

# ON SYMMETRIES AND INVARIANT SOLUTIONS OF A COUPLED KdV SYSTEM WITH VARIABLE COEFFICIENTS

K. SINGH AND R. K. GUPTA

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We investigate symmetries and reductions of a coupled KdV system with variable coefficients. The infinitesimals of the group of transformations which leaves the KdV system invariant and the admissible forms of the coefficients are obtained using the generalized symmetry method based on the Fréchet derivative of the differential operators. An optimal system of conjugacy inequivalent subgroups is then identified with the adjoint action of the symmetry group. For each basic vector field in the optimal system, the KdV system is reduced to a system of ordinary differential equations, which is further studied with the aim of deriving certain exact solutions.

## 1. Introduction

The last few decades have witnessed a tremendous growth in research interests related to the applications of group theoretical methods while deriving exact solutions to a wide range of problems describing nonlinear phenomena. The most effective methods, for finding the symmetry reduction and constructing exact solutions, include the classical Lie approach [7, 16, 21, 26], the nonclassical approach [2, 6, 13, 25], the direct method [11, 12], the modified direct method [22], the generalized conditional symmetry method [14, 31], nonlocal symmetry method [1, 8], the truncated Painlevé approach [9, 24], the transformation method [18, 19], and the ansatz-based methods [10, 17, 20, 27, 30]. There are in fact, many different directions of the mathematical and physical theory to concern their exact solutions and various properties.

Herein, we study the symmetries and reductions of a coupled KdV system, with variable coefficients, of the form

$$\begin{aligned}u_t + \alpha_1(t)uu_x + \beta_1(t)vv_x + \gamma_1(t)u_{xxx} &= 0, \\v_t + \alpha_2(t)vu_x + \beta_2(t)uv_x + \gamma_2(t)v_{xxx} &= 0.\end{aligned}\tag{1.1}$$

Equations (1.1) are an extension of the KdV system examined by Zhou et al. [32] for periodic wave solutions using the  $F$ -expansion method, an overall generalization of the Jacobi elliptic function expansion method. The constant coefficient version of (1.1) was

earlier discussed by Lei et al. [28], wherein the author had derived the classical periodic solutions using Fourier expansion on an incomplete basis.

To determine the underlying symmetry group for (1.1), we utilize a method which is based on the Fréchet derivative of the differential operators associated with the system (1.1). The technique has earlier been used to obtain the exact solutions of various nonlinear partial differential equations (refer to [3, 4, 5, 23]). However, for the sake of completeness, and in order to help the reader relate it to the familiar one-parameter group of transformations, we provide here some basic knowledge about the approach.

Let the system (1.1) be defined in terms of the nonlinear operators  $N_1$  and  $N_2$  as follows:

$$\begin{aligned} N_1(u, v) &\equiv u_t + \alpha_1(t)uu_x + \beta_1(t)vv_x + \gamma_1(t)u_{xxx} = 0, \\ N_2(u, v) &\equiv v_t + \alpha_2(t)vu_x + \beta_2(t)uv_x + \gamma_2(t)v_{xxx} = 0. \end{aligned} \tag{1.2}$$

Next, we define an operator called a symmetry operator for the system (1.2) given by

$$\bar{S} = (S_1, S_2), \tag{1.3}$$

where

$$\begin{aligned} S_1(u) &\equiv A(\bar{X}, \bar{\eta}) \frac{\partial u}{\partial t} + B(\bar{X}, \bar{\eta}) \frac{\partial u}{\partial x} + C_1(\bar{X}, \bar{\eta}), \\ S_2(v) &\equiv A(\bar{X}, \bar{\eta}) \frac{\partial v}{\partial t} + B(\bar{X}, \bar{\eta}) \frac{\partial v}{\partial x} + C_2(\bar{X}, \bar{\eta}) \end{aligned} \tag{1.4}$$

with  $\bar{X} = (t, x)$ ,  $\bar{\eta} = (u, v)$ .

