COEFFICIENT ESTIMATES AND SUBORDINATION PROPERTIES ASSOCIATED WITH CERTAIN CLASSES OF FUNCTIONS

R. K. RAINA AND DEEPAK BANSAL

Received 8 June 2005 and in revised form 13 September 2005

Dedicated to Professor H. M. Srivastava, University of Victoria, Canada, on his 65th Birth Anniversary

This note investigates coefficient estimates and subordination properties for certain classes of normalized functions (which are essentially defined by means of a Hadamard product of two analytic functions). We exhibit several interesting consequences of our main results, and in the process, we are also led to the corrected forms of the results given by Owa and Nishiwaki (2002).

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ normalized by $f(0) = f'(0) - 1 = 0$, and analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C},\ |z| < 1\}$; then $f(z)$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let us denote by $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ two subclasses of the class $\mathcal{A}$, which are defined (for $\alpha > 1$) as follows:

$$\begin{aligned} \mathcal{M}(\alpha) &= \left\{ f : f \in \mathcal{A};\ \Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha\ (z \in \mathcal{U};\ \alpha > 1) \right\}, \\ \mathcal{N}(\alpha) &= \left\{ f : f \in \mathcal{A};\ \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \alpha\ (z \in \mathcal{U};\ \alpha > 1) \right\}. \end{aligned} \tag{1.2}$$

The classes $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ were studied recently by Owa and Nishiwaki [3] and also by Owa and Srivastava [4]. In fact, for $1 < \alpha \leqq 4/3$, these classes were investigated earlier by Uralegaddi et al. [7].

It follows from (1.2) that

$$f(z) \in \mathcal{N}(\alpha) \Longleftrightarrow zf'(z) \in \mathcal{M}(\alpha). \tag{1.3}$$

International Journal of Mathematics and Mathematical Sciences 2005:22 (2005) 3685–3695
DOI: 10.1155/IJMMS.2005.3685

If $f,g \in \mathcal{A}$, where $f(z)$ is given by (1.1), and $g(z)$ is defined by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.4}$$

then their Hadamard product (or convolution) $f * g$ is defined (as usual) by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z). \tag{1.5}$$

For two functions f and g analytic in $\mathcal{U}$, we say that the function f is subordinate to g in $\mathcal{U}$ (denoted by $f \prec g$) if there exists a Schwarz function $w(z)$, analytic in $\mathcal{U}$ with $w(0) = 0$, and $|w(z)| < 1$ $(z \in \mathcal{U})$, such that $f(z) = g(w(z))$.

We introduce here a class $\mathcal{S}_\alpha(\phi,\psi)$ which is defined as follows: suppose the functions $\phi(z)$ and $\psi(z)$ are given by

$$\begin{aligned} \phi(z) &= z + \sum_{n=2}^{\infty} \lambda_n z^n, \\ \psi(z) &= z + \sum_{n=2}^{\infty} \mu_n z^n, \end{aligned} \tag{1.6}$$

where $\lambda_n \geqq \mu_n \geqq 0$ (for all $n \in \mathbb{N}\backslash\{1\}$). We say that $f \in \mathcal{A}$ is in $\mathcal{S}_\alpha(\phi,\psi)$ provided that $(f * \psi)(z) \neq 0$ and

$$\Re\left\{\frac{(f * \phi)(z)}{(f * \psi)(z)}\right\} < \alpha \quad (\alpha > 1;\ z \in \mathcal{U}). \tag{1.7}$$

Several new and known subclasses can be obtained from the class $\mathcal{S}_\alpha(\phi,\psi)$ by suitably choosing the functions $\phi(z)$ and $\psi(z)$. We mention below some of these subclasses of $\mathcal{S}_\alpha(\phi,\psi)$ consisting of functions $f(z) \in \mathcal{A}$. The Ruscheweyh derivative operator (see [5]): $\mathcal{D}^\mu : \mathcal{A} \to \mathcal{A}$, is defined by

$$\mathcal{D}^\mu\{f(z)\} = \frac{z}{(1-z)^{1+\mu}} * f(z) \quad (\mu > -1;\ f(z) \in \mathcal{A}). \tag{1.8}$$

