COMPLETELY REGULAR FUZZIFYING TOPOLOGICAL SPACES

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Some of the properties of the completely regular fuzzifying topological spaces are investigated. It is shown that a fuzzifying topology τ is completely regular if and only if it is induced by some fuzzy uniformity or equivalently by some fuzzifying proximity. Also, τ is completely regular if and only if it is generated by a family of probabilistic pseudometrics.

1. Introduction

The concept of a fuzzifying topology was given in [1] under the name L-fuzzy topology. Ying studied in [9, 10, 11] the fuzzifying topologies in the case of $L = [0,1]$. A classical topology is a special case of a fuzzifying topology. In a fuzzifying topology τ on a set X, every subset A of X has a degree $\tau(A)$ of belonging to τ, $0 \le \tau(A) \le 1$. In [4], we defined the degrees of compactness, of local compactness, Hausdorffnes, and so forth in a fuzzifying topological space (X,τ). We also introduced the fuzzifying proximities. Every fuzzifying proximity δ induces a fuzzifying topology τ_δ. In [6], we studied the level classical topologies τ^θ, $0 \le \theta < 1$, corresponding to a fuzzifying topology τ. In the same paper, we studied connectedness and local connectedness in fuzzifying topological spaces as well as the so-called sequential fuzzifying topologies. In [5], we introduced the fuzzifying syntopogenous structures. We also proved that every fuzzy uniformity $\mathcal{U}$, as it is defined by Lowen in [7], induces a fuzzifying proximity $\delta_{\mathcal{U}}$, and that for every fuzzifying proximity δ, there exists at least one fuzzy uniformity $\mathcal{U}$ with $\delta = \delta_{\mathcal{U}}$. Some of the results contained in papers [4, 6] are closely related to those which appeared in the papers [12, 13].

In this paper, we continue with the investigation of fuzzifying topologies. In particular, we study the completely regular fuzzifying topologies, that is, those fuzzifying topologies τ for which each level topology τ^θ is completely regular. As in the classical case, we prove that for a fuzzifying topology τ on X, the following properties are equivalent: (1) τ is completely regular; (2) τ is uniformizable, that is, it is induced by some fuzzy uniformity; (3) τ is proximizable, that is, it is induced by some fuzzifying proximity; and (4) τ is generated by a family of so-called probabilistic pseudometrics on X. We also give a characterization of completely regular fuzzifying spaces in terms of continuous functions. Many Theorems on classical topologies follow as special cases of results obtained in the paper.

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2. Preliminaries

A fuzzifying topology on a set X (see [1, 9, 10, 11]) is a map $\tau : 2^X \to [0,1]$ (where 2^X is the power set of X) satisfying the following conditions:

(FT1) $\tau(X) = \tau(\varnothing) = 1$;

(FT2) $\tau(A_1 \cap A_2) \geq \tau(A_1) \wedge \tau(A_2)$;

(FT3) $\tau(\bigcup A_i) \geq \inf_i \tau(A_i)$.

If τ is a fuzzifying topology on X and $x \in X$, then the τ-neighborhood system of x is the function

$$N_x = N_x^\tau : 2^X \longrightarrow [0,1], \qquad N_x(A) = \sup\{\tau(B) : x \in B \subset A\}. \tag{2.1}$$

By [9, Theorem 3.2], we have that $\tau(A) = \inf_{x\in A} N_x(A)$.

The following theorem is contained in [9] (see also [3, 13]).

Theorem 2.1. *If τ is a fuzzifying topology on a set X, then the map $x \to N_x = N_x^\tau$, from X to the fuzzy power set $\mathcal{F}(2^X)$ of 2^X, has the following properties:*

(FN1) $N_x(X) = 1$ *and* $N_x(A) = 0$ *if* $x \notin A$;

(FN2) $N_x(A_1 \cap A_2) = N_x(A_1) \wedge N_x(A_2)$;

(FN3) $N_x(A) \leq \sup_{x\in D\subset A} \inf_{y\in D} N_y(D)$.

Conversely, if a map $x \to N_x$, from X to $\mathcal{F}(2^X)$, satisfies (FN1)–(FN3), *then the map*

$$\tau : 2^X \longrightarrow [0,1], \qquad \tau(A) = \inf_{x\in A} N_x(A), \tag{2.2}$$

is a fuzzifying topology and $N_x = N_x^\tau$ for every $x \in X$.

Let now (X,τ) be a fuzzifying topological space. To every subset A of X corrersponds a fuzzy subset $\bar{A} = \bar{A}^\tau$ of X defined by $\bar{A}(x) = 1 - N_x(A^c)$ (see [13, Remark 3.16]). A function f, from a fuzzifying topological space (X,τ_1) to another one (Y,τ_2), is said to be continuous at some $x \in X$ (see [4, 13]) if $N_x(f^{-1}(A)) \geq N_{f(x)}(A)$ for every subset A of Y. If f is continuous at every point of X, then it is said that (τ_1,τ_2)-continuous. As it is shown in [4], f is continuous if and only if $\tau_2(A) \leq \tau_1(f^{-1}(A))$ for every subset A of Y. For $f : X \to Y$ a function and τ a fuzzifying topology on Y, $f^{-1}(\tau)$ is defined to be the weakest fuzzifying topology on X for which f is continuous. By [4], $f^{-1}(\tau)$ is given by the neighborhood structure $N_x(A) = N_{f(x)}(Y \setminus f(A^c))$. If $(\tau_i)_{i\in I}$ is a family of fuzzifying topologies on X, we will denote by $\bigvee_{i\in I}\tau_i$, or by $\sup\tau_i$, the weakest of all fuzzifying topologies on X which are finer than each τ_i. As it is proved in [4], $\bigvee_{i\in I}\tau_i$ is given by the neighborhood structure

$$N_x(A) = \sup\left\{\inf_{i\in J} N_x^{\tau_i}(A_i) : x \in \bigcap_{i\in J} A_i \subset A\right\}, \tag{2.3}$$

where the infimum is taken over the family of all finite subsets J of I and all $A_i \subset X$, $i \in J$. For Y a subset of a fuzzifying topological space (X,τ), $\tau|_Y$ will be the fuzzifying topology induced on Y by τ, that is, the fuzzifying topology $f^{-1}(\tau)$, where $f : Y \to X$ is the inclusion map. For a family $(X_i,\tau_i)_{i\in I}$ of fuzzifying topological spaces, the product fuzzifying topology $\tau = \prod \tau_i$ on $X = \prod X_i$ is the weakest fuzzifying topology on X for which each projection $\pi_i : X \to X_i$ is continuous. Thus, $\tau = \bigvee_i \pi_i^{-1}(\tau_i)$ and it is given by the neighborhood structure

$$N_x(A) = \sup\left\{\inf_{i\in J} N_{x_i}(A_i) : x \in \bigcap_{i\in J} \pi_i^{-1}(A_i) \subset A\right\}, \tag{2.4}$$

where the supremum is taken over the family of all finite subsets J of I and $A_i \subset X_i$, for $i \in J$ (see [4]).

The degree of convergence to an $x \in X$, of a net (x_δ) in a fuzzifying topological space (X,τ), is the number $c(x_\delta \to x) = c^\tau(x_\delta \to x)$ defined by

$$c(x_\delta \longrightarrow x) = \inf\{1 - N_x(A) : A \subset X,\ (x_\delta) \text{ frequently in } A^c\}. \tag{2.5}$$

As it is shown in [6], for $A \subset X$ and $x \in X$, we have

$$\bar{A}(x) = \max\{c(x_\delta \longrightarrow x) : (x_\delta) \text{ net in } A\}. \tag{2.6}$$

The degree of Hausdorffness of X (see [4]) is defined by

$$T_2(X) = 1 - \sup_{x\neq y}\sup\{c(x_\delta \longrightarrow x) \wedge c(x_\delta \longrightarrow y) : (x_\delta) \text{ net in } X\}. \tag{2.7}$$

Also, the degree of X being T_1 is defined by

$$T_1(X) = \inf_x \inf_{y\neq x} \sup\{N_x(B) : y \notin B\}. \tag{2.8}$$

Let now (X,τ) be a fuzzifying topological space. For each $0 \le \theta < 1$, the family $B_\theta^\tau = \{A \subset X : \tau(A) > \theta\}$ is a base for a classical topology τ^θ on X (see [5]). It is easy to see that a subset B of X is a τ^θ-neighborhood of x if and only if $N_x(B) > \theta$. By [6], $T_2(X)$ (resp., $T_1(X)$) is the supremum of all $0 \le \theta < 1$ for which τ^θ is T_2 (resp., T_1). Also, for $\tau = \vee \tau_i$, we have that $\tau^\theta = \sup_i \tau_i^\theta$ (see [6, Theorem 3.5]). If $\tau = \prod \tau_i$ is a product fuzzifying topology, then $\tau^\theta = \prod \tau_i^\theta$ (see [6, Theorem 3.5]). If Y is a subspace of (X,τ) and $\tau_1 = \tau|Y$, then $\tau_1^\theta = \tau^\theta|Y$. By [6, Theorem 3.10], for a fuzzifying topological space (X,τ), $\mathrm{co}(X)$ coincides with the supremum of all $0 < \theta < 1$ for which $\tau^{1-\theta}$ is compact.