The Fréchet derivative of  $\bar{N}(\bar{\eta}) = (N_1, N_2)$  in the direction of  $\bar{\eta}_1 = (u_1, v_1)$  is given by

$$\bar{F}(\bar{N}, \bar{\eta}, \bar{\eta}_1) = \frac{d}{d\varepsilon} [\bar{N}(\bar{\eta} + \varepsilon\bar{\eta}_1)] \Big|_{\varepsilon=0}. \tag{1.5}$$

For invariance of (1.1), we require that the Fréchet derivative (1.5) must vanish on the solution set  $\Gamma$  of (1.1) in the direction of the symmetry operator  $\bar{S}$ . That is, we must have

$$\bar{F}(\bar{N}, \bar{\eta}, \bar{S}) \Big|_{\Gamma} = \bar{O}. \tag{1.6}$$

The associated Lie algebra of infinitesimal symmetries of (1.1) is then the set of vector fields of the form

$$V = A(\bar{X}, \bar{\eta}) \frac{\partial}{\partial t} + B(\bar{X}, \bar{\eta}) \frac{\partial}{\partial x} - C_1(\bar{X}, \bar{\eta}) \frac{\partial}{\partial u} - C_2(\bar{X}, \bar{\eta}) \frac{\partial}{\partial v}. \tag{1.7}$$

Or, equivalently the one-parameter group of point transformations of (1.1) is as follows:

$$\begin{aligned} t^* &= t + \varepsilon A(\bar{X}, \bar{\eta}) + O(\varepsilon^2), \\ x^* &= x + \varepsilon B(\bar{X}, \bar{\eta}) + O(\varepsilon^2), \\ \bar{\eta}^* &= \bar{\eta} - \varepsilon \bar{C}(\bar{X}, \bar{\eta}) + O(\varepsilon^2), \end{aligned} \tag{1.8}$$

where  $\bar{C} = (C_1, C_2)$ ,  $\bar{\eta}^* = (u^*, v^*)$ .

The advantage in our approach is that it, not only furnishes the group infinitesimals comparatively easily but, also often renders symmetries more generalized than the classical Lie method (refer to [4], e.g.). It is precisely due to this fact that we prefer to call the said approach the generalized symmetry method. Besides demonstrating the importance and efficacy of the method, the motivation for the present study lies in the physical importance of the model KdV system (1.1) and the need to have some exact solutions. To have an insight, the explicit analytic solutions of the system (1.1) may enable one to better understand the physical phenomena which it describes. The exact solutions, which are accurate and explicit, may help physicists and engineers to discuss and examine the sensitivity of the model with variable coefficients to several important physical parameters. To our knowledge, a detailed analysis that leads to an exact analytic solution for (1.1) has not been performed and is therefore desirable.

The paper has been organized as follows. Section 2 is entirely devoted to showing how the powerful generalized symmetry method can be used to generate various symmetries of the KdV system. In Section 3, we present the reduced systems of ordinary differential equations (ODEs) and their exact solutions. The final section contains the discussion and concluding remarks.

## 2. The symmetry group and the optimal system

The method for determining the symmetry group of (1.1) mainly consists of finding the coefficients  $A, B, C_1,$  and  $C_2$  in the two symmetry operators  $S_1$  and  $S_2$  as defined by (1.4). These coefficients are to be determined from the invariance condition (1.6). Accordingly, we first find the Fréchet derivative

$$\bar{F}(\bar{N}, \bar{\eta}, \bar{\eta}_1) = (F_1(N_1, \bar{\eta}, \bar{\eta}_1), F_2(N_2, \bar{\eta}, \bar{\eta}_1)), \tag{2.1}$$

where

$$F_1(N_1, \bar{\eta}, \bar{\eta}_1) = \left. \frac{d}{d\varepsilon} [N_1(\bar{\eta} + \varepsilon\bar{\eta}_1)] \right|_{\varepsilon=0}, \tag{2.2}$$

$$F_2(N_2, \bar{\eta}, \bar{\eta}_1) = \left. \frac{d}{d\varepsilon} [N_2(\bar{\eta} + \varepsilon\bar{\eta}_1)] \right|_{\varepsilon=0}.$$

With the help of (2.2) the Fréchet derivatives are obtained and then  $\bar{\eta}_1$  is substituted by  $\bar{S} = (S_1, S_2)$  in order to evaluate them in the direction of the symmetry operator. This leads to the following

$$F_1(N_1, \bar{\eta}, \bar{S}) = [S_1]_t + \alpha_1 [S_1]u_x + \alpha_1 [S_1]_x u + \beta_1 v [S_2]_x + \beta_1 [S_2]v_x + \gamma_1 [S_1]_{xxx}, \tag{2.3}$$

$$F_2(N_2, \bar{\eta}, \bar{S}) = [S_2]_t + \alpha_2 [S_2]u_x + \alpha_2 [S_1]_x v + \beta_2 u [S_2]_x + \beta_2 [S_1]v_x + \gamma_2 [S_2]_{xxx}.$$