Indeed, we have

$$\begin{aligned} &\mathcal{S}_\alpha\left(\frac{z}{(1-z)^{\lambda+2}}, \frac{z}{(1-z)^{\lambda+1}}\right) \\ &\quad = \left\{f : f \in \mathcal{A};\ \Re\left(\frac{D^{\lambda+1} f(z)}{D^\lambda f(z)}\right) < \alpha\ (z \in \mathcal{U};\ \alpha > 1;\ \lambda > -1)\right\} \equiv \Lambda_\alpha(\lambda), \end{aligned} \tag{1.9}$$

and the relationships

$$\mathcal{S}_\alpha\left(\frac{z}{(1-z)^2}, \frac{z}{(1-z)}\right) \equiv \mathcal{M}(\alpha) \quad (\alpha > 1),$$
$$\mathcal{S}_\alpha\left(\frac{z+z^2}{(1-z)^3}, \frac{z}{(1-z)^2}\right) \equiv \mathcal{N}(\alpha) \quad (\alpha > 1). \tag{1.10}$$

A similar type of a subclass $\Lambda_\alpha(\lambda)$ involving the Ruscheweyh operator D^λ (defined by (1.9) above) was also considered by Ali et al. [1]; whereas, the subclasses $\mathcal{M}(\alpha)$ and $\mathcal{N}(\alpha)$ are the known classes (see [3]) defined by (1.2), respectively.

In our present investigation, we require the following definition and also a related result due to Wilf [8].

Definition 1.1. A sequence $\{b_n\}_1^\infty$ of complex numbers is said to be a subordination factor sequence if whenever $f(z)$ given by (1.1) is univalent and convex in $\mathcal{U}$, then

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z) \quad \text{in } \mathcal{U}. \tag{1.11}$$

Lemma 1.2. *The sequence $\{b_n\}_1^\infty$ is a subordinating factor sequence if and only if*

$$\Re\left[1 + 2\sum_{n=1}^{\infty} b_n z^n\right] > 0 \quad (z \in \mathcal{U}). \tag{1.12}$$

The main purpose of this note is first to investigate the coefficient estimates and related properties of certain classes of normalized functions, defined as a Hadamard product (or convolution) of two analytic functions, in the open unit disk. Additionally, by defining another analogous class of analytic functions, we develop a subordination theorem for this class and consider various interesting consequences (including the corrected forms of the results obtained in [3]) from our main results.

2. Coefficient inequalities and related properties

We first derive a sufficient condition for the function $f(z)$ to belong to the aforementioned class $\mathcal{S}_\alpha(\phi,\psi)$. The result is contained in the following.

Theorem 2.1. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{(\lambda_n - k\mu_n) + |\lambda_n + (k-2\alpha)\mu_n|\}\, |a_n| \leqq 2(\alpha - 1), \tag{2.1}$$

for some k $(0 \leqq k \leqq 1)$ and some α $(\alpha > 1)$, then $f(z) \in \mathcal{S}_\alpha(\phi,\psi)$.

Proof. Let condition (2.1) be satisfied for the function $f(z) \in \mathcal{A}$. It is sufficient to show that

$$\left|\frac{(f*\phi)(z)/(f*\psi)(z) - k}{(f*\phi)(z)/(f*\psi)(z) - (2\alpha - k)}\right| < 1 \quad (z \in \mathcal{U}). \tag{2.2}$$

We note that

$$\begin{aligned}
&\left| \frac{(f*\phi)(z)/(f*\psi)(z)-k}{(f*\phi)(z)/(f*\psi)(z)-(2\alpha-k)} \right| \\
&\quad = \left| \frac{(1-k)+\sum_{n=2}^{\infty}(\lambda_n-k\mu_n)a_n z^{n-1}}{(1-2\alpha+k)+\sum_{n=2}^{\infty}\{\lambda_n-(2\alpha-k)\mu_n\}a_n z^{n-1}} \right| \\
&\quad \leqq \frac{(1-k)+\sum_{n=2}^{\infty}(\lambda_n-k\mu_n)\,|a_n|\,|z|^{n-1}}{(2\alpha-k-1)-\sum_{n=2}^{\infty}|\lambda_n-(2\alpha-k)\mu_n|\,|a_n|\,|z|^{n-1}} \\
&\quad < \frac{(1-k)+\sum_{n=2}^{\infty}(\lambda_n-k\mu_n)\,|a_n|}{(2\alpha-k-1)-\sum_{n=2}^{\infty}|\lambda_n-(2\alpha-k)\mu_n|\,|a_n|}.
\end{aligned} \tag{2.3}$$