Next, we will recall the notion of a fuzzifying proximity given in [4]. A fuzzifying proximity on a set X is a map $\delta : 2^X \times 2^X \to [0,1]$ satisfying the following conditions:

(FP1) $\delta(A,B) = 1$ if the A, B are not disjoint;
(FP2) $\delta(A,B) = \delta(B,A)$;
(FP3) $\delta(\varnothing,B) = 0$;
(FP4) $\delta(A_1 \cup A_2,B) = \delta(A_1,B) \vee \delta(A_2,B)$;
(FP5) $\delta(A,B) = \inf\{\delta(A,D) \vee \delta(D^c,B) : D \subset X\}$.

Every fuzzifying proximity δ induces a fuzzifying topology τ_δ given by the neighborhood structure $N_x(A) = 1 - \delta(x, A^c)$. A fuzzifying proximity δ_1 is said to be finer than another one δ_2 if $\delta_1(A,B) \le \delta_2(A,B)$ for all subsets A, B of X. For $f : X \to Y$ a function and δ a fuzzifying proximity on Y, the function

$$f^{-1}(\delta) : 2^X \times 2^X \longrightarrow [0,1], \qquad f^{-1}(\delta)(A,B) = \delta\big(f(A), f(B)\big), \tag{2.9}$$

is a fuzzifying proximity on X (see [4]) and it is the weakest of all fuzzifying proximities δ_1 on X for which f is (δ_1,δ)-proximally continuous, that is, it satisfies $\delta_1(A,B) \le \delta(f(A), f(B))$ for all subsets A, B of X. As it is shown in [4], $\tau_{f^{-1}(\delta)} = f^{-1}(\tau_\delta)$.

Let now $(\delta_\lambda)_{\lambda\in\Lambda}$ be a family of fuzzifying proximities on a set X. We will denote by $\delta = \bigvee_\lambda \delta_\lambda$, or by $\sup \delta_\lambda$, the weakest fuzzifying proximity on X which is finer than each δ_λ. By [4, Theorem 8.10], δ is given by

$$\delta(A,B) = \inf\left\{\sup_{i,j}\inf_{\lambda\in\Lambda} \delta_\lambda(A_i, B_j)\right\}, \tag{2.10}$$

where the infimum is taken over all finite collections (A_i), (B_j) of subsets of X with $A = \bigcup A_i$, $B = \bigcup B_j$. Moreover, $\tau_\delta = \bigvee \tau_{\delta_\lambda}$ (see [4]).

Finally, we will recall the definition of a fuzzy uniformity introduced by Lowen in [7]. For a set X, let Ω_X be the collection of all functions $\alpha : X \times X \to [0,1]$ such that $\alpha(x,x) = 1$ for all $x \in X$. For $\alpha,\beta \in \Omega_X$, the $\alpha\wedge\beta$, $\alpha\circ\beta$ and α^{-1} are defined by $\alpha\wedge\beta(x,y) = \alpha(x,y)\wedge\beta(x,y)$, $\alpha\circ\beta(x,y) = \sup_z \beta(x,z)\wedge\alpha(z,y)$, $\alpha^{-1}(x,y) = \alpha(y,x)$. If $\alpha = \alpha^{-1}$, then α is called symmetric. A fuzzy uniformity on X is a nonempty subset $\mathcal{U}$ of Ω_X satisfying the following conditions.

(FU1) If $\alpha,\beta \in \mathcal{U}$, then $\alpha\wedge\beta \in \mathcal{U}$.

(FU2) If $\alpha \in \mathcal{U}$ is such that, for every $\epsilon > 0$, there exists a $\beta \in \mathcal{U}$ with $\beta \le \alpha + \epsilon$, then $\alpha \in \mathcal{U}$.

(FU3) For each $\alpha \in \mathcal{U}$ and each $\epsilon > 0$, there exists a $\beta \in \mathcal{U}$ with $\beta\circ\beta \le \alpha + \epsilon$.

(FU4) If $\alpha \in \mathcal{U}$, then $\alpha^{-1} \in \mathcal{U}$.

A subset $\mathcal{B}$, of a fuzzy uniformity $\mathcal{U}$, is a base for $\mathcal{U}$ if for each $\alpha \in \mathcal{U}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta \le \alpha + \epsilon$. It is easy to see that for a subset $\mathcal{B}$ of Ω_X, the following are equivalent.

(1) $\mathcal{B}$ is a base for a fuzzy uniformity on X.

(2) (a) If $\alpha,\beta \in \mathcal{B}$ and $\epsilon > 0$, then there exists $\gamma \in \mathcal{B}$ with $\gamma \le \alpha\wedge\beta + \epsilon$.

(b) For each $\alpha \in \mathcal{B}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta\circ\beta \le \alpha + \epsilon$.

(c) For each $\alpha \in \mathcal{B}$ and each $\epsilon > 0$, there exists $\beta \in \mathcal{B}$ with $\beta \le \alpha^{-1} + \epsilon$.

In case (2) is satisfied, the fuzzy uniformity $\mathcal{U}$ for which $\mathcal{B}$ is a base consists of all $\alpha \in \Omega_X$ such that for each $\epsilon > 0$, there exists a $\beta \in \mathcal{B}$ with $\beta \le \alpha + \epsilon$.

By [5], every fuzzy uniformity $\mathcal{U}$ on X induces a fuzzifying proximity $\delta_{\mathcal{U}}$ defined by

$$\delta_{\mathcal{U}}(A,B) = \inf_{\alpha\in\mathcal{U}} \sup_{x\in A, y\in B} \alpha(x,y). \tag{2.11}$$

In case $\mathcal{B}$ is a base for $\mathcal{U}$, then

$$\delta_{\mathcal{U}}(A,B) = \inf_{\alpha \in \mathcal{B}} \sup_{x \in A,\, y \in B} \alpha(x,y). \tag{2.12}$$

Every fuzzy uniformity $\mathcal{U}$ induces a fuzzifying topology $\tau_{\mathcal{U}}$ given by the neighborhood structure

$$N_x(A) = 1 - \delta_{\mathcal{U}}(x, A^c) = 1 - \inf_{\alpha \in \mathcal{U}} \sup_{y \notin A} \alpha(x,y). \tag{2.13}$$

For every fuzzifying proximity δ, there exists at least one compatible fuzzy uniformity, that is, a fuzzy uniformity $\mathcal{U}$ with $\delta_{\mathcal{U}} = \delta$ (see [5, Theorem 11.4]).

3. Probabilistic pseudometrics

A fuzzy real number is a fuzzy subset u of the real numbers $\mathbb{R}$ which is increasing, left continuous, and such that $\lim_{t\to+\infty} u(t) = 1$, $\lim_{t\to-\infty} u(t) = 0$. A fuzzy real number u is said to be nonnegative if $u(t) = 0$ if $t \le 0$. We will denote by $\mathbb{R}_\phi^+$ the collection of all nonnegative fuzzy real numbers. To every real number r corresponds a fuzzy real number $\bar{r}$, where $\bar{r}(t) = 0$ if $t \le r$ and $\bar{r}(t) = 1$ if $t > r$. For $u, v \in \mathbb{R}_\phi^+$, we define $u \preceq v$ if and only if $v(t) \le u(t)$ for all $t \in \mathbb{R}$. If $\mathcal{A}$ is a nonempty subset of $\mathbb{R}_\phi^+$ and if $u_o \in \mathbb{R}_\phi^+$ is defined by $u_o(t) = \sup_{v\in\mathcal{A}} v(t)$, then u_o is the biggest of all $u \in \mathbb{R}_\phi^+$ with $u \preceq v$ for all $v \in \mathcal{A}$. We will denote u_o by $\inf \mathcal{A}$ or by $\bigwedge \mathcal{A}$. For $u_1, u_2 \in \mathbb{R}_\phi^+$, we define $u = u_1 \oplus u_2 \in \mathbb{R}_\phi^+$ by $u(t) = \sup\{u_1(t_1) \wedge u_2(t_2) : t = t_1 + t_2\}$. Also, for $u \in \mathbb{R}_\phi^+$ and $\lambda > 0$, we define λu by $(\lambda u)(t) = u(\lambda^{-1} t)$. It is easy to see that for $u \in \mathbb{R}_\phi^+$ and $\lambda > 0$, we have $(\bar{\lambda} \oplus u)(t) = u(t - \lambda)$.

