(x)}{\phi'(x)}\right)' + \bar{B}(x,0)\frac{\phi''(x)}{\phi'(x)} - \left(\frac{\phi''(x)}{\phi'(x)}\right)^2 = \bar{B}_x(x,0), \tag{3.30}$$

which is a Riccati equation in $\phi''(x)/\phi'(x)$, whose solution is

$$\frac{\phi''(x)}{\phi'(x)} = \bar{b}(x) - \frac{\exp\left(\int \bar{b}(x)dx\right)}{\int \exp\left(\int \bar{b}(x)dx\right)dx}, \quad \bar{b}(x) = \bar{B}(x,0), \tag{3.31}$$

or equivalently, $\phi(x) = \log[\int \exp(\int \bar{b}(x)dx)dx]$. Substituting this expression into (3.29), we have

$$\begin{aligned}0 &= \phi'\bar{B}_x + y'(\bar{b} - \phi')\bar{B}_{y'} + (\phi'^2 - \phi'\bar{b})\bar{B} \\ &\quad + (\phi' - \phi'^2)\bar{b}^2 - (\phi' + \phi'^2)\bar{b}' + a_1'y'.\end{aligned} \tag{3.32}$$

As above, this quasi-linear differential equation can be integrated elementarily and its general solution is

$$\bar{B} = \phi'\int \frac{\bar{b}'(1+\phi') - \bar{b}^2(1-\phi')}{\phi'}dx - \frac{\phi'}{u}y'\int \frac{a_1'u}{\phi'}dx + \varphi(y'u^{-1}), \tag{3.33}$$

where $u = \exp(1/\phi')$. Hence, $F(x,y,y') = (a_1(x) + a_2(y))y'^2 + y'\bar{B}(x,y')$. By imposing that F satisfies the condition (3.5), we deduce that $\phi' = \bar{b} = 1$, and a_1 is a constant, thus finishing the proof. □

Remark 3.3. The ODE $y'' = f(y,y')$ admits the point symmetry $X = d/dx$, which is mapped by the transformation (3.2) into $X = \phi'(\phi^{-1}(x))(d/dx)$. Thus, the problem of reducing (3.1) to an autonomous equation is equivalent to finding a point symmetry of the form $X = \xi(x)(d/dx)$. Such problem is easily solvable using the classical algorithm for finding point symmetries (see, e.g., references [6, 43]) and it yields the same solvability conditions (3.10)–(3.12) above.

Remark 3.4. The conditions in Proposition 3.2 also allow one to reduce (3.1) to an equation of second homogeneous type (cf. [33]),

$$\frac{d^2y}{d\bar{x}^2} = \bar{x}^{-2}f\left(y,\bar{x}\frac{dy}{d\bar{x}}\right), \tag{3.34}$$

as the change of variable $\tilde{x} = \phi(\bar{x}) = \ln(\bar{x})$ transforms (3.34) into the autonomous equation:

$$\frac{d^2 y}{d\tilde{x}^2} = f\left(y, \frac{dy}{d\tilde{x}}\right) + \frac{dy}{d\tilde{x}}. \tag{3.35}$$

Proposition 3.5. *The ODE (3.1) can be reduced to*

$$\frac{d^2 y}{d\bar{x}^2} = f(\bar{x}, y) \tag{3.36}$$

by means of (3.2) if and only if

$$\begin{aligned} 0 &= I^{\sigma}_{010} + 2I^{\sigma}, \\ 0 &= J^{\sigma}_{01} + 2J^{\sigma}. \end{aligned} \tag{3.37}$$

In this case, the change of variables is

$$\begin{aligned} \bar{x} &= \int \exp\left(\int F_{y'}\, dx\right) dx, \\ \bar{y} &= y. \end{aligned} \tag{3.38}$$

Proof. It is easy to check that (3.1) is reducible to the form of the statement under the change of variable (3.2) if and only if

$$F(x, y, z) = z\frac{\phi''(x)}{\phi'(x)} + \phi'(x)^2 f(\phi(x), y). \tag{3.39}$$

Moreover, a simple computation shows that $I^{\sigma}_{010} + 2I^{\sigma} = 0$ if and only if $F_{yy'} = 0$, $J^{\sigma}_{01} + 2J^{\sigma} = 0$ if and only if $F_{y'y'} = 0$. It is obvious that (3.39) implies $F_{yy'} = F_{y'y'} = 0$. Conversely, if these conditions hold, then, $F(x, y, y') = h(x)y' + \phi'(x)^2 g(\phi(x), y)$, with $\phi(x) = \int \exp(\int h(x)dx)dx$. □