For invariance of system (1.1), the following conditions must be satisfied:

$$F_1(N_1, \bar{\eta}, \bar{S}) \Big|_{(N_1, N_2) = \bar{0}} = 0, \tag{2.4a}$$

$$F_2(N_2, \bar{\eta}, \bar{S}) \Big|_{(N_1, N_2) = \bar{0}} = 0. \tag{2.4b}$$

Equations (2.4), when expanded, result in the polynomial expressions in various partial derivatives of  $u(t, x)$  and  $v(t, x)$  with respect to the spatial variable. The calculations involved are tedious, however, to keep the interest of the reader we list here a much simplified set of determining equations for the group infinitesimals  $A$ ,  $B$ ,  $C_1$ , and  $C_2$  which we get from (2.4a) after equating the coefficients of various derivative terms to zero:

$$\begin{aligned}
 A_x &= A_u = A_v = 0, \\
 B_u &= B_v = 0, \\
 C_{1uu} &= C_{1vv} = C_{1uv} = C_{1xv} = 0, \\
 -\gamma_1 A_t - A\gamma_{1t} + 3\gamma_1 B_x &= 0, \\
 (\gamma_1 - \gamma_2) C_{1v} &= 0, \\
 3\gamma_1 (B_{xx} + C_{1xu}) &= 0, \\
 -\alpha_1 u A_t - A\alpha_{1t} u + B_t - \alpha_2 v C_{1v} + \alpha_1 C_1 + \alpha_1 u B_x + \beta_1 v C_{2u} + \gamma_1 B_{xxx} + 3\gamma_1 C_{1xxu} &= 0, \\
 -\beta_1 v A_t - A\beta_{1t} v - \beta_1 v C_{1u} - \beta_2 u C_{1v} + \alpha_1 u C_{1v} + \beta_1 C_2 + \beta_1 v B_x + \beta_1 v C_{2v} &= 0, \\
 C_{1t} + \alpha_1 u C_{1x} + \beta_1 v C_{2x} + \gamma_1 C_{1xxx} &= 0.
 \end{aligned} \tag{2.5}$$

Similarly, (2.4b) brings in the following additional equations. It is being mentioned here that these equations have been obtained keeping in view the consequences on the infinitesimals as effected by the set of equations (2.5):

$$\begin{aligned}
 C_{2uu} &= C_{2vv} = C_{2uv} = C_{2xu} = 0, \\
 (\gamma_2 - \gamma_1) C_{2u} &= 0, \\
 3\gamma_2 (B_{xx} + C_{2xv}) &= 0, \\
 -\gamma_2 A_t - A\gamma_{2t} + 3\gamma_2 B_x &= 0, \\
 -\alpha_2 v A_t - A\alpha_{2t} v - \alpha_1 u C_{2u} - \alpha_2 v C_{2v} + \alpha_2 C_2 + \alpha_2 v B_x + \alpha_2 v C_{1u} + \beta_2 u C_{2u} &= 0, \\
 -\beta_2 u A_t - A\beta_{2t} u + B_t - \beta_1 v C_{2u} + \alpha_2 v C_{1v} + \beta_2 C_1 + \beta_2 u B_x &= 0, \\
 C_{2t} + \alpha_2 v C_{1x} + \beta_2 u C_{2x} + \gamma_2 C_{2xxx} &= 0.
 \end{aligned} \tag{2.6}$$