Definition 3.1. A probabilistic pseudometric on a set X (see [2]) is a mapping $F : X \times X \to \mathbb{R}_\phi^+$ such that for all x, y, $z \in X$,

$$F(x,x) = \bar{0}, \qquad F(x,y) = F(y,x), \qquad F(x,z) \preceq F(x,y) \oplus F(y,z). \tag{3.1}$$

If in addition $F(x,y)(0+) = 0$ when $x \ne y$, then F is called a probabilistic metric.

If r_1, r_2 are nonnegative real numbers, then $\overline{r_1} \preceq \overline{r_2}$ if and only if $r_1 \le r_2$. Also, for $r = |r_1 - r_2|$, we have that

$$\bar{r} = \wedge\{u \in \mathbb{R}_\phi^+ : \bar{r_2} \preceq u \oplus \overline{r_1},\ \overline{r_1} \preceq u \oplus \bar{r_2}\}. \tag{3.2}$$

In fact, let $u_o = \wedge\{u \in \mathbb{R}_\phi^+ : \bar{r_2} \preceq u \oplus \overline{r_1}$ and $\overline{r_1} \preceq u \oplus \overline{r_2}\}$ and assume that (say) $r_1 \ge r_2$. Let $u \in \mathcal{R}_\phi^+$ be such that $\bar{r_2} \preceq u \oplus \overline{r_1}$, $\overline{r_1} \preceq u \oplus \overline{r_2}$. Then $\overline{r_1}(t) \ge (u \oplus \overline{r_2})(t) = u(t - r_2)$ for all t. If $s < r_1$, then $0 = \overline{r_1}(s) \ge u(s - r_2)$ and so $u(r_1 - r_2) = \sup_{s<r_1} u(s - r_2) = 0$ which implies that $\bar{r} \preceq u$. Thus $\bar{r} \preceq u_o$. On the other hand, we have $\bar{r} \oplus \overline{r_2} = \overline{r_1}$ and $\bar{r} \oplus \overline{r_1} = \overline{2r_1 - r_2}$. Since $\overline{r_2} \preceq \overline{2r_1 - r_2}$, it follows that $u_o \preceq \bar{r}$, and hence $\bar{r} = u_o$. Motivated by the above, we define the following distance function on $\mathcal{R}_\phi^+$:

$$D : \mathcal{R}_\phi^+ \times \mathcal{R}_\phi^+ \longrightarrow \mathcal{R}_\phi^+, \qquad D(u_1, u_2) = \wedge\{u \in \mathcal{R}_\phi^+ : u_1 \preceq u_2 \oplus u,\ u_2 \preceq u \oplus u_1\}. \tag{3.3}$$

Then D is a probabilistic pseudometric on $\mathscr{R}_\phi^+$. In fact, it is clear that $D(u_1,u_2) = D(u_2,u_1)$. Also, since $u = u \oplus \bar{0}$, when $u \in \mathscr{R}_\phi^+$, we have that $D(u,u) = \bar{0}$. Finally, let $D(u_1,u_2)(t_1) \wedge D(u_2,u_3)(t_2) > \theta > 0$. There are $v_1, v_2 \in \mathscr{R}_\phi^+$ with $u_1 \preceq v_1 \oplus u_2$, $u_2 \preceq v_1 \oplus u_1$, $u_3 \preceq v_2 \oplus u_2$, $u_2 \preceq v_2 \oplus u_3$, $v_1(t_1) > \theta$, $v_2(t_2) > \theta$. Now $u_1 \preceq v_1 \oplus u_2 \preceq v_1 \oplus (v_2 \oplus u_3) = (v_1 \oplus v_2) \oplus u_3$ and $u_3 \preceq v_2 \oplus u_2 \preceq v_2 \oplus (v_1 \oplus u_1) = (v_1 \oplus v_2) \oplus u_1$. Thus, $D(u_1,u_3) \preceq v_1 \oplus v_2$ and $D(u_1,u_3)(t_1+t_2) \geq v_1(t_1) \wedge v_2(t_2) > \theta$. This proves that $D(u_1,u_3) \preceq D(u_1,u_2) \oplus D(u_2,u_3)$ and the claim follows. We will refer to D as the usual probabilistic pseudometric on $\mathscr{R}_\phi^+$.

Let now F be a probabilistic pseudometric on X. For $t > 0$, let $u_{F,t}$ be defined on X^2 by $u_{F,t}(x,y) = F(x,y)(t)$. The family $\mathscr{B}_F = \{u_{F,t} : t > 0\}$ is a base for a fuzzy uniformity $\mathscr{U}_F$ on X. Let τ_F be the fuzzifying topology induced by $\mathscr{U}_F$.

In the rest of the paper, we will consider on $\mathscr{R}_\phi^+$ the fuzzifying topology induced by the usual probabilistic pseudometric D.

Theorem 3.2. *A probabilistic pseudometric F, on a fuzzifying topological space (X,τ), is $\tau \times \tau$ continuous if and only if $\tau_F \leq \tau$.*

Proof. Assume that $\tau_F \leq \tau$ and let G be a subset of $\mathbb{R}_\phi^+$ and $u = F(x_o,y_o)$ with $N_u(G) > \theta > 0$. There exists a $t > 0$ such that $1 - \sup_{v \notin G} D(v,u)(t) > \theta$. For $x, y \in X$, we have

$$F(x,y) \preceq F(x,x_o) \oplus (x_o,y_o) \oplus F(y_o,y) = [F(x,x_o) \oplus F(y,y_o)] \oplus F(x_o,y_o). \tag{3.4}$$

Similarly, $F(x_o,y_o) \preceq [F(x,x_o) \oplus F(y,y_o)] \oplus F(x,y)$. Thus,

$$D(F(x,y),F(x_o,y_o)) \preceq F(x,x_o) \oplus F(y,y_o). \tag{3.5}$$

Let

$$A_1 = \left\{x \in X : F(x,x_o)\left(\frac{t}{2}\right) \geq 1-\theta\right\}, \qquad A_2 = \left\{x \in X : F(y,y_o)\left(\frac{t}{2}\right) \geq 1-\theta\right\}. \tag{3.6}$$

If $x \in A_1$, $y \in A_2$, then

$$D(F(x,y),F(x_o,y_o))(t) \geq F(x,x_o)\left(\frac{t}{2}\right) \wedge F(y,y_o)\left(\frac{t}{2}\right) \geq 1-\theta, \tag{3.7}$$

and so $F(x,y) \in G$. Also, $N_{x_o}^{\tau}(A_1) \geq N_{x_o}^{\tau_F}(A_1) \geq 1 - \sup_{x \notin A_1} F(x,x_o)(t/2) \geq \theta$ and $N_{y_o}^{\tau}(A_2) \geq \theta$. Therefore,

$$N_{(x_o,y_o)}^{\tau\times\tau}(F^{-1}(G)) \geq N_{x_o}^{\tau}(A_1) \wedge N_{y_o}^{\tau}(A_1) \geq \theta, \tag{3.8}$$

which proves that $N_{(x_o,y_o)}^{\tau\times\tau}(F^{-1}(G)) \geq N_{F(x_o,y_o)}(G)$ and so F is $\tau \times \tau$ continuous. Conversely, assume that F is $\tau \times \tau$ continuous and let $N_{x_o}^{\tau_F}(A) > \theta > 0$. Choose $\epsilon > 0$ such that $N_{x_o}^{\tau_F}(A) > \theta + \epsilon$. There exists a $t > 0$ such that $1 - \sup_{x \notin A} F(x,x_o)(t) > \theta + \epsilon$. If

$$Z = \{u \in \mathbb{R}_\phi^+ : D(u,\bar{0})(t) = u(t) > 1-\theta-\epsilon\}, \tag{3.9}$$

then

$$N_{\bar{0}}(Z) \geq 1 - \sup_{u \notin Z} D(u,\bar{0})(t) \geq \theta + \epsilon > \theta. \tag{3.10}$$

Since F is $\tau \times \tau$ continuous and $F(x_o,x_o) = \bar{0}$, there exists a subset A_1 of X containing x_o such that $A_1 \times A_1 \subset F^{-1}(Z)$ and $N_{x_o}(A_1) > \theta$. If $x \in A_1$, then $F(x,x_o) \in Z$ and so $F(x,x_o)(t) > 1-\theta-\epsilon$, which implies that $x \in A$. Thus, $A_1 \subset A$ and so $N_{x_o}(A) \ge N_{x_o}^{\tau_F}(A)$ for every subset A of X and every $x_o \in X$. Hence, $\tau_F \le \tau$ and the result follows. □

THEOREM 3.3. *Let F be a probabilistic pseudometric on a set X, $\tau = \tau_F$, $(x_\delta)_{\delta\in\Delta}$ a net in X, and $x \in X$. Then*

$$c(x_\delta \longrightarrow x) = \inf_{t>0} \liminf_{\delta} F(x_\delta,x)(t). \tag{3.11}$$

Proof. Let $d = \inf_{t>0}\liminf_\delta F(x_\delta,x)(t)$ and assume that $d < \theta < 1$. There exists a $t > 0$ such that $\liminf_\delta F(x_\delta,x)(t) < \theta$. Let $A = \{y : F(y,x)(t) > \theta\}$. Then (x_δ) is not eventually in A, and so $c(x_\delta \to x) \le 1 - N_x(A) \le \sup_{y\notin A} F(y,x)(t) \le \theta$, which proves that $c(x_\delta \to x) \le d$. On the other hand, let $c(x_\delta \to x) < r < 1$. There exists a subset B of X such that (x_δ) is not eventually in B and $1 - N_x(B) < r$. Let $s > 0$ be such that $1 - \sup_{y\notin B} F(y,x)(s) > 1 - r$. For each $\delta \in \Delta$, there exists $\delta' \ge \delta$ with $x_{\delta'} \notin B$, and so $F(x_{\delta'},x)(s) \le \sup_{y\notin B} F(y,x)(s)$. Thus, $d \le \liminf_\delta F(x_\delta,x)(s) < r$, which proves that $d \le c(x_\delta \to x)$ and the result follows. □

THEOREM 3.4. *Let $F_1,F_2,\ldots,F_n$ be probabilistic pseudometrics on X and define F by*

$$F(x,y)(t) = \min_{1\le k\le n} F_k(x,y)(t). \tag{3.12}$$

Then F is a probabilistic pseudometric and $\tau_F = \bigvee_{k+1}^{n} \tau_{F_k}$.