3.2. Painlevé transcendents. The search for nonlinear ordinary differential equations with solutions without moving critical points (critical points, the location of which depends on the initial conditions to the differential equation)—so named equations with Painlevé property—was an important mathematical problem in 19th century. For the equations of the form $y'' = F(x, y, y')$, which are rational in y', algebraic in y and analytic in x, Painlevé and Gambier found fifty types of differential equations satisfying Painlevé property (see [13, 19, 46, 47]), six of them have solutions in terms of the Painlevé transcendents. The first three of these equations were due to Painlevé:

$$y'' = 6y^2 + x, \tag{3.40}$$

$$y'' = 2y^3 + xy + \alpha, \tag{3.41}$$

$$y'' = \frac{y'^2}{y} - \frac{y'}{x} + \frac{(\alpha y^2 + \beta)}{x} + \gamma y^3 + \frac{\delta}{y}. \tag{3.42}$$

The role of such equations in several aspects of physics (statistical mechanics, theory of solitons, and integrable dynamical systems, etc.) is fundamental. In order to characterize (3.40)–(3.42), a similar result to Lemma 3.1 is obtained.

Lemma 3.6. *An arbitrary second-order ODE (3.1) can be reduced to the first Painlevé transcendent (3.40) by means of a change of coordinates of the independent variable (3.2) if and only if*

$$F(x,y,y') = \varphi(x)\varphi'(x)^2 + 6\varphi'(x)^2y^2 + \frac{\varphi''(x)}{\varphi'(x)}y'. \tag{3.43}$$

The condition for the second Painlevé transcendent is

$$F(x,y,y') = 2\varphi'(x)^2y^3 + \varphi(x)\varphi'(x)^2y + \alpha\varphi'(x)^2 + \frac{\varphi''(x)}{\varphi'(x)}y', \tag{3.44}$$

and, finally, the necessary and sufficient condition for the third Painlevé transcendent is as follows:

$$\begin{aligned} F(x,y,y') = {}& \beta\frac{\varphi'(x)^2}{\varphi(x)} + \frac{\delta\varphi'(x)^2}{y} + \alpha\frac{\varphi'(x)^2}{\varphi(x)}y^2 + \gamma\varphi'(x)^2y^3 \\ & + \left(\frac{\varphi''(x)}{\varphi'(x)} - \frac{\varphi'(x)}{\varphi(x)}\right)y' + \frac{y'^2}{y}. \end{aligned} \tag{3.45}$$

Consequently, the following characterization criteria for first, second, and third Painlevé transcendents are obtained.

Proposition 3.7. *An arbitrary second-order ODE (3.1) is reducible to the first Painlevé transcendent under a change of coordinates of the independent variable (3.2) if and only if the following relations hold:*

(1) $2I^\sigma J^\sigma + I^\sigma_{001} = 0$,
(2) $2I^\sigma + I^\sigma_{010} = 0$,
(3) $2J^\sigma + J^\sigma_{10} = 0$,
(4) $I^\sigma_{200} = 0$,
(5) $J^\sigma_{02} + 6J^{\sigma,3} + 7J^\sigma J^\sigma_{01} = 0$.

Proof. Suppose that $\sigma \equiv (y'' = F(x,y,y'))$ is reduced to the first Painlevé transcendent (3.40) by means of the change of coordinates (3.2), then applying Lemma 3.6, we obtain

$$F(x,y,y') = \varphi(x)\varphi'(x)^2 + 6\varphi'(x)^2y^2 + \frac{\varphi''(x)}{\varphi'(x)}y'. \tag{3.46}$$

Consequently, it yields

$$\begin{gathered}
I^{\sigma}=\frac{12\varphi'(x)y^{2}}{y'^{2}}, \qquad J^{\sigma}=\frac{\varphi'(x)^{2}(\varphi(x)+6y^{2})}{y'^{2}},\\
I^{\sigma}_{010}=-\frac{24\varphi'(x)^{2}y}{y'^{2}}, \qquad I^{\sigma}_{001}=-\frac{24\varphi'(x)^{4}y(6y^{2}+\varphi(x))}{y'^{4}},\\
J^{\sigma}_{01}=\frac{\varphi'(x)^{3}[y'-2\varphi'(x)^{2}(6y^{2}+\varphi(x))]}{y'^{4}},\\
I^{\sigma}_{200}=0, \qquad J^{\sigma}_{10}=-\frac{2\varphi'(x)^{2}(6y^{2}+\varphi(x))}{y'^{2}},\\
J^{\sigma}_{02}=\frac{(6y^{2}+\varphi(x))\varphi'(x)^{5}[8(6y^{2}+\varphi(x))^{2}\varphi'(x)-7y']}{y'^{6}},
\end{gathered} \tag{3.47}$$