The two sets of equations (2.5) and (2.6) are further simplified to the extent possible and we eventually arrive at the following version of the determining equations:

$$\begin{aligned}
 A &= A(t), & B &= B_1(t)x + B_2(t), & C_1 &= \phi_1(t)u + \phi_2(x, t), \\
 C_2 &= \varphi(t)v, & \phi_{1t} + \alpha_1 \phi_{2x} &= 0, & \phi_{2t} + \gamma_1 \phi_{2xxx} &= 0, \\
 \varphi_t + \alpha_2 \phi_{2x} &= 0, & B_{1t}x + B_{2t} + \alpha_1 \phi_2 &= 0, & B_{1t}x + B_{2t} + \beta_2 \phi_2 &= 0, \\
 -\alpha_1 A_t - A\alpha_{1t} + \alpha_1 \phi_1 + \alpha_1 B_1 &= 0, & -\gamma_1 A_t - A\gamma_{1t} + 3\gamma_1 B_1 &= 0, \\
 -\gamma_2 A_t - A\gamma_{2t} + 3\gamma_2 B_1 &= 0, & -\beta_1 A_t - A\beta_{1t} - \beta_1 \phi_1 + 2\beta_1 \varphi + \beta_1 B_1 &= 0, \\
 -\alpha_2 A_t - A\alpha_{2t} + \alpha_2 B_1 + \alpha_2 \phi_1 &= 0, & -\beta_2 A_t - A\beta_{2t} + \beta_2 B_1 + \beta_2 \phi_1 &= 0.
 \end{aligned} \tag{2.7}$$

Equations (2.7) enable us to derive the infinitesimals  $A, B, C_1,$  and  $C_2,$  along with the admissible forms of the various coefficient functions of system (1.1). Without presenting any calculations, we provide the results obtained as follows:

$$\begin{aligned}
 A &= \frac{1}{\alpha_1(t)} \left[ (a_1 + a_2) \int \alpha_1(t) dt + a_4 \right], \\
 B &= a_1x + a_5, \quad C_1 = a_2u, \quad C_2 = a_3v,
 \end{aligned}
 \tag{2.8}$$

where  $a_j, j = 1, 2, \dots, 5$  are arbitrary constants. The functions  $\alpha_1(t), \beta_1(t), \gamma_1(t), \alpha_2(t), \beta_2(t),$  and  $\gamma_2(t)$  are governed by the following relations:

$$\begin{aligned}
 \frac{d}{dt} [A(t)\beta_1(t)] + (a_2 - 2a_3 - a_1)\beta_1(t) &= 0, \\
 \frac{d}{dt} [A(t)\gamma_i(t)] - 3a_1\gamma_i(t) &= 0, \quad i = 1, 2, \\
 \alpha_2(t) = k_1\alpha_1(t), \quad \beta_2(t) = k_2\alpha_1(t),
 \end{aligned}
 \tag{2.9}$$

where  $k_1$  and  $k_2$  are arbitrary constants. The symmetries under which (1.1) is invariant can be spanned by the following five linearly independent infinitesimal generators:

$$\begin{aligned}
 V_1 &= \left[ \frac{1}{\alpha_1(t)} \int \alpha_1(t) dt \right] \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, & V_2 &= \left[ \frac{1}{\alpha_1(t)} \int \alpha_1(t) dt \right] \frac{\partial}{\partial t} - u \frac{\partial}{\partial u}, \\
 V_3 &= -v \frac{\partial}{\partial v}, & V_4 &= \frac{1}{\alpha_1(t)} \frac{\partial}{\partial t}, & V_5 &= \frac{\partial}{\partial x}.
 \end{aligned}
 \tag{2.10}$$

Using these generators one can obtain a reduction of (1.1) to a system of ODEs after getting the similarity variable and the form by solving the characteristic equations

$$\frac{dt}{A} = \frac{dx}{B} = -\frac{du}{C_1} = -\frac{dv}{C_2}.
 \tag{2.11}$$

In general, one may obtain the reduced system of ODEs from any linear combination of generators  $V_j, j = 1, 2, \dots, 5.$  Since there exist infinite possibilities for such combinations, a systematic procedure to classify these reductions is based on the property that the transformations of the symmetry group will transform solutions of (1.1) into other solutions. Therefore, we classify the symmetry algebra of the KdV system into conjugacy inequivalent subalgebra under the adjoint action of the symmetry group. We thus deduce the following basic fields which form an optimal system for the KdV system, and from which every other solution can be derived:

- (i)  $V_1 + aV_2 + bV_3,$
- (ii)  $V_2 + cV_3,$
- (iii\_1)  $V_2 + dV_3 + V_5,$
- (iii\_2)  $V_2 + dV_3 - V_5,$
- (iv)  $V_3 + \mu V_4,$