Proof. Using induction on n, it suffices to prove the result in the case of $n = 2$. It follows easily that F is a probabilistic pseudometric. Since F_1, $F_2 \preceq F$, it follows that τ_{F_1}, $\tau_{F_2} \le \tau_F$ and so $\tau_o = \tau_{F_1} \vee \tau_{F_2} \le \tau_F$. On the other hand, let $N_x^{\tau_F}(A) > \theta > 0$. There exists a $t > 0$ such that $1 - \sup_{y\notin A} F(y,x)(t) > \theta$. Let $B_i = \{y \in A^c : F_i(y,x)(t) < 1-\theta\}$, $i = 1,2$. Then $A^c = B_1 \cup B_2$ and so $A = A_1 \cap A_2$, $A_i = B_i^c$. Moreover $N_x^{\tau_{F_i}}(A_i) \ge 1 - \sup_{y\in B_i} F_i(y,x)(t) \ge \theta$, and thus

$$N_x^{\tau_o}(A) \ge N_x^{\tau_o}(A_1) \bigwedge N_x^{\tau_o}(A_2) \ge N_x^{\tau_{F_1}}(A_1) \bigwedge N_x^{\tau_{F_2}}(A_2) \ge \theta. \tag{3.13}$$

This proves that $N_x^{\tau_o}(A) \ge N_x^{\tau_F}(A)$ and the result follows. □

For $\mathcal{F}$ a family of probabilistic pseudometrics on a set X, we will denote by $\tau_{\mathcal{F}}$ the supremum of the fuzzifying topologies τ_F, $F \in \mathcal{F}$, that is, $\tau_{\mathcal{F}} = \bigvee_{F\in\mathcal{F}} \tau_F$.

THEOREM 3.5. *If $\tau = \tau_{\mathcal{F}}$, where $\mathcal{F}$ is a family of probabilistic pseudometrics on a set X, then $T_2(X) = T_1(X) = 1 - \sup_{y\neq x}\inf_{F\in\mathcal{F}} F(x,y)(0+)$.*

Proof. Let $d = 1 - \sup_{y\neq x}\inf_{F\in\mathcal{F}} F(x,y)(0+)$. It is always true that $T_2(X) \le T_1(X)$. Suppose that $T_1(X) > r > 0$ and let $x \neq y$. Since τ^r is T_1, there exists a τ^r-neighborhood A of x not containing y. Now $N_x(A) > r$, and hence there are subsets $A_1,\ldots,A_n$ of X and $F_1,\ldots,F_n \in \mathcal{F}$ such that $\bigcap A_k \subset A$, $N_x^{\tau_{F_k}}(A_k) > r$. Since y is not in A, there exists a k with

$y \notin A_k$. Let $t > 0$ be such that

$$1 - \sup_{z \notin A_k} F_k(z,x)(t) > r \text{ and so } \inf_{F \in \mathcal{F}} F(x,y)(t)(0+) \le F_k(x,y)(t) < 1 - r, \tag{3.14}$$

which proves that $d \ge r$. Thus $d \ge T_1(X)$. On the other hand, assume that $d > \theta > 0$ and let $x \ne y$. Choose $\epsilon > 0$ such that $d > \theta + \epsilon$. There exists $F \in \mathcal{F}$ with $F(x,y)(0+) < 1 - \theta - \epsilon$, and hence $F(x,y)(t) < 1 - \theta - \epsilon$ for some $t > 0$. Let

$$A = \left\{ z : F(z,x)\left(\frac{t}{2}\right) > 1 - \theta - \epsilon \right\}, \qquad B = \left\{ z : F(z,y)\left(\frac{t}{2}\right) > 1 - \theta - \epsilon \right\}. \tag{3.15}$$

Clearly $x \in A$, $y \in B$. If $z \in A \cap B$, then

$$F(x,y)(t) \ge F(x,z)\left(\frac{t}{2}\right) \wedge F(z,y)\left(\frac{t}{2}\right) > 1 - \theta - \epsilon, \tag{3.16}$$

a contradiction. Thus $A \cap B = \varnothing$. Moreover

$$N_x(A) \ge N_x^{\tau_F}(A) \ge 1 - \sup_{z \notin A} F(x,z)\left(\frac{t}{2}\right) \ge \theta + \epsilon > \theta, \qquad N_y(A) > \theta. \tag{3.17}$$

It follows that $T_2(X) \ge d$ and the proof is complete. □

Let us say that a fuzzifying topology τ on a set X is pseudometrizable if there exists a probabilistic pseudometric F on X with $\tau = \tau_F$.

Theorem 3.6. *A fuzzifying topology τ on X is pseudometrizable if and only if each level topology τ^θ, $0 \le \theta < 1$, is pseudometrizable.*

Proof. Assume that $\tau = \tau_F$ for some probabilistic pseudometric F and let $0 \le \theta < 1$. For each positive integer n, with $n > 1/(1-\theta)$, let

$$A_n = \left\{ (x,y) \in X^2 : F(x,y)\left(\frac{1}{n}\right) > 1 - \theta - \frac{1}{n} \right\}. \tag{3.18}$$

Then $A_{n+1} \subset A_n$ and the family $\mathcal{D} = \{A_n : n \in \mathbb{N},\ n > 1/(1-\theta)\}$ is a base for a uniformity $\mathcal{U}$ on X. The topology σ_θ induced by $\mathcal{U}$ is pseudometrizable since $\mathcal{D}$ is countable. Moreover $\sigma_\theta = \tau^\theta$. Indeed, let A be a σ_θ-neighborhood of x. There exists $n \in \mathbb{N}$, $n > 1/(1-\theta)$, such that $B = \{y : F(x,y)(1/n) > 1 - \theta - 1/n\} \subset A$. Now

$$N_x^\tau(A) \ge N_x^\tau(B) \ge 1 - \sup_{y \notin B} F(x,y)\left(\frac{1}{n}\right) \ge \theta + \frac{1}{n} > \theta, \tag{3.19}$$

and so A is a τ^θ-neighborhood of x. Conversely, assume that A is a τ^θ-neighborhood of x. There exists $\epsilon > 0$ with $N_x(A) > \theta + \epsilon$. Now there exists a positive integer $n > 1/\epsilon$ such that $1-\sup_{y\notin A}F(x,y)(1/n)>\theta+1/n$. Hence

$$\left\{y:F(x,y)\left(\frac{1}{n}\right)>1-\theta-\frac{1}{n}\right\}\subset A, \tag{3.20}$$