and, as a simple calculation shows, the conditions (2.5)–(2.12) of the statement hold. On the other hand, if (2.5)–(2.12) hold, then, for $\sigma\equiv(y''=F(x,y,y'))$, it is

$$0=F_{xy}-2F_{y}F_{y'}, \tag{3.48}$$

$$0=F_{yy'}, \tag{3.49}$$

$$0=F_{y'y'}, \tag{3.50}$$

$$0=F_{yyy}, \tag{3.51}$$

$$0=F_{xx}+F(6F_{y'}^{2}-2F_{xy'})-5F_{x}F_{y'}-y'(6F_{y'}^{3}-7F_{y'}F_{xy'}+F_{xxy'}). \tag{3.52}$$

Now, using (3.49)-(3.50), we obtain

$$F(x,y,y')=f(x)y'+g(x,y), \quad f\in C^{\infty}(\mathbb{R}),\ g\in C^{\infty}(\mathbb{R}^{2}), \tag{3.53}$$

and, for (3.51)–(3.53), it yields

$$F(x,y,y')=f(x)y'+\alpha(x)y^{2}+\beta(x)y+\gamma(x), \quad \alpha,\beta,\gamma\in C^{\infty}(\mathbb{R}). \tag{3.54}$$

For (3.48)–(3.54), it is easy to obtain the explicit expression for coefficient $f(x)$,

$$f(x)=\frac{\alpha'(x)}{2\alpha(x)}=\frac{\beta'(x)}{2\beta(x)}\Longrightarrow\beta(x)=\lambda\alpha(x), \quad \lambda\in\mathbb{R}, \tag{3.55}$$

and, consequently,

$$F(x,y,y')=\frac{\alpha'(x)}{2\alpha(x)}y'+\alpha(x)y^{2}+\lambda\alpha(x)y+\gamma(x). \tag{3.56}$$

Finally, taking (3.52)–(3.56) into account, we obtain a linear second-order ODE, which must satisfy the coefficient $\gamma(x)$,

$$\gamma''(x)-\frac{5}{2}\frac{\alpha'(x)}{\alpha(x)}\gamma'(x)+\frac{5\alpha'(x)^{2}-2\alpha(x)\alpha''(x)}{2\alpha(x)^{2}}\gamma(x)=0, \tag{3.57}$$

thus,

$$y(x) = \frac{\alpha(x)}{6\sqrt{6}} \int \alpha(x)^{1/2} dx. \tag{3.58}$$

As a consequence, the explicit expression for the function that defines the second-order ODE satisfies Lemma 3.6, since

$$F(x, y, y') = \frac{\alpha'(x)}{2\alpha(x)} y' + \alpha(x) y^2 + \lambda \alpha(x) y + \frac{\alpha(x)}{6\sqrt{6}} \int \alpha(x)^{1/2} dx, \tag{3.59}$$

where $\alpha \in C^\infty(\mathbb{R})$, $\lambda \in \mathbb{R}$, thus finishing. □

Similar arguments prove the following two results regarding second and third Painlevé transcendents.

Proposition 3.8. *An arbitrary second-order ODE (3.1) is reducible to the second Painlevé transcendent under a change of coordinates of the independent variable (3.2) if and only if the following equations hold:*

(1) $2I^\sigma + I^\sigma_{010} = 0$,
(2) $2J^\sigma + J^\sigma_{10} = 0$,
(3) $I^\sigma_{100} - yI^\sigma_{200} = 0$,
(4) $2J^\sigma I^\sigma_{100} + 2I^{\sigma,2} + I^\sigma_{101} = 0$,
(5) $y(I^\sigma_{001} + 2I^\sigma J^\sigma) - 2J^{\sigma,2} - J^\sigma_{01} = 0$,
(6) $1728y^3(I^\sigma_{001} + 2I^\sigma J^\sigma)^2 - I^{\sigma,2}_{100} = 0$.