The extreme-right-side expression of the above inequality would remain bounded by 1 if

$$(1-k)+\sum_{n=2}^{\infty}(\lambda_n-k\mu_n)\,|a_n| \leqq (2\alpha-k-1)-\sum_{n=2}^{\infty}|\lambda_n-(2\alpha-k)\mu_n|\,|a_n|, \tag{2.4}$$

which leads to the desired inequality (2.1). This completes the proof. □

Taking $k=1$ in (2.1), we observe that the sequence

$$\omega_n(\alpha)=\lambda_n+(1-2\alpha)\mu_n \quad (\alpha>1;\ \lambda_n>\mu_n>0;\ n\in\mathbb{N}\setminus\{1\}) \tag{2.5}$$

remains nonnegative whenever the sequences $\langle\mu_n\rangle$ and $\langle\lambda_n/\mu_n\rangle$ are nondecreasing, and α satisfies the inequality $1<\alpha\leqq(1/2)(1+\lambda_2/\mu_2)$. Using this in Theorem 2.1, we obtain the following.

Corollary 2.2. *If $f(z)\in\mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}(\lambda_n-\alpha\mu_n)\,|a_n| \leqq (\alpha-1), \tag{2.6}$$

provided that $\lambda_n>\mu_n>0$, $\langle\mu_n\rangle$ and $\langle\lambda_n/\mu_n\rangle$ are nondecreasing sequences, and α is such that $1<\alpha\leqq(1/2)(1+\lambda_2/\mu_2)$, then $f(z)\in\mathcal{S}_\alpha(\phi,\psi)$.

By appealing to (1.9), (1.10), when the functions $\phi(z)$ and $\psi(z)$ in (1.7) are selected appropriately, then Theorem 2.1 yields the following results.

Corollary 2.3. *If $f(z)\in\mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}\frac{\Gamma(\lambda+n)}{\Gamma(\lambda+2)\Gamma(n)}\left[\{n+(1-k)\lambda-k\}+|n+\lambda(1+k-2\alpha)+k-2\alpha|\right]|a_n| \leqq 2(\alpha-1) \tag{2.7}$$

for some k $(0\leqq k\leqq 1)$ and some α $(\alpha>1)$, then $f(z)\in\Lambda_\alpha(\lambda)$.

Corollary 2.4. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{(n-k) + |n+k-2\alpha|\} |a_n| \leqq 2(\alpha - 1) \tag{2.8}$$

for some k $(0 \leqq k \leqq 1)$ and some α $(\alpha > 1)$, then $f(z) \in \mathcal{M}(\alpha)$.

Corollary 2.5. *If $f(z) \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \{(n-k) + |n+k-2\alpha|\} n |a_n| \leqq 2(\alpha - 1) \tag{2.9}$$

for some k $(0 \leqq k \leqq 1)$ and some α $(\alpha > 1)$, then $f(z) \in \mathcal{N}(\alpha)$.

Remark 2.6. Corollary 2.3 is a new result, whereas, Corollaries 2.4 and 2.5 are the same assertions as obtained earlier by Owa and Nishiwaki [3, Theorems 2.1 and 2.3, pages 2-3].

We next mention here another known class, $E(\phi, \psi; \beta)$ (see [2]), consisting of the function $f(z) \in \mathcal{A}$ defined analogously to the class $\mathcal{S}_\alpha(\phi, \psi)$, which satisfies the condition

$$\Re\left\{\frac{(f * \phi)(z)}{(f * \psi)(z)}\right\} > \beta \quad (0 \leqq \beta < 1;\ z \in \mathcal{U}). \tag{2.10}$$

We prove the following result.