which implies that A is a σ_θ-neighborhood of x. Thus $\tau^\theta=\sigma_\theta$, and therefore each τ^θ is pseudometrizable. Conversely, suppose that each τ^θ is pseudometrizable. By an argument analogous to the one used in the proof of [6, Theorem 3.3], we show that there exists a family $\{d_\theta:0\le\theta<1\}$ of pseudometrics on X such that $d_\theta=\sup_{\theta_1>\theta}d_{\theta_1}$, for each $0\le\theta<1$, and τ^θ coincides with the topology induced by the pseudometric d_θ. Now, for $x,y\in X$, define $F(x,y):\mathbb{R}\to[0,1]$ by $F(x,y)(t)=0$ if $t\le 0$ and $F(x,y)(t)=\sup\{\theta: 0<\theta\le 1, d_{1-\theta}(x,y)<t\}$ if $t>0$. It is clear that $F(x,y)$ is increasing and left continuous. For $0<r<1$ and $t>d_{1-r}(x,y)$, we have that $F(x,y)(t)\ge r$, and so $\lim_{t\to\infty}F(x,y)(t)=1$. Also $F(x,x)(t)=1$ for every x and every $t>0$. To show that F is a probabilistic pseudometric on X, we must prove that it satisfies the triangle inequality. So, let $F(x,y)(t_1)\wedge F(y,z)(t_2)>\theta>0$. Then $d_{1-\theta}(x,y)<t_1$, $d_{1-\theta}(y,z)<t_2$, and so $d_{1-\theta}(x,z)<t_1+t_2$, which implies that $F(x,z)(t_1+t_2)\ge\theta$. Thus the triangle inequality is satisfied and F is a probabilistic pseudometric. We will finish the proof by showing that $\tau_F=\tau$. So let $N_x^{\tau_F}>\theta>0$ and choose $t>0$ such that $1-\sup_{y\notin A}F(y,x)(t)>\theta$. If now $d_\theta(x,y)<t$, then $F(x,y)(t)\ge 1-\theta$, and thus $y\in A$, which proves that A is a $\sigma_\theta=\tau^\theta$ neighborhood of x. Hence $\tau\ge\tau_F$. On the other hand, let B be a τ^θ-neighborhood of x. There exists $\theta_1>\theta$ such that $N_x(B)>\theta_1$. Now B is a τ_{θ_1}-neighborhood of x, and so there exists $t>0$ such that $\{y:d_{\theta_1}(x,y)<t\}\subset B$. If $F(x,y)(t)>1-\theta_1$, then there exists $\alpha>1-\theta_1$ such that $d_{1-\alpha}(x,y)<t$ and so $d_{\theta_1}(x,y)<t$. Thus $\{y:F(x,y)(t)>1-\theta_1\}\subset B$, and therefore

$$N_x^{\tau_F}(B)\ge 1-\sup_{y\notin B}F(x,y)(t)\ge\theta_1>\theta. \tag{3.21}$$

Thus, $\tau_F\ge\tau$ and the result follows. □

Theorem 3.7. *Let (X,F) be a probabilistic pseudometric space, $A\subset X$, and $x\in X$. Let*

$$\begin{aligned}\alpha&=\sup\left\{\inf_{t>0}\liminf_n F(x_n,x)(t):(x_n)\ \textit{sequence in } A\right\},\\ \beta&=\sup\left\{\liminf_n F(x_n,x)(t_n):t_n\longrightarrow 0+,\ (x_n)\ \textit{sequence in } A\right\},\\ \gamma&=\sup\left\{\liminf_n F(x_n,x)(1/n):(x_n)\ \textit{sequence in } A\right\}.\end{aligned} \tag{3.22}$$

Then $\alpha=\beta=\gamma=\bar{A}(x)$.

Proof. If $(x_n)\subset A$, then

$$\bar{A}(x)\ge c(x_n\longrightarrow x)=\inf_{t>0}\liminf_n F(x_n,x)(t), \tag{3.23}$$

and so $\bar{A}(x)\ge\alpha$. Assume that $\beta>\theta>0$. There exist a sequence $(x_n)\in A$ and a sequence (t_n) of positive real numbers, with $t_n\to 0+$, such that $\liminf_n F(x_n,x)(t_n)>\theta$. Let $t>0$

and choose k such that $t_n < t$ when $n \geq k$. For $m \geq k$, we have $\inf_{n>m} F(x_n,x)(t) \geq \inf_{n\geq m} F(x_n,x)(t_n) > \theta$. Thus $\liminf_n F(x_n,x)(t) > \theta$ for each $t > 0$ and so $\alpha \geq \theta$, which proves that $\alpha \geq \beta$. Clearly $\beta \geq \gamma$. Finally, $N_x(A^c) \geq 1 - sup_{y\in A} F(y,x)(1/n)$, and so $\sup_{y\in A} F(y,x)(1/n) \geq 1 - N_x(A^c) = \bar{A}(x) > \bar{A}(x) - 1/n$. Hence, for each $n \in \mathbb{N}$, there exists $x_n \in A$ with $F(x_n,x)(1/n) > \bar{A}(x) - 1/n$. Consequently,

$$\gamma \geq \liminf_n F(x_n,x)\left(\frac{1}{n}\right) \geq \liminf_n \left(\bar{A}(x) - \frac{1}{n}\right) = \bar{A}(x), \tag{3.24}$$

and so $\gamma \geq \bar{A}(x) \geq \alpha \geq \beta \geq \gamma$, which completes the proof. $\square$

In view of [6, Theorem 4.14], we have the following corollary.

Corollary 3.8. *Every pseudometrizable fuzzifying topological space is $\mathbb{N}$-sequential and hence sequential.*

Theorem 3.9. *If (F_n) is a sequence of probabilistic pseudometrics on a set X, then there exists a probabilistic pseudometric F such that $\tau_F = \bigvee_n \tau_{F_n}$.*

Proof. If F is a probabilistic pseudometric on X and if $\bar{F}$ is defined by $\bar{F}(x,y)(t) = F(x,y)(t)$ if $t \leq 1$ and $\bar{F}(x,y)(t) = 1$ if $t > 1$, then $\bar{F}$ is a probabilistic pseudometric on X and $\tau_{\bar{F}} = \tau_F$. Hence, we may assume that $F_n(x,y)(t) = 1$, for all n, if $t > 1$. For $x,y\in X$, define $F(x,y)$ on $\mathbb{R}$ by $F(x,y)(t) = 0$ if $t \leq 0$ and $F(x,y)(t) = \inf_n[(1/n)F_n(x,y)](t)$ if $t > 0$. Clearly, $F(x,y)$ is increasing and $F(x,y)(t) = 1$ if $t > 1$. Also, $F(x,y)$ is left continuous. In fact, let $F(x,y)(t) > \theta > 0$ and choose n such that $(n+1)t > 1$. There exists $0 < s_1 < t$ such that $F_k(x,y)(ks_1) > \theta$ for $k = 1,\ldots,n$. Choose $s_1 < s < t$ such that $(n+1)s > 1$. Now, $F_m(x,y)(ms) = 1$ if $m > n$. Thus

$$F(x,y)(s) = \min_{1\leq k\leq n}\left[\frac{1}{k}F_k(x,y)\right](s) > \theta, \tag{3.25}$$

which proves that $F(x,y)$ is in $\mathbb{R}^+_\phi$. It is clear that $F(x,x) = \bar{0}$. We need to prove that F satisfies the triangle inequality. So assume that $F(x,y)(t_1) \wedge F(y,z)(t_2) > \theta > 0$. If m is such that $(m+1)(t_1+t_2) > 1$, then

$$F(x,z)(t_1+t_2) = \min_{1\leq k\leq m} F_k(x,z)(k(t_1+t_2)). \tag{3.26}$$

Since

$$F_k(x,z)(k(t_1+t_2)) \geq F_k(x,y)(kt_1) \wedge F_k(y,z)(kt_2) > \theta, \tag{3.27}$$

it follows that $F(x,z)(t_1+t_2) > \theta$, and so F satisfies the triangle inequality. We will finish the proof by showing that $\tau_F = \bigvee \tau_{F_n}$. To see this, we first observe that $(1/n)F_n \preceq F$, which implies that $\tau_{F_n} = \tau_{(1/n)F_n} \leq \tau_F$, and so $\tau_o = \bigvee_n \tau_{(1/n)F_n} \leq \tau_F$. On the other hand, let $N_x^{\tau_F}(A) > \theta$ and choose $\epsilon > 0$ such that $N_x^{\tau_F}(A) > \theta + \epsilon$. Let $t > 0$ be such that $1 - \sup_{y\notin A} F(y,x)(t) > \theta + \epsilon$. If $(m+1)t > 1$, then

$$F(y,z)(t) = \min_{1\leq k\leq m} F_k(y,z)(kt). \tag{3.28}$$

Let $A_k = \{y : F_k(y,x)(kt) \geq 1 - \theta - \epsilon\}$. Then

$$N_x^{\tau_o}(A_k) \geq N_x^{\tau_{F_k}}(A_k) \geq 1 - \sup_{z \notin A_k} F_k(z,x)(kt) \geq \theta + \epsilon > \theta \tag{3.29}$$

and $\bigcap_{k=1}^m A_k \subset A$. Hence, $N_x^{\tau_o}(A) \geq \min_{1 \leq k \leq m} N_x^{\tau_o}(A_k) > \theta$. This proves that $\tau_F \leq \tau_o$ and the result follows. □

Theorem 3.10. *Let $f : X \to Y$ be a function and let F be a probabilistic pseudometric on Y. Then the function*

$$f^{-1}(F) : X^2 \longrightarrow \mathbb{R}_\phi^{+1}, \qquad f^{-1}(F)(x,y) = F(f(x), f(y)), \tag{3.30}$$

is a probabilistic pseudometric on X and $\tau_{f^{-1}(F)} = f^{-1}(\tau_F)$.