Proposition 3.9. *An arbitrary second-order ODE (3.1) is reducible to the third Painlevé transcendent under a change of coordinates of the independent variable (3.2) if and only if the following equations hold:*

(1) $J^\sigma_{20} + 2J^\sigma_{10} = 0$,
(2) $J^\sigma_{11} + 3J^\sigma_{01} + J^\sigma J^\sigma_{10} = 0$,
(3) $2I^\sigma_{110} + 4I^\sigma_{100} + (2J^\sigma + J^\sigma_{10})^3 = 0$,
(4) $4I^\sigma + (2J^\sigma + J^\sigma_{10})^2 - I^\sigma_{020} = 0$,
(5) $y^3 I^\sigma_{300} + 4I^\sigma - 4yI^\sigma_{100} + 2y^2 I^\sigma_{200} = 0$,
(6)

$$\begin{aligned} 0 = {} & 144 + 168yJ^\sigma + 24y^2 I^\sigma + 60y^2 J^{\sigma,2} + 36y^3 I^\sigma J^\sigma - 8y^4 I^{\sigma,2} - 24y^3 I^\sigma_{100} - 18y^4 I^\sigma_{100} J^\sigma \\ & + 10y^5 I^\sigma I^\sigma_{100} - 18y^6 I^\sigma J^\sigma I^\sigma_{100} + 6y^7 I^{\sigma,2} I^\sigma_{100} - 2y^6 I^{\sigma,2}_{100} - 6y^8 I^\sigma I^{\sigma,2}_{100} + 24y^4 I^\sigma_{200} \\ & + 28y^5 I^\sigma_{200} J^\sigma + 4y^6 I^\sigma I^\sigma_{200} + 10y^6 I^\sigma_{200} J^{\sigma,2} + 2y^7 I^\sigma I^\sigma_{200} J^\sigma - 4y^7 I^\sigma_{100} I^\sigma_{200} \\ & - 3y^6 I^\sigma_{201} J^\sigma + y^7 I^\sigma I^\sigma_{201} - y^8 I^\sigma_{100} I^\sigma_{201} - 12y^2 J^\sigma_{11} - 2y^6 I^\sigma_{200} J^\sigma_{11}, \end{aligned} \tag{3.60}$$

(7) $3J^\sigma - yI^\sigma + y^2 I^\sigma_{100} \neq 0$.

4. Linearizable equations for $\mathrm{Aut}^\nu p$

The problem of linearization of second-order ODEs has extensively been dealt with in the literature. Two methods have been used by the authors to solve it: Cartan's equivalence method (see [2, 17, 18, 20, 21, 23, 26, 27, 34, 35, 44, 48]), and the theory of symmetries (see [21, 22, 23, 27, 44, 48, 49]). As in the previous section, we propose an alternative way: the use of differential invariants. Let us consider an arbitrary second-order ordinary differential equation (3.1) and the vertical change of variables

$$\begin{aligned} \bar{x} &= x, \\ \bar{y} &= \psi(x,y), \qquad \psi_y \neq 0. \end{aligned} \tag{4.1}$$

A straightforward calculation yields the following known result.

Lemma 4.1 ([32, Chapter 12],[44, Chapter 12],[51, Chapter 14]). *The second-order ODE (3.1) can be reduced to a linear ODE under the change of variables (4.1) if and only if there exist $A, B, C \in C^\infty(\mathbb{R})$ such that,*

$$F(x,y,z) = f(x,y) + g(x,y)y' + h(x,y)y'^2, \tag{4.2}$$

$$\begin{aligned} f(x,y) &= \frac{1}{\psi_y}\left(A(x)\psi_x + B(x)\psi + C(x) - \psi_{xx}\right), \\ g(x,y) &= A(x) - 2\frac{\psi_{xy}}{\psi_y}, \\ h(x,y) &= -\frac{\psi_{yy}}{\psi_y}. \end{aligned} \tag{4.3}$$

As a consequence, we obtain the following criterion for linearization in terms of relative invariants.

Theorem 4.2. *A second-order ODE (3.1) can be reduced to a linear one under (4.1) if and only if the following three conditions hold:*

$$R_1^\sigma = 0, \qquad R_2^\sigma = 0, \qquad R_3^\sigma = 0. \tag{4.4}$$

In this case, the change of variables is given by

$$\begin{aligned} \bar{x} &= x, \\ \bar{y} &= \int \exp\left(-\frac{1}{2}\int F_{y'y'}\,dy\right)dy. \end{aligned} \tag{4.5}$$

Proof. If $y'' = F(x,y,y')$ reduces to a linear ODE under (4.1), then it is readily checked that (4.2) and (4.3) hold. Furthermore, if $F(x,y,y')$ is a polynomial of degree two in y', then $R_1^\sigma = 0$. Now, taking (4.3) into account and the expressions for K_{100} and K_{001} given in (2.12), we have

$$\begin{aligned} R_2^\sigma &= g_y - 2h_x = 0, \\ R_3^\sigma &= -4f_y h + 2y' g_y h + 2y'^2 h h_y + 3g g_y + 2y' g h_y - 2f h_y \\ &\quad + 4f_{yy} + 2y' g_{yy} - 2g h_x - 4y' h h_x - 2g_{xy} - 4y' h_{xy} = 0. \end{aligned} \tag{4.6}$$