Table 2.1

Type of essential field	Similarity variable $\xi$	Similarity solution $(u, v)$	Forms of coefficient functions
(i)	$x[\int \alpha_1(t)dt]^{1/(1+a)}$	$[\int \alpha_1(t)dt]^{-a/(1+a)}F(\xi),$ $[\int \alpha_1(t)dt]^{-b/(1+a)}G(\xi)$	$\beta_1(t) = k_3\alpha_1(t)[\int \alpha_1(t)dt]^{2(b-a)/(1+a)},$ $\alpha_2(t) = k_1\alpha_1(t), \beta_2(t) = k_2\alpha_1(t),$ $\gamma_i(t) = \sigma_i\alpha_1(t)[\int \alpha_1(t)dt]^{(2-a)/(1+a)},$ $i = 1, 2$
(ii)	$x$	$[\int \alpha_1(t)dt]^{-1}F(\xi),$ $[\int \alpha_1(t)dt]^{-c}G(\xi)$	$\beta_1(t) = k_3\alpha_1(t)[\int \alpha_1(t)dt]^{2(c-1)},$ $\alpha_2(t) = k_1\alpha_1(t), \beta_2(t) = k_2\alpha_1(t),$ $\gamma_i(t) = \sigma_i\alpha_1(t)[\int \alpha_1(t)dt]^{-1}, i = 1, 2$
(iii)	$\ln[\int \alpha_1(t)dt] - x$	$[\int \alpha_1(t)dt]^{-1}F(\xi),$ $[\int \alpha_1(t)dt]^{-d}G(\xi)$	$\beta_1(t) = k_3\alpha_1(t)[\int \alpha_1(t)dt]^{2(d-1)},$ $\alpha_2(t) = k_1\alpha_1(t), \beta_2(t) = k_2\alpha_1(t),$ $\gamma_i(t) = \sigma_i\alpha_1(t)[\int \alpha_1(t)dt]^{-1}, i = 1, 2$
(iv)	$x$	$F(\xi),$ $\exp\left[-\frac{1}{\mu}\int \alpha_1(t)dt\right]G(\xi)$	$\beta_1(t) = \frac{k_3}{\mu}\alpha_1(t)\exp\left[\frac{2}{\mu}\int \alpha_1(t)dt\right],$ $\alpha_2(t) = k_1\alpha_1(t), \beta_2(t) = k_2\alpha_1(t),$ $\gamma_i(t) = \sigma_i\alpha_1(t), i = 1, 2$
(v)	$x - \frac{1}{\lambda}\int \alpha_1(t)dt$	$F(\xi),$ $\exp\left[-\frac{1}{\lambda}\int \alpha_1(t)dt\right]G(\xi)$	$\beta_1(t) = \frac{k_3}{\lambda}\alpha_1(t)\exp\left[\frac{2}{\lambda}\int \alpha_1(t)dt\right],$ $\alpha_2(t) = k_1\alpha_1(t), \beta_2(t) = k_2\alpha_1(t),$ $\gamma_i(t) = \sigma_i\alpha_1(t), i = 1, 2$
(vi)	$x - \int \alpha_1(t)dt$	$F(\xi), G(\xi)$	$\beta_1(t) = k_3\alpha_1(t), \alpha_2(t) = k_1\alpha_1(t),$ $\beta_2(t) = k_2\alpha_1(t),$ $\gamma_i(t) = \sigma_i\alpha_1(t), i = 1, 2$
(vii)	$x$	$F(\xi), G(\xi)$	$\beta_1(t) = k_3, \alpha_2(t) = k_1,$ $\beta_2(t) = k_2, \alpha_1(t) = k_0,$ $\gamma_i(t) = \sigma_i, i = 1, 2$
(viii)	$t$	$F(\xi), G(\xi)$	Any arbitrary function of $t$

$k_0, k_1, k_2, k_3, \sigma_1, \sigma_2, a, b, c, d, \mu,$  and  $\lambda$  are arbitrary constants. In rows (i)–(vi),  $\alpha_1(t)$  is arbitrary.