Proof. It follows easily that $f^1(F)$ is a probabilistic pseudometric on X. Let $x \in X$ and $B \subset X$. If $D = Y \setminus f(B^c)$, then

$$\begin{aligned} N_x^{\tau_{f^{-1}(F)}}(B) &= \inf_{t>0} \left[1 - \sup_{y \notin B} F(f(y), f(x))(t) \right] \\ &= \inf_{t>0} \left[1 - \sup_{z \in D^c} F(z, f(x))(t) \right] \\ &= N_{f(x)}^{\tau_F}(D) = N_x^{f^{-1}(\tau_F)}(B), \end{aligned} \tag{3.31}$$

which clearly completes the proof. □

Corollary 3.11. *If F is a probabilistic pseudometric on a set X and $Y \subset X$, then $\tau_F|_Y$ is induced by the probabilistic pseudometric $G = F|_{Y \times Y}$, $G(x,y) = F(x,y)$.*

Corollary 3.12. *If (X_n, τ_n) is a sequence of pseudometrizable fuzzifying topological spaces, then the Cartesian product $(X, \tau) = (\prod X_n, \prod \tau_n)$ is pseudometrizable.*

Proof. Let F_n be a probabilistic pseudometric on X_n inducing τ_n. If $G_n = \pi_n^{-1}(F_n)$, then $\tau_{G_n} = \pi_n^{-1}(\tau_n)$, and so $\tau = \bigvee_n \pi_n^{-1}(\tau_n)$ is pseudometrizable. □

4. Level proximities

Let δ be a fuzzifying proximity on a set X. For each $0 < d \leq 1$, let δ^d be the binary relation on 2^X defined by $A \delta^d B$ if and only if $\delta(A,B) \geq d$. It is easy to see that δ^d is a classical proximity on X. We will show that the classical topology σ_d induced by δ^d coincides with τ^{1-d}. In fact, let $x \in A \in \sigma_d$. Then, x is not in the σ_d-closure of A^c, which implies that $x \not\delta^d A^c$, that is, $\delta(x, A^c) < d$, and so $N_x^\tau(A) = 1 - \delta(x, A^c) > 1 - d$. This proves that $A \in \tau^{1-d}$. Conversely, if $x \in B \in \tau^{1-d}$, then $N_x^\tau(A) > 1 - d$, and thus $\delta(x, A^c) < d$, which implies that x is not in the σ_d-closure of B^c. Hence B^c is σ_d-closed, and so B is σ_d-open.

Theorem 4.1. *If δ is a fuzzifying proximity on a set X and $0 < d \leq 1$, then*

$$\delta^d = \bigvee_{0<\theta<d} \delta^\theta. \tag{4.1}$$

Proof. If $0 < \theta < d$, then δ^θ is coarser than δ^d, and so $\delta_o = \bigvee_{0<\theta<d} \delta^\theta$ is coarser than δ^d. On the other hand, let $A\delta_o B$. Since δ_o is finer than δ^θ (for $0 < \theta < d$), we have that $A\delta^\theta B$ and so $\delta(A,B) \geq \theta$, for each $0 < \theta < d$, which implies that $\delta(A,B) \geq d$, that is, $A\delta^d B$. So δ_o is finer than δ^d and the result follows. □

Theorem 4.2. *For a family $\{\gamma_d : 0 < d \leq 1\}$ of classical proximities on a set X, the following are equivalent.*

(1) *There exists a fuzzifying proximity δ on X such that $\delta^d = \gamma_d$ for all d.*

(2) *$\gamma_d = \bigvee_{0<\theta<d} \gamma_\theta$ for each $0 < d \leq 1$.*

Proof. In view of the preceding theorem, (1) implies (2). Assume now that (2) is satisfied and define δ on $2^X \times 2^X$ by $\delta(A,B) = \sup\{d : A\gamma_d B\}$ (the supremum over the empty family is taken to be zero). It is clear that $\delta(A,B) = 1$ if the A,B are not disjoint. Also, $\delta(A,B) = \delta(A,B)$ and $\delta(A,B \geq \delta(A_1,B_1)$ if $A_1 \subset A$, $B_1 \subset B$. Let now $\delta(A,B) < d < 1$. Then $A \not\gamma_d B$, and so there exists a subset D of X such that $A \not\gamma_d D$ and $D^c \not\gamma_d B$. Since $A \not\gamma_d D$, we have that $\delta(A,D) \leq d$. Similarly $\delta(D^c,B) \leq d$, and so $\inf\{\delta(A,D) \wedge \delta(D^c,B)\} \leq \delta(A,B)$. On the other hand, if $\delta(A,D) \wedge \delta(D^c,B) < \theta < 1$, then $A \subset D^c$, and so $\delta(A,B) \leq \delta(D^c,B) < \theta$. This proves that δ is a fuzzifying proximity on X. We will finish the proof by showing that $\delta^d = \gamma_d$ for all d. Indeed, if $A\gamma_d B$, then $\delta(A,B) \geq d$, that is, $A\delta^d B$. On the other hand, let $A\delta^d B$ and let (A_i), (B_j) be finite families of subsets of X with $A = \bigcup_i$, $B = \bigcup B_j$. Since $\delta(A,B) = \bigvee_{i,j} \delta(A_i,B_j) \geq d$, there exists a pair (i,j) such that $\delta(A_i,B_j) \geq d$. If now $0 < \theta < d$, then there exists $r > \theta$ with $A_i\gamma_r B_j$, and so $A_i\gamma_\theta B_j$. This proves that $A\gamma_d B$ since $\gamma_d = \bigvee_{0<\theta<d} \gamma_\theta$. This completes the proof. □

Theorem 4.3. *Let (X,δ_1), $(Y\delta_2)$ be fuzzifying proximity spaces and let $f : X \to Y$ be a function. Then f is proximally continuous if and only if $f : (X,\delta_1^d) \to (Y,\delta_2^d)$ is proximally continuous for each $0 < d \leq 1$.*

Proof. It follows immediately from the definitions. □

Theorem 4.4. *Let $(X_\lambda,\delta_\lambda)_{\lambda\in\Lambda}$ be a family of fuzzifying proximity spaces and let $(X,\delta) = (\prod X_\lambda, \prod \delta_\lambda)$ be the product fuzzifying proximity space. Then $\delta^d = \prod \delta_\lambda^d$ for all $0 < d \leq 1$.*

Proof. Since each projection $\pi_\lambda : (X,\delta^d) \to (X_\lambda,\delta_\lambda^d)$ is proximally continuous, it follows that δ^d is finer than $\sigma = \prod \delta_\lambda^d$. On the other hand, let $A\sigma B$. We need to show that $\delta(A,B) \geq d$. In fact, let (A_i), (B_j) be finite families of subsets of X such that $A = \bigcup A_i$, $B = \bigcup B_j$. Since $A\sigma B$ and $\sigma = \bigvee_\lambda \pi_\lambda^{-1}(\delta_\lambda^d)$, there exists a pair (i,j) such that $A_i\pi_\lambda^{-1}(\delta^d)B_j$, that is, $\delta_\lambda(\pi_\lambda(A_i),\pi_\lambda(B_j)) \geq d$. In view of [4, Theorem 8.9], we conclude that $\delta(A,B) \geq d$. Hence, $\sigma = \delta^d$ and the proof is complete. □

We have the following easily established theorem.

Theorem 4.5. *Let (Y,δ) be a fuzzifying proximity space and let $f : X \to Y$. Then $f^{-1}(\delta)^d = f^{-1}(\delta^d)$ for each $0 < d \leq 1$.*

Theorem 4.6. *Let $(\delta_\lambda)_{\lambda\in\Lambda}$ be a family of fuzzifying proximities on a set X and $\delta = \vee_\lambda \delta_\lambda$. Then $\delta^d = \bigvee_\lambda \delta_\lambda^d$ for each $0 < d \le 1$.*

Proof. Let $\sigma = \bigvee_\lambda \delta_\lambda^d$. Since δ is finer than each δ_λ, it follows that δ^d is finer than each δ_λ^d, and so δ^d is finer than σ. On the other hand, let $A\sigma B$ and let (A_i), (B_j) be finite families of subsets of X such that $A = \bigcup A_i$, $B = \bigcup B_j$. There exists a pair (i,j) such that $A_i\sigma B_j$. Since σ is finer than each δ_λ^d, we have that $A_i\delta_\lambda^d B_j$, that is, $\delta_\lambda(A_i,B_j) \ge d$. In view of [4, Theorem 8.10], we get that $\delta(A,B) \ge d$, that is, $A\delta^d B$. So σ is finer than δ^d and the proof is complete. □

5. Completely regular fuzzifying spaces

Definition 5.1. A fuzzifying topological space (X,τ) is called completely regular if each of the classical level topologies τ^d, $0 \le d < 1$, is completely regular.

Definition 5.2. A fuzzifying proximity δ on a set X is said to be compatible with a fuzzifying topology τ if τ coincides with the fuzzifying topology τ_δ induced by δ.

We have the following easily established theorem.