Moreover, if $y'' = F(x,y,y')$ satisfies (4.4), then, as a simple calculation shows, we have

$$F(x,y,y') = \alpha(x,y) + \beta(x,y)y' + \gamma(x,y)y'^2, \tag{4.7}$$

where $\alpha,\beta,\gamma \in C(\mathbb{R}^2)$, and

$$\begin{aligned} 0 &= \beta_y - 2\gamma_x, \\ 0 &= \left(\gamma\alpha - \alpha_y\right)_y + \gamma_{xx} - \beta\gamma_x. \end{aligned} \tag{4.8}$$

If we take the change of coordinates

$$\begin{aligned} \bar{x} &= x, \\ \bar{y} &= \psi(x,y) = \int \exp\left(-\int \gamma\, dy\right) dy, \end{aligned} \tag{4.9}$$

then it is easy to check that

$$\gamma = -\frac{\psi_{yy}}{\psi_y}. \tag{4.10}$$

Substituting this result in (4.8), we obtain, respectively,

$$\begin{aligned} \beta &= -2\frac{\psi_{yy}}{\psi_y} + \xi(x), \quad \xi \in C^\infty(\mathbb{R}), \\ \alpha &= \frac{\xi(x)\psi_x + \eta(x)\psi + \tau(x) - \psi_{xx}}{\psi_y}, \quad \eta,\tau \in C^\infty(\mathbb{R}). \end{aligned} \tag{4.11}$$

Accordingly, (4.7) can be written as follows:

$$F(x,y,y') = \frac{\xi(x)\psi_x + \eta(x)\psi + \tau(x) - \psi_{xx}}{\psi_y} + \left(\xi(x) - 2\frac{\psi_{xy}}{\psi_y}\right)y' - \frac{\psi_{yy}}{\psi_y}y'^2, \tag{4.12}$$

and Lemma 4.1 is applied. □

This result allows us to characterize the ordinary differential equations which reduce to $y'' = 0$. A first known result in this direction is the following.

Lemma 4.3 ([32, Chapter 12], [44, Chapter 12], [51, Chapter 14]). *A linear second-order ODE $y'' = A(x)y' + B(x)y + C(x)$ can be reduced to*

$$\frac{d^2\bar{y}}{dx^2} = 0 \tag{4.13}$$

under (4.1) if and only if $K^\sigma = 0$. In this case, the change of variables is

$$\begin{aligned} \bar{x} &= x, \\ \bar{y} &= y\exp\left(-\frac{1}{2}\int A(x)dx\right) + h(x), \quad h \in C^\infty(\mathbb{R}). \end{aligned} \tag{4.14}$$

As a consequence, we can state the following criterion.

Theorem 4.4. *A second-order ODE (3.1) is reducible to*

$$\frac{d^2\bar{y}}{dx^2} = 0 \tag{4.15}$$

under (4.1) if and only if

$$R_1^\sigma = 0, \qquad K^\sigma = 0. \tag{4.16}$$

Proof. Let us consider the ODE (3.1). By Theorem 4.2, this equation can be reduced to a linear ODE under (4.1) if and only if $R_i^\sigma = 0$ for $i = 1,2,3$. Furthermore, by Lemma 4.3, every second-order ODE σ can be reduced to $d^2\bar{y}/dx^2 = 0$ under (4.1) if and only if $K^\sigma = 0$. As a consequence, (3.1) can be reduced to $d^2\bar{y}/dx^2 = 0$ by means of (4.1) if and only if $K^\sigma = R_i^\sigma = 0$ for $i = 1,2,3$. Now, it is easily checked that $K^\sigma = 0$ if and only if $R_2^\sigma = R_3^\sigma = 0$. Hence, the statement follows. □