Theorem 2.7. *If $f(z) \in \mathcal{A}$ satisfies (2.6) for some α $(1 < \alpha \leqq 2\lambda_2/(\lambda_2 + \mu_2))$, then $f(z) \in E(\phi, \psi; \beta)$ $(\beta = (2\lambda_2 - \alpha\mu_2 - \alpha\lambda_2)/(\mu_2 + \lambda_2 - 2\alpha\mu_2))$.*

Proof. If α satisfies the inequality $(1 < \alpha \leqq (1/2)(1 + \lambda_2/\mu_2))$, then (2.6) of Corollary 2.2 implies

$$\sum_{n=2}^{\infty} \frac{(\lambda_n - \alpha\mu_n)}{(\alpha - 1)} |a_n| \leqq 1. \tag{2.11}$$

Now using the following result [2, Theorem 1, page 72] which determines the sufficiency condition for the function $f(z) \in \mathcal{A}$ to be in the class $E(\phi, \psi; \beta)$:

$$\sum_{n=2}^{\infty} \frac{(\lambda_n - \beta\mu_n)}{(1 - \beta)} |a_n| \leqq 1 \quad (0 \leqq \beta < 1), \tag{2.12}$$

we need to find the smallest positive β such that

$$\sum_{n=2}^{\infty} \frac{\lambda_n - \beta\mu_n}{1 - \beta} |a_n| \leqq \sum_{n=2}^{\infty} \frac{\lambda_n - \alpha\mu_n}{\alpha - 1} |a_n| \leqq 1. \tag{2.13}$$

This gives

$$\beta \leqq \frac{2\lambda_n - \alpha\mu_n - \alpha\lambda_n}{\mu_n + \lambda_n - 2\alpha\mu_n} \quad (n \in \mathbb{N} \setminus \{1\}). \tag{2.14}$$

Let us put

$$F(n)=\frac{2\lambda_n-\alpha\mu_n-\alpha\lambda_n}{\mu_n+\lambda_n-2\alpha\mu_n}$$
$$\left(\lambda_n>\mu_n>0;\, n\in\mathbb{N}\backslash\{1\};\, 1<\alpha<\frac{1}{2}\left(1+\frac{\lambda_2}{\mu_2}\right);\, \left\langle\frac{\lambda_n}{\mu_n}\right\rangle \text{is nondecreasing}\right). \tag{2.15}$$

We will show that $F(n)$ is a nondecreasing function of n. Elementary calculations give

$$F(n+1)-F(n)=\frac{2(\alpha-1)^2(\lambda_{n+1}\mu_n-\mu_{n+1}\lambda_n)}{\omega_{n+1}(\alpha)\omega_n(\alpha)}, \tag{2.16}$$

which is observed to be positive, under the constraints stated with (2.15), where $\omega_n(\alpha)$, given by (2.5), is a nonvanishing sequence. Hence, we conclude from (2.14) that

$$\beta\leqq\frac{2\lambda_2-\alpha\mu_2-\alpha\lambda_2}{\mu_2+\lambda_2-2\alpha\mu_2}, \tag{2.17}$$

which proves that $f(z)\in E(\phi,\psi;\beta)$ $(\beta=(2\lambda_2-\alpha\mu_2-\alpha\lambda_2)/(\mu_2+\lambda_2-2\alpha\mu_2))$. □

Setting $\beta=1/2$ in (2.10) and choosing $\phi(z)$ and $\psi(z)$ appropriately (as mentioned in (1.9)), we obtain the class $\mathbb{K}_\lambda$ involving the Ruschweyh derivative D^λ (see [5]) which is defined by

$$E\left(\frac{z}{(1-z)^{\lambda+2}},\frac{z}{(1-z)^{\lambda+1}};\frac{1}{2}\right)$$
$$=\left\{f:f\in\mathcal{A};\,\Re\left(\frac{D^{\lambda+1}f(z)}{D^\lambda f(z)}\right)>\frac{1}{2}\ (z\in\mathcal{U};\,\lambda>-1)\right\}\equiv\mathbb{K}_\lambda. \tag{2.18}$$

Also, the class $E(\phi,\psi;\beta)$ includes among others the familiar subclasses of $\mathcal{A}$ which consist of starlike functions $\mathcal{S}^*(\beta)$ of order β $(0\leqq\beta<1)$, and convex functions $\mathcal{K}(\beta)$ of order β $(0\leqq\beta<1)$. If we select the functions $\phi(z)$ and $\psi(z)$ according to (1.9), (1.10) in Theorem 2.7, then we obtain the following results.