Theorem 5.3. *Subspaces and Cartesian products of completely regular fuzzifying spaces are completely regular.*

Theorem 5.4. *Let (X,τ) be a completely regular fuzzifying topological space and define $\delta = \delta(\tau) : 2^X \times 2^X \to [0,1]$ by*

$$\delta(A,B) = 1 - \sup\{d : 0 \le d < 1,\ \exists f : (X,\tau^d) \longrightarrow [0,1] \text{ continuous } f(A) = 0,\ f(B) = 1\}. \tag{5.1}$$

Then, (1) *δ is a fuzzifying proximity on X compatible with τ;*

(2) *if δ_1 is any fuzzifying proximity on X compatible with τ, then δ is finer than δ_1.*

Proof. It is easy to see that δ satisfies (FP1), (FP2), (FP3), and (FP5). We will prove that δ satisfies (FP4). Let

$$\alpha = \inf\{\delta(A,D) \vee \delta(D^c,B) : D \subset X\}. \tag{5.2}$$

If $\delta(A,D) \vee \delta(D^c,B) < \theta$, then $A \subset D^c$, and so $\delta(A,B) \le \delta(D^c,B) < \theta$, which proves that $\delta(A,B) \le \alpha$. On the other hand, assume that $\delta(A,B) < r < 1$. There exist a d, $1 - r < d < 1$, and $f : X \to [0,1]\tau^d$-continuous such that $f(A) = 0$, $f(B) = 1$. Let $D = \{x \in X : 1/2 \le f(x) \le 1\}$ and define $h_1, h_2 : [0,1] \to [0,1]$, $h_1(t) = 2t$, $h_2(t) = 0$ if $0 \le t \le 1/2$ and $h_1(t) = 1$, $h_2(t) = 2t - 1$ if $1/2 < t \le 1$. If $g_i = h_i \circ f$, $i = 1,2$, then $g_1(A) = 0$, $g_1(D) = 1$, $g_2(D^c) = 0$, $g_2(B) = 1$. Thus, $\delta(A,D) \le 1 - d < r$, $\delta(D^c,B) < r$, which proves that $\alpha \le \delta(A,B)$. Hence, δ is a fuzzifying proximity on X. We need to show that $\tau = \tau_\delta$. So, let $\tau(A) > \theta > 0$. Since τ^θ is completely regular, given $x \in A$, there exist $f_x : X \to [0,1]$, τ^θ-continuous, $f_x(x) = 0$, $f_x(A^c) = 1$. Thus $\delta(x,A^c) \le 1 - \theta$, and so $N_x^{\tau_\delta}(A) = 1 - \delta(x,A^c) \ge \theta$. It follows that $\tau_\delta(A) = \inf_{x\in A} N_x^{\tau_\delta}(A) \ge \theta$, which proves that $\tau_\delta \ge \tau$. On the other hand, assume that $\tau_\delta(A) > r > 0$. If $x \in A$, then $\delta(x,A^c) = 1 - N_x^{\tau_\delta}(A) < 1 - r$, and therefore there exist a d, $0 < 1 - d < 1 - r$ and $f : X \to [0,1]\tau^d$-continuous such that $f(x) = 0$, $f(A^c) = 1$. The set

$G = \{y : f(y) < 1/2\}$ is in τ^d and $x \in G \subset A$. Thus,

$$N_x^\tau(A) \geq N_x^\tau(G) \geq d > r. \tag{5.3}$$

This proves that $\tau(A) \geq r$ and so $\tau \geq \tau_\delta$, which completes the proof of (1).

Let δ_1 be a fuzzifying proximity on X compatible with τ and let A, B be subsets of X with $\delta_1(A,B) < \theta < 1$. If $d = 1-\theta$, then δ_1^θ is compatible with τ^d. Since $A \not\delta_1^\theta B$, there exists (by [8, Remark 3.15]) an $f : X \to [0,1]\tau_d$-continuous, with $f(A) = 0$, $f(B) = 1$, and so $\delta(A,B) \leq 1-d = \theta$, which proves that $\delta(A,B) \leq \delta_1(A,B)$, and therefore δ is finer than δ_1. This completes the proof. □

Theorem 5.5. *For a fuzzifying topological space (X,τ), the following are equivalent.*
(1) *(X,τ) is completely regular.*
(2) *There exists a fuzzifying proximity δ on X compatible with τ.*
(3) *(X,τ) is fuzzy uniformizable, that is, there exists a fuzzy uniformity $\mathcal{U}$ on X such that τ coincides with the fuzzifying topology $\tau_{\mathcal{U}}$ induced by $\mathcal{U}$.*

Proof. By [5], (2) is equivalent to (3). Also (1) implies (2) in view of the preceding theorem. Assume now that $\tau = \tau_\delta$ for some fuzzifying proximity δ. For each $0 < d \leq 1$, δ^d is a classical proximity compatible with τ^{1-d}, and so τ^{1-d} is completely regular. This completes the proof. □

Theorem 5.6. *Every pseudometrizable fuzzy topological space (X,τ) is completely regular.*

Proof. If τ is pseudometrizable, then each τ^d, $0 \leq d < 1$, is pseudometrizable, and hence τ^d is completely regular. □

Theorem 5.7. *For a fuzzifying topological space (X,τ), the following are equivalent.*
(1) *(X,τ) is completely regular.*
(2) *If $\mathcal{F} = \mathcal{F}_\tau$ is the family of all probabilistic pseudometrics on X which are $\tau \times \tau$ continuous as functions from X^2 to $\mathbb{R}_\phi^+$, then $\tau = \tau_{\mathcal{F}_\tau}$.*
(3) *There exists a family $\mathcal{F}$ of probabilistic pseudometrics on X such that $\tau = \tau_{\mathcal{F}}$.*

Proof. (1)⇒(2). For each $F \in \mathcal{F}_\tau$, we have that $\tau_F \leq \tau$ (by Theorem 3.2), and so $\tau_{\mathcal{F}_\tau} \leq \tau$. Let now $A \subset X$ and $x_o \in X$ with $N_{x_o}^\tau(A) > \theta > 0$. Since τ^θ is completely regular, there exists a τ^θ-continuous function f from X to [0,1] such that $f(x_o) = 0$, $f(A^c) = 1$. For $x, y \in X$, define $F(x,y)$ on $\mathbb{R}$ by

$$F(x,y)(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1-\theta & \text{if } |f(x)-f(y)| \geq t > 0, \\ 1 & \text{if } |f(x)-f(y)| < t. \end{cases} \tag{5.4}$$

Clearly, $F(x,y) = F(y,x) \in \mathbb{R}_\phi^+$ and $F(x,x) = \bar{0}$. We will prove that F satisfies the triangle inequality. So, assume that $F(x,y)(t_1) \wedge F(y,z)(t_2) > F(x,z)(t_1+t_2)$. Then, $t_1, t_2 > 0$, $F(x,z)(t_1+t_2) = 1-\theta$, $F(x,y)(t_1) = F(y,z)(t_2) = 1$. Thus, $t_1 > |f(x)-f(y)|$, $t_2 > |f(y)-f(z)|$, and hence $|f(x)-f(z)| < t_1+t_2$, which implies that $F(x,z)(t_1+t_2) = 1$, a contradiction. So F is a probabilistic pseudometric on X. Next we show that F is $\tau \times \tau$ continuous, or equivalently that $\tau_F \leq \tau$. So assume that $N_x^{\tau_F}(B) > r > 0$. Let $\theta_1 > r$ be such that $N_x^{\tau_F}(B) > \theta_1$. Choose $t > 0$ such that $1 - \sup_{y \notin B} F(x,y)(t) > \theta_1$, and so $F(x,y)(t) = 1 - \theta$ and $|f(x) - f(y)| \geq t$ if $y \notin B$. Thus, $\{y : |f(x) - f(y)| < t\} \subset B$. This shows that B is a τ^θ-neighborhood of x. As $r < \theta$, B is a τ^r-neighborhood of x, that is, $N_x^\tau(B) > r$, and so $\tau_F \leq \tau$. Finally if $y \notin A$, then $|f(y) - f(x_o)| = 1$, and so $F(y,x_o)(1/2) = 1 - \theta$, which implies that

$$N_{x_o}^{\tau_{\mathcal{F}}}(A) \geq N_{x_o}^{\tau_F}(A) \geq 1 - \sup_{y \notin A} F(y,x_o)\left(\frac{1}{2}\right) \geq \theta. \tag{5.5}$$

This shows that $N_{x_o}^{\tau_{\mathcal{F}}} \geq N_{x_o}^\tau$, and so $\tau \leq \tau_{\mathcal{F}}$, which completes the proof of the implication (1)⇒(2).

(3)⇒(1). Assume that $\tau = \tau_{\mathcal{F}}$ for some family $\mathcal{F}$ of probabilistic pseudometrics on X. For each $F \in \mathcal{F}$, τ_F is completely regular and so $\tau_{\mathcal{F}}$ is completely regular since $\tau_{\mathcal{F}}^d = \bigvee_{F \in \mathcal{F}} \tau_F^d$ for each $0 \leq d < 1$. Hence the result follows. □

We will denote by $[0,1]_\phi$ the subspace of $\mathbb{R}_\phi^+$ consisting of all $u \in \mathbb{R}_\phi^+$ with $u(t) = 1$ if $t > 1$.