5. Computational implementation

As it is mentioned above, the criteria for characterization of different types of second-order ODEs studied can be implemented easily in most computer algebra systems. In this work, we will use the computational package Mathematica. Note that the characterization conditions stated above can be written in terms of the differential function F, which defines the second-order ODE, as follows: for autonomous and second homogeneous-type differential equations,

$$\begin{gathered} 0 = \left(\frac{F_{xy}}{G}\right)_y, \\ 0 = \left(\frac{F_{xy}}{G}\right)_{y'}, \\ 0 = y'\left(\left(\frac{F_{xy}}{G}\right)_x + \left(\frac{F_{xy}}{G}\right)^2\right) + \frac{F_{xy}(2F - y'F_{y'})}{G} + F_x, \end{gathered} \tag{5.1}$$

where $G = y'F_{yy'} - 2F_y$. For special differential equations, these conditions are

$$0 = F_{yy'}, \qquad 0 = F_{y'y'}. \tag{5.2}$$

For the first Painlevé transcendent, they are

$$\begin{gathered} 0 = F_{xy} - 2F_yF_{y'}, \qquad 0 = F_{yy'}, \qquad 0 = F_{y'y'}, \qquad 0 = F_{yyy}, \\ 0 = F_{xx} + F(6F_{y'}^2 - 2F_{xy'}) - 5F_xF_{y'} - y'(6F_{y'}^3 - 7F_{y'}F_{xy'} + F_{xxy'}). \end{gathered} \tag{5.3}$$

For the second Painlevé transcendent, they are

$$\begin{gathered} 0 = F_{yy'}, \qquad 0 = F_{y'y'}, \qquad 0 = F_{yy} - yF_{yyy}, \qquad 0 = F_{xyy} - 2F_{y'}F_{yy}, \\ 0 = 2FF_{y'} - 2y'F_{y'}^2 - 2yF_yF_{y'} - F_x + y'F_{xy'} + yF_{xy}, \\ 0 = 1728y^3(F_{xy} - 2F_yF_{y'})^2 - F_{yy}^2. \end{gathered} \tag{5.4}$$

For the third Painlevé transcendent, they are

$$
\begin{aligned}
&0 = F_{y'y'y'}, \qquad 0 = F_{xy'y'}, \qquad 0 = 2F_{yyy'} - y'F^3_{y'y'},\\
&0 = y'F^2_{y'y'} + 3F_{yy'} - y'F_{yy'y'}, \qquad 0 = y^3F_{yyyy} + 4F_y - 4yF_{yy} + 2y^2F_{yyy},\\
&0 = 144y'^5 - 8y'^5y^4 + 24y'^3y^2F_y + y'F_y{}^2 - 24y'^3y^3F_{yy} + 10y'y^5F_yF_{yy}\\
&\quad + y^7F_yF_{yy'}F_{yy} - 2y'y^6F^2_{yy} - y^8F_{yy'}F + 2y^7F^2_yF_{yyy'}\\
&\quad - 2y^8F_yF_{yy}F_{yyy'} + 24y'^3y^4F_{yyy} + 4y'y^6F_yF_{yyy} - 4y'y^7F_{yy}F_{yyy}\\
&\quad + 6F^2_{y'}(6y'^3y^2 + y'y^6F_{yyy}) - 3F^2(12y'y^2 + y^6F_{yyyy'})\\
&\quad + 36y'^2y^2F_x + 6y^6F_{yyy}F_x - 36y'^3y^2F_{xy'}\\
&\quad - 6y'y^6F_{yyy}F_{xy'} + y^7F_yF_{xyyy} - y^8F_{yy}F_{xyyy}\\
&\quad + yF(168y'^3 + 36y'y^2F_y - 18y'y^3F_{yy} - 3y^5F_{yy'}F_{yy} - 6y^5F_yF_{yyy'}\\
&\qquad + 28y'y^4F_{yyy} - 6y^5F_{y'}F_{yyy} - 8F_{y'y'}(6y'^3y + y'y^5F_{yyy})\\
&\qquad + 3y'y^5F_{y'}F_{yyyy'} + y^6F_yF_{yyyy'} - y^7F_{yy}F_{yyyy'} - 3y^5F_{xyyy})\\
&\quad + yF_{y'}(-168y'^4 + 18y'^2y^3F_{yy} + 3y'y^5F_{yy'}F_{yy}\\
&\qquad - 28y'^2y^4F_{yyy} + 2y^7F_{yy}F_{yyy} + 2F_{y'y'}(6y'^4y + y'^2y^5F_{yyy})\\
&\qquad - 2F_y(18y'^2y^2 - 3y'y^5F_{yyy'} + y^6F_{yyy}) + 3y'y^5F_{xyyy}),\\
&\qquad\qquad 0 \neq 3F - 3y'F_{y'} - yF_y + y^2F_{yy}.
\end{aligned}
\tag{5.5}
$$

For linear differential equations, the following conditions are

$$
\begin{aligned}
&0 = F_{y'y'y'}, \qquad 0 = -FF_{y'y'y'} + F_{yy'} - y'F_{yy'y'} - F_{xy'y'},\\
&0 = -FF_{y'}F_{y'y'y'} - 2F_yF_{y'y'} + 3F_{y'}F_{yy'} - 2FF_{yy'y'}\\
&\qquad - y'F_{y'}F_{yy'y'} + 4F_{yy} - 2y'F_{yyy'} - F_{y'}F_{xy'y'} - 2F_{xyy'}.
\end{aligned}
\tag{5.6}
$$