Corollary 2.8. *If $f(z)\in\mathcal{A}$ satisfies the coefficient inequality (2.7) for $k=1$ and some α $(1<\alpha\leqq(2\lambda+5)/2(\lambda+2))$, then $f(z)\in\mathbb{K}_\lambda$.*

Corollary 2.9. *If $f(z)\in\mathcal{A}$ satisfies the coefficient inequality (2.8) for $k=1$ and some α $(1<\alpha\leqq 4/3)$, then $f(z)\in\mathcal{S}^*(\beta)$ $(\beta=(4-3\alpha)/(3-2\alpha))$.*

Corollary 2.10. *If $f(z)\in\mathcal{A}$ satisfies the coefficient inequality (2.9) for $k=1$ and some α $(1<\alpha\leqq 4/3)$, then $f(z)\in\mathcal{K}(\beta)$ $(\beta=(4-3\alpha)/(3-2\alpha))$.*

Remark 2.11. Corollary 2.8 is a new result, and Corollaries 2.9 and 2.10 are the *corrected forms* of the known assertions stated in [3, Theorem 2.5, page 3].

3. Subordination theorem

Before stating and proving our subordination theorem, we define a class $G_\alpha(\phi,\psi)$ as follows.

We denote $G_\alpha(\phi,\psi)$ to be a class of functions $f \in \mathcal{S}_\alpha(\phi,\psi)$ whose coefficients satisfy the condition (2.1). Evidently, we have

$$\begin{gathered} G_\alpha\left(\frac{z}{(1-z)^{\lambda+2}}, \frac{z}{(1-z)^{\lambda+1}}\right) \equiv \Lambda_\alpha^*(\lambda), \\ G_\alpha\left(\frac{z}{(1-z)^{2}}, \frac{z}{1-z}\right) \equiv \mathcal{M}^*(\alpha), \\ G_\alpha\left(\frac{z+z^2}{(1-z)^{3}}, \frac{z}{(1-z)^2}\right) \equiv \mathcal{N}^*(\alpha), \end{gathered} \tag{3.1}$$

where $\Lambda_\alpha^*(\lambda)$, $\mathcal{M}^*(\alpha)$, and $\mathcal{N}^*(\alpha)$ are, respectively, the subclasses of $\Lambda_\alpha(\lambda)$, $\mathcal{M}(\alpha)$, and $\mathcal{N}(\alpha)$ consisting of functions $f(z) \in \mathcal{A}$ satisfying inequalities (2.7), (2.8), and (2.9), respectively. Again, it is obvious that

$$\Lambda_\alpha^*(\lambda) \subset \Lambda_\alpha(\lambda); \qquad \mathcal{M}^*(\alpha) \subset \mathcal{M}(\alpha); \qquad \mathcal{N}^*(\alpha) \subset \mathcal{N}(\alpha). \tag{3.2}$$

Theorem 3.1. *Let $f \in G_\alpha(\phi,\psi)$, and let the sequences $\langle\lambda_n\rangle$, $\langle\mu_n\rangle$, and $\langle\lambda_n/\mu_n\rangle$ be nondecreasing; then*

$$\begin{gathered} \frac{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|}{2\{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|\}}(f * g)(z) \prec g(z) \\ (\lambda_n > \mu_n > 0;\ \alpha > 1;\ 0 \leqq k \leqq 1;\ z \in \mathcal{U}) \end{gathered} \tag{3.3}$$

for every function $g \in \mathcal{K}$. In particular,

$$\Re\{f(z)\} > -\frac{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|}{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|} \qquad (z \in \mathcal{U}). \tag{3.4}$$

The constant factor

$$\frac{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|}{2\{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|\}} \tag{3.5}$$

in the subordination result (3.3) cannot be replaced by any larger one.

Proof. Let $f(z)$ defined by (1.1) belong to the class $G_\alpha(\phi,\psi)$, and let $g(z)$ defined by (1.4) be any function in the class $\mathcal{K}$. It follows then that

$$\begin{aligned} &\frac{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|}{2\{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|\}}(f * g)(z) \\ &\quad = \frac{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|}{2\{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|\}}\left(z + \sum_{n=2}^{\infty} a_n b_n z^n\right). \end{aligned} \tag{3.6}$$

By invoking Definition 1.1, the subordination (3.3) of our theorem will hold true if the sequence

$$\left\{\frac{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|}{2\{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|\}}a_n\right\}_{n=1}^{\infty} \tag{3.7}$$

is a subordinating factor sequence. By virtue of Lemma 1.2, this is equivalent to the inequality

$$\Re\left(1 + 2\sum_{n=1}^{\infty}\frac{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|}{2\{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|\}}a_n z^n\right) > 0 \quad (z \in \mathcal{U}). \tag{3.8}$$