Theorem 5.8. *A fuzzifying topological space (X,τ) is completely regular if and only if the following condition is satisfied. If $N_{x_o}(A) > \theta > 0$, then there exists $f : X \to [0,1]_\phi$ continuous such that $f(x_o) = \bar{0}$ and $f(y)(t) = 1 - \theta$ if $y \notin A$ and $0 < t < 1$.*

Proof. Assume that (X,τ) is completely regular and let $N_{x_o}(A) > \theta > 0$. Since τ^θ is completely regular, there exists $h : (X,\tau^\theta) \to [0,1]$ continuous, $h(x_o) = 0$, $h(y) = 1$ if $y \notin A$. For $x, y \in X$, define $F(x,y)$ on $\mathbb{R}$ by

$$F(x,y)(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 - \theta & \text{if } |h(x) - h(x_o)| \geq t > 0, \\ 1 & \text{if } |h(x) - h(x_o)| < t. \end{cases} \tag{5.6}$$

Clearly, $F(x,y) \in [0,1]_\phi$. Also, $F(x,z) \preceq F(x,y) \oplus F(y,z)$. In fact, assume that $F(x,y)(t_1) \wedge F(y,z)(t_2) > r > F(x,z)(t_1 + t_2)$. Then t_1, $t_2 > 0$, $F(x,y)(t_1) = F(y,z)(t_2) = 1$. Now, $|h(x) - h(y)| < t_1$, $|h(y) - h(z)| < t_2$, and so $|h(x) - h(z)| < t_1 + t_2$, which implies that $F(x,z)(t_1 + t_2) = 1$, a contradiction. So F is a probabilistic pseudometric. Moreover, F is $\tau \times \tau$ continuous, or equivalently $\tau_F \leq \tau$. In fact, let $N_x^{\tau_F}(B) > r > 0$. There exists a $t > 0$ such that $1 - \sup_{z \notin B} F(z,x)(t) > r$. If $z \notin B$, then $F(z,x)(t) < 1 - r < 1$, and so $F(z,x)(t) = 1 - \theta < 1 - r$, that is, $r < \theta$, and $|h(z) - h(x)| \geq t$. Hence

$$M = \{z : |h(z) - h(x)| < t\} \subset B. \tag{5.7}$$

The set M is a τ^θ-neighborhood of x, and hence a τ^r-neighborhood, that is, $N_x^\tau(B) > r$. Thus $\tau \geq \tau_F$. Finally, define $f : X \to [0,1]_\phi$, $f(y) = F(y,x_o)$. Then f is τ-continuous, $f(x_o) = \bar{0}$. For $y \notin A$ and $0 < t < 1$, we have that $f(y)(t) = F(y,x_o)(t) = 1 - \theta$ (since $|h(x) - h(x_o| = 1 \geq t)$. Conversely, assume that the condition is satisfied and let $\mathcal{F}$ be the family of all $\tau \times \tau$ continuous pseudometrics on X. Then $\tau_{\mathcal{F}} \leq \tau$. Let $N_{x_o}^\tau(A) > \theta$. There exists a $\theta_1 > \theta$ such that $N_{x_o}^{\tau}(A) > \theta_1$. By our hypothesis, there exists $f : X \to [0,1]_\phi$ continuous such that $f(x_o) = \bar{0}$ and $f(y)(t) = 1 - \theta_1$ if $y \notin A$ and $0 < t < 1$. Define $F(x,y) = D(f(x), f(y))$. Then F is $\tau \times \tau$ continuous and

$$\begin{aligned} N_{x_o}^{\tau_{\mathcal{F}}}(A) \geq N_{x_o}^{\tau_F}(A) &\geq 1 - \sup_{y \notin A} F(x_o, y)(1) \\ &= 1 - \sup_{y \notin A} D(\bar{0}, f(y))(1) \\ &= 1 - \sup_{y \notin A} f(y)(1) \geq \theta_1 > \theta. \end{aligned} \tag{5.8}$$

Thus $N_{x_o}^{\tau_{\mathcal{F}}}(A) \geq N_{x_o}^{\tau}(A)$, for every subset A of X, and so $\tau \leq \tau_{\mathcal{F}}$. Therefore, $\tau = \tau_{\mathcal{F}}$, and so τ is completely regular. □

For a fuzzifying topological space X, we will denote by $C(X, [0,1]_\phi)$ the family of all continuous functions from X to $[0,1]_\phi$.

Theorem 5.9. *A fuzzifying topological space (X, τ) is completely regular if and only if τ coincides with the weakest of all fuzzifying topologies τ_1 on X for which each $f \in C(X, [0,1]_\phi)$ is continuous.*

Proof. Assume that (X, τ) is completely regular and let τ_1 be the weakest of all fuzzifying topologies on X for which each $f \in C(X, [0,1]_\phi)$ is continuous. Clearly $\tau_1 \leq \tau$. On the other hand, let τ_2 be a fuzzifying topology on X for which each $f \in C(X, [0,1]_\phi)$ is continuous. Let $N_x^{\tau}(A) > \theta > 0$. In view of the preceding theorem, there exists an $f \in C(X, [0,1]_\phi)$ such that $f(x) = \bar{0}$, $f(y)(t) = 1 - \theta$ if $y \notin A$ and $0 < t < 1$. Let

$$G = \left\{ u \in \mathbb{R}_\phi^+ : D(f(x), u)\left(\frac{1}{2}\right) = u\left(\frac{1}{2}\right) > 1 - \theta \right\}. \tag{5.9}$$

Then

$$N_{\bar{0}}(G) \geq 1 - \sup_{u \notin G} D(f(x), u)\left(\frac{1}{2}\right) \geq \theta. \tag{5.10}$$

Since f is τ_2-continuous, we have that $N_x^{\tau_2}(f^{-1}(G)) \geq \theta$. But $f^{-1}(G) \subset A$ since, for $y \notin A$, we have that $f(y)(1/2) = 1 - \theta$. Thus $N_x^{\tau_2}(A) \geq \theta$. This proves that $N_x^{\tau_2}(A) \geq N_x^{\tau}(A)$, for every subset A of X, and so $\tau_2 \geq \tau$. This clearly proves that $\tau_1 = \tau$. Conversely, assume that $\tau_1 = \tau$. If σ is the usual fuzzifying topology of $\mathbb{R}_\phi^+$, then

$$\tau = \tau_1 = \bigvee_{f \in C(X, [0,1]_\phi)} f^{-1}(\sigma). \tag{5.11}$$

Since σ is completely regular, each $f^{-1}(\sigma)$ is completely regular, and so τ is completely regular. This completes the proof. □

References

[1] U. Höhle, *Upper semicontinuous fuzzy sets and applications*, J. Math. Anal. Appl. **78** (1980), no. 2, 659–673.

[2] ———, *Probabilistic metrization of fuzzy uniformities*, Fuzzy Sets and Systems **8** (1982), no. 1, 63–69.

[3] ———, *Characterization of L-topologies by L-valued neighborhoods*, Mathematics of Fuzzy Sets: Logic, Topology, and Measure Theory, Handb. Fuzzy Sets Ser., vol. 3, Kluwer Academic, Massachusetts, 1999, pp. 389–432.

[4] A. K. Katsaras and C. G. Petalas, *Fuzzifying topologies and fuzzifying proximities*, J. Fuzzy Math. **11** (2003), no. 2, 411–436.

[5] ———, *Fuzzifying syntopogenous structures*, J. Fuzzy Math. **12** (2004), no. 1, 77–108.

[6] ———, *On fuzzifying topological spaces*, J. Fuzzy Math. (2005).

[7] R. Lowen, *Convergence in fuzzy topological spaces*, General Topology Appl. **10** (1979), no. 2, 147–160.

[8] S. A. Naimpally and B. D. Warrack, *Proximity Spaces*, Cambridge Tracts in Mathematics and Mathematical Physics, no. 59, Cambridge University Press, London, 1970.

[9] M. S. Ying, *A new approach for fuzzy topology. I*, Fuzzy Sets and Systems **39** (1991), no. 3, 303–321.

[10] ———, *A new approach for fuzzy topology. II*, Fuzzy Sets and Systems **47** (1992), no. 2, 221–232.

[11] ———, *A new approach for fuzzy topology. III*, Fuzzy Sets and Systems **55** (1993), no. 2, 193–207.

[12] ———, *On the method of neighborhood systems in fuzzy topology*, Fuzzy Sets and Systems **68** (1994), no. 2, 227–238.

[13] D. Zhang and L. Xu, *Categories isomorphic to FNS*, Fuzzy Sets and Systems **104** (1999), no. 3, 373–380.

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