And, finally, for $y'' = 0$, the conditions are

$$
0 = F_{y'y'y'}, \qquad 0 = \frac{F^2_{y'}}{2} - FF_{y'y'} + 2F_y - y'F_{yy'} - F_{xy'}. \tag{5.7}
$$

As an example, we implement the characterization conditions for autonomous and second homogeneous-type differential equations. The notation used in the Mathematica code is as follows: the variables x, y, and p stand for the coordinates x, y, and y', respectively. So, first of all, we have to input the function that defines the differential equation by means of

$$
\text{In[1]} := \texttt{F=Input[``ODE to study'']}. \tag{5.8}
$$

Subsequently, we must input in the variable G the function G,

$$
\text{In[2]} := \texttt{G=p} * \texttt{D[F,\{y,1\},\{p,1\}]} - 2 * \texttt{D[F,y]}. \tag{5.9}
$$

Finally, we are ready to check the vanishing of the three conditions (5.1):

$$
\begin{aligned}
&\text{In }[3] := \texttt{Simplify }[\texttt{Expand}[\texttt{D}[\texttt{D }[\texttt{F},\{\texttt{x},1\},\{\texttt{y},1\}]/\texttt{G},\texttt{y}]]],\\
&\text{In }[4] := \texttt{Simplify}[\texttt{Expand}[\texttt{D}[\texttt{D }[\texttt{F},\{\texttt{x},1\},\{\texttt{y},1\}]/\texttt{G},\texttt{p}]]],\\
&\text{In }[5] := \texttt{Simplify}[\texttt{Expand}[\texttt{p}*(\texttt{D}[\texttt{D}[\texttt{F},\{\texttt{x},1\},\{\texttt{y},1\}]/\ \texttt{G},\texttt{x}]\\
&\qquad\qquad +(\texttt{D}[\texttt{F},\{\texttt{x},1\},\{\texttt{y},1\}]/\texttt{G})\hat{\ }2) + (\texttt{D}[F,\{x,1\},\{y,1\}]/\texttt{G})\\
&\qquad\qquad * (2 * F - p * D[F,p]) + D[F,x]]].
\end{aligned}
\tag{5.10}
$$

Appendix

In this appendix, the computational implementation of the algorithms generating horizontal and vertical invariants (see [38]) is described. As it is mentioned in Section 5, the computer algebra system used is Mathematica. If the variables `x`, `y`, `p`, and `k[a,b,c]` stands for x, y, y', and y''_{abc}, respectively, then the total derivatives D_x, D_y, and $D_{y'}$ can be implemented as the functions `dtx`, `dty`, and `dtp` as follows:

$$
\begin{aligned}
&\text{In}[1] := \texttt{dtx}[\texttt{w\_}] := \texttt{D}[\texttt{w},\texttt{x}]\\
&\qquad + \texttt{Sum}[\texttt{k}[\texttt{a}+1,\texttt{b},\texttt{c}] * \texttt{D}[\texttt{w},\texttt{k}[\texttt{a},\texttt{b},\texttt{c}]],\{\texttt{a},0,\texttt{r}+2\},\{\texttt{b},0,\texttt{r}+2\},\{\texttt{c},0,\texttt{r}+2\}],\\
&\text{In}[2] := \texttt{dty}[\texttt{w\_}] := \texttt{D}[\texttt{w},\texttt{y}]\\
&\qquad + \texttt{Sum}[\texttt{k}[\texttt{a},\texttt{b}+1,\texttt{c}] * \texttt{D}[\texttt{w},\texttt{k}[\texttt{a},\texttt{b},\texttt{c}]],\{\texttt{a},0,\texttt{r}+2\},\{\texttt{b},0,\texttt{r}+2\},\{\texttt{c},0,\texttt{r}+2\}],\\
&\text{In}[3] := \texttt{dtp}[\texttt{w\_}] := \texttt{D}[\texttt{w},\texttt{p}]\\
&\qquad + \texttt{Sum}[\texttt{k}[\texttt{a},\texttt{b},\texttt{c}+1] * \texttt{D}[\texttt{w},\texttt{k}[\texttt{a},\texttt{b},\texttt{c}]],\{\texttt{a},0,\texttt{r}+2\},\{\texttt{b},0,\texttt{r}+2\},\{\texttt{c},0,\texttt{r}+2\}],
\end{aligned}
\tag{A.1}
$$