Let us put

$$\sigma(n) = (\lambda_n - k\mu_n) + |\lambda_n + (k-2\alpha)\mu_n|, \tag{3.9}$$

and we write

$$\sigma(n) = \mu_n\left\{\left(\frac{\lambda_n}{\mu_n} - k\right) + \left|\frac{\lambda_n}{\mu_n} + (k-2\alpha)\right|\right\}. \tag{3.10}$$

It is observed that the sequence $\sigma(n)$ is a nondecreasing function of n under the constraints

$$(\lambda_n \geqq \mu_n > 0;\ \alpha > 1;\ 0 \leqq k \leqq 1;\ n \in \mathbb{N}\backslash\{1\}). \tag{3.11}$$

In particular (under the same condition)

$$(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2| \leqq (\lambda_n - k\mu_n) + |\lambda_n + (k-2\alpha)\mu_n|. \tag{3.12}$$

Therefore, for $|z| = r$ $(r < 1)$, we obtain

$$\begin{aligned}
&\Re\left(1 + \sum_{n=1}^{\infty}\frac{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|}{\{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|\}}a_n z^n\right)\\
&\quad = \Re\Bigg(1 + \frac{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|}{\{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|\}}z\\
&\qquad + \sum_{n=2}^{\infty}\frac{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|}{\{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|\}}a_n z^n\Bigg)\\
&\quad \geqq 1 - \frac{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|}{\{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|\}}r\\
&\qquad - \sum_{n=2}^{\infty}\frac{(\lambda_n - k\mu_n) + |\lambda_n + (k-2\alpha)\mu_n|}{\{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|\}}|a_n| r^n\\
&\quad > 1 - \frac{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|}{\{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|\}}r\\
&\qquad - \frac{2(\alpha - 1)}{\{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|\}}r > 0.
\end{aligned} \tag{3.13}$$

This evidently establishes the inequality (3.8), and consequently the subordination relation (3.3) of our Theorem 3.1 is proved. The assertion (3.4) follows readily from (3.3) when the function $g(z)$ is selected as

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in \mathcal{K}. \tag{3.14}$$

The sharpness of the multiplying factor in (3.3) can be established by considering a function $h(z)$ defined by

$$h(z) = z - \frac{2(\alpha-1)}{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|} z^2, \tag{3.15}$$

which belongs to the class $G_\alpha(\phi,\psi)$. Using (3.3), we infer that

$$\frac{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|}{2\{(2\alpha - 2 + \lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|\}} h(z) \prec \frac{z}{1-z}. \tag{3.16}$$

It can easily be verified that

$$\min_{|z|\leqq 1} \left\{ \Re\left(\frac{(\lambda_2 - k\mu_2) + |\lambda_2 + (k-2\alpha)\mu_2|}{2\{2\alpha - 2 + \lambda_2 - k\mu_2 + |\lambda_2 + (k-2\alpha)\mu_2|\}} h(z) \right) \right\} = -\frac{1}{2}, \tag{3.17}$$

which completes the proof of Theorem 3.1. □

On choosing the arbitrary functions $\phi(z)$ and $\psi(z)$, in accordance with the subclasses defined by (3.1), we arrive at the following results.

Corollary 3.2. *Let $f(z) \in \Lambda_\alpha^*(\lambda)$; then for every function g in $\mathcal{K}$,*

$$\frac{(\lambda(1-k) + 2 - k) + |\lambda(1+k-2\alpha) + 2 + k - 2\alpha|}{2\{\lambda(1-k) + 2\alpha - k + |\lambda(1+k-2\alpha) + 2 + k - 2\alpha|\}} (f * g)(z) \prec g(z)$$
$$(\alpha > 1;\ 0 \leqq k \leqq 1;\ \lambda > -1;\ z \in \mathcal{U}), \tag{3.18}$$
$$\Re\{f(z)\} > -\frac{\lambda(1-k) + 2\alpha - k + |\lambda(1+k-2\alpha) + 2 + k - 2\alpha|}{(\lambda(1-k) + 2 - k) + |\lambda(1+k-2\alpha) + 2 + k - 2\alpha|} \quad (z \in \mathcal{U}).$$

The following constant factor:

$$\frac{\lambda(1-k) + |\lambda(1+k-2\alpha) + 2 + k - 2\alpha|}{2\{\lambda(1-k) + 2\alpha - k + |\lambda(1+k-2\alpha) + 2 + k - 2\alpha|\}} \tag{3.19}$$

cannot be replaced by a larger one.