where `r` is the maximum order of the differential invariants to be calculated. As a consequence, the operators, given in (2.5) and (2.10), are easily implemented

$$
\begin{gathered}
\text{In}[4] := \texttt{Y}[1][\texttt{w\_}] := \texttt{dty}[\texttt{w}],\\
\text{In}[5] := \texttt{Y}[2][\texttt{w\_}] := \texttt{p} * \texttt{dtp}[\texttt{w}],\\
\text{In}[6] := \texttt{Y}[3][\texttt{w\_}] := \texttt{dtx}[\texttt{w}]/\texttt{p} + (\texttt{k}[0,0,0]/\texttt{p}) * \texttt{dtp}[\texttt{w}],\\
\text{In}[7] := \texttt{Z}[1][\texttt{w\_}] := \texttt{dtp}[\texttt{w}]/\texttt{Sqrt}[\texttt{k}[0,0,3]],\\
\text{In}[8] := \texttt{Z}[2][\texttt{w\_}] := \texttt{dtx}[\texttt{w}] + \texttt{p} * \texttt{dty}[\texttt{w}] + \texttt{k}[0,0,0] * \texttt{dtp}[\texttt{w}],\\
\text{In}[9] := \texttt{Z}[3][\texttt{w\_}] := (2 * \texttt{dty}[\texttt{w}] + \texttt{k}[0,0,1] * \texttt{dtp}[\texttt{w}])/\texttt{Sqrt}[\texttt{k}[0,0,3]].
\end{gathered}
\tag{A.2}
$$

Moreover, the horizontal invariants, I, J, and the vertical invariants, K, V, needed to calculate the corresponding base are introduced as follows:

$$\begin{aligned}
&\text{In}[10] := \texttt{Inv[0,0,0]} = \texttt{k[0,1,0]/p\^{}2},\\
&\text{In}[11] := \texttt{Jnv[0,0]} = (\texttt{k[0,0,0]} - \texttt{p} * \texttt{k[0,0,1]})\texttt{/p\^{}2},\\
&\text{In}[12] := \texttt{Knv[0,0,0]}\\
&\qquad = \texttt{k[0,0,1]\^{}2/2} - \texttt{k[0,0,0]} * \texttt{k[0,0,2]} + 2 * \texttt{k[0,1,0]-p*k[0,1,1]-k[1,0,1]},\\
&\text{In}[13] := \texttt{V} = \texttt{k[0,0,4]/k[0,0,3]\^{}(3/2)}.
\end{aligned} \tag{A.3}$$

Consequently, the basis for rth order horizontal differential invariants can be calculated as follows:

$$\begin{aligned}
&\text{In}[14] := \texttt{Do[Inv[a,b,c]}\\
&\qquad = \texttt{Nest[Y[1],Nest[Y[2],Nest[Y[3],Inv[0,0,0],c],b],a]},\\
&\qquad \texttt{\{a,0,r} - \texttt{1\},\{b,0,r} - \texttt{1} - \texttt{a\},\{c,0,r} - \texttt{1} - \texttt{a} - \texttt{b\}]},\\
&\texttt{Do[Jnv[b,c]} = \texttt{Nest[Y[2],Nest[Y[3],Jnv[0,0],c],b]},\\
&\qquad \texttt{\{b,0,r} - \texttt{1\},\{c,0,r} - \texttt{1} - \texttt{b\}]},
\end{aligned} \tag{A.4}$$

where `Inv[a,b,c]` and `Jnv[b,c]` stand for the invariants I_{abc} and J_{bc}, respectively. Furthermore, the basis for vertical differential invariants is calculated as follows:

$$\begin{aligned}
&\text{In}[15] := \texttt{Do[Knv[a,b,c]Nest[Z[1],Nest[Z[2],Nest[Z[3],Knv[0,0,0],c],b],a]},\\
&\qquad \texttt{\{a,0,r} - \texttt{2\},\{b,0,r} - \texttt{2} - \texttt{a\},\{c,0,r} - \texttt{2} - \texttt{a} - \texttt{b\}]},\\
&\texttt{Do[KKnv[t,s]} = \texttt{Nest[Z[3],Nest[Z[2],Z[1][Knv[0,0,0]],s],t]},\\
&\qquad \texttt{\{s,0,n} - \texttt{3\},\{t,1,n} - \texttt{3} - \texttt{s\}]},
\end{aligned} \tag{A.5}$$

where `Knv[a,b,c]` and `KKnv[t,s]` stand for the invariants K_{abc} and K_{ts}, respectively.

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