Corollary 3.3. *Let $f(z) \in \mathcal{M}^*(\alpha)$; then for every function g in $\mathcal{K}$,*

$$\frac{(2-k)+|2+k-2\alpha|}{2\{(2\alpha-k)+|2+k-2\alpha|\}}(f*g)(z) \prec g(z),$$
$$\Re\{f(z)\} > -\frac{(2\alpha-k)+|2+k-2\alpha|}{(2-k)+|2+k-2\alpha|} \quad (z \in \mathcal{U}). \tag{3.20}$$

The following constant factor:

$$\frac{(2-k)+|2+k-2\alpha|}{2\{(2\alpha-k)+|2+k-2\alpha|\}} \tag{3.21}$$

cannot be replaced by a larger one.

Corollary 3.4. *Let $f(z) \in \mathcal{N}^*(\alpha)$; then for every function g in $\mathcal{K}$,*

$$\frac{(2-k)+|2+k-2\alpha|}{2\{(\alpha+1-k)+|2+k-2\alpha|\}}(f*g)(z) \prec g(z),$$
$$\Re\{f(z)\} > -\frac{(\alpha+1-k)+|2+k-2\alpha|}{(2-k)+|2+k-2\alpha|} \quad (z \in \mathcal{U}). \tag{3.22}$$

The following constant factor

$$\frac{(2-k)+|2+k-2\alpha|}{2\{(\alpha+1-k)+|2+k-2\alpha|\}} \tag{3.23}$$

cannot be replaced by a larger one.

Remark 3.5. Corollary 3.2 is a new result, and Corollaries 3.3 and 3.4 correspond to the known results due to Srivastava and Attiya [6].

Acknowledgment

The authors would like to express their sincerest thanks to the referees for useful comments. The present investigation was supported by AICTE (Government of India), New Delhi.

References

[1] R. M. Ali, M. Hussain Khan, V. Ravichandran, and K. G. Subramanian, *A class of multivalent functions with positive coefficients defined by convolution*, JIPAM. J. Inequal. Pure Appl. Math. **6** (2005), no. 1, art. 22.
[2] O. P. Juneja, T. R. Reddy, and M. L. Mogra, *A convolution approach for analytic functions with negative coefficients*, Soochow J. Math. **11** (1985), 69–81.
[3] S. Owa and J. Nishiwaki, *Coefficient estimates for certain classes of analytic functions*, JIPAM. J. Inequal. Pure Appl. Math. **3** (2002), no. 5, art. 72.
[4] S. Owa and H. M. Srivastava, *Some generalized convolution properties associated with certain subclasses of analytic functions*, JIPAM. J. Inequal. Pure Appl. Math. **3** (2002), no. 3, art. 42.
[5] S. Ruscheweyh, *New criteria for univalent functions*, Proc. Amer. Math. Soc. **49** (1975), no. 1, 109–115.

[6] H. M. Srivastava and A. A. Attiya, *Some subordination results associated with certain subclasses of analytic functions*, JIPAM. J. Inequal. Pure Appl. Math. **5** (2004), no. 4, art. 82.

[7] B. A. Uralegaddi, M. D. Ganigi, and S. M. Sarangi, *Univalent functions with positive coefficients*, Tamkang J. Math. **25** (1994), no. 3, 225–230.

[8] H. S. Wilf, *Subordinating factor sequences for convex maps of the unit circle*, Proc. Amer. Math. Soc. **12** (1961), no. 5, 689–693.

R. K. Raina: Department of Mathematics, College of Technology and Engineering, M. P. University of Agriculture and Technology, Udaipur-313001, Rajasthan, India
E-mail address: rainark_7@hotmail.com

Deepak Bansal: Department of Mathematics, College of Science, M. L. Sukhadia University, Udaipur-313001, Rajasthan, India
E-mail address: deepakbansal_79@yahoo.com