- (v\_1)  $V_3 + \lambda V_4 + V_5,$
- (v\_2)  $V_3 + \lambda V_4 - V_5,$
- (vi\_1)  $V_4 + V_5,$
- (vi\_2)  $V_4 - V_5,$
- (vii)  $V_4,$
- (viii)  $V_5.$

Because the discrete symmetry  $(t, x, u, v) \mapsto (t, -x, u, v)$  will map (iii\_1), (v\_1), and (vi\_1) to (iii\_2), (v\_2), and (vi\_2), respectively, in the optimal system we confine ourselves to eight generators only, while neglecting the other three.

In Table 2.1, we now list the similarity variable and the form which we get from (2.11), and also the coefficient functions of the KdV system, for the essential fields in the optimal system.

### 3. The reduced systems and the exact solutions

In the following, we utilize the similarity variable and the form of the similarity solution, as listed in Table 2.1, to obtain the reduced system of ODEs for (1.1). Some exact solutions for the reduced systems are studied for each type of essential fields mentioned in Table 2.1.

Type (i). The reduced system of ODEs in this case is as follows:

$$\begin{aligned} (1+a)[\sigma_1 F''' + k_3 GG' + FF'] - \xi F' - aF &= 0, \\ (1+a)[\sigma_2 G''' + k_2 FG' + k_1 GF'] - \xi G' - bG &= 0, \end{aligned} \tag{3.1}$$

where prime denotes the differentiation with respect to the variable  $\xi$ .

To system (3.1) we seek a special solution in the following form:

$$\begin{aligned} F &= A_1 \xi^{-2} + B_1 \xi, \\ G &= A_2 \xi^{-2} + B_2 \xi. \end{aligned} \tag{3.2}$$

On substituting these expressions for  $F$  and  $G$  in (3.1), we arrive at the following relations among the various parameters involved:

$$\begin{aligned} B_1 &= \frac{2(1+a)(1+b) - 3(2-a)}{(2b-1)(1+a) + 3k_2(2a-1)(1+a)}, \\ B_2 &= \pm \left[ \frac{B_1 - B_1^2}{k_3} \right]^{1/2}, \quad k_1 = \frac{(1+b)}{(1+a)B_1} - k_2, \\ A_1 &= -\frac{12\sigma_2}{(k_1 + k_2)}, \quad A_2 = \frac{[(2-a) - (1+a)B_1]}{k_3(1+a)B_2} A_1, \quad \sigma_1 = -\frac{(A_1^2 + k_3 A_2^2)}{12A_1}, \end{aligned} \tag{3.3}$$

where  $a, b, k_2, k_3,$  and  $\sigma_2$  are arbitrary constants.

Thus, on using the back transformation, we get the solution of the system (1.1) as

$$\begin{aligned} u &= A_1 x^{-2} \left[ \int \alpha_1(t) dt \right]^{(2-a)/(1+a)} + B_1 x \left[ \int \alpha_1(t) dt \right]^{-1}, \\ v &= A_2 x^{-2} \left[ \int \alpha_1(t) dt \right]^{(2-b)/(1+a)} + B_2 x \left[ \int \alpha_1(t) dt \right]^{-(1+b)/(1+a)}. \end{aligned} \tag{3.4}$$

Type (ii). The reduced system of ODEs is

$$\begin{aligned} \sigma_1 F''' + k_3 GG' + FF' - F &= 0, \\ \sigma_2 G''' + k_2 FG' + k_1 GF' - cG &= 0. \end{aligned} \tag{3.5}$$

Herein, we will look for a special solution in terms of exponential functions, that is,  $F = a_1 \exp(m\xi)$  and  $G = a_2 \exp(m\xi)$ , for (3.5). The following exact solution of system (1.1)

can be readily derived:

$$\begin{aligned} u &= a_1 \left[ \int \alpha_1(t) dt \right]^{-1} \exp(mx), \\ v &= a_2 \left[ \int \alpha_1(t) dt \right]^{-c} \exp(mx), \end{aligned} \tag{3.6}$$

where  $a_1, a_2,$  and  $m$  are arbitrary constants and

$$\sigma_1 = \frac{1}{m^3}, \quad k_3 = -\frac{a_1^2}{a_2^2}, \quad k_2 = -k_1, \quad c = \sigma_2 m^3. \tag{3.7}$$

Type (iii). In this case, we have

$$\begin{aligned} \sigma_1 F''' + k_3 GG' + FF' - F' + F &= 0, \\ \sigma_2 G''' + k_2 FG' + k_1 GF' - G' + dG &= 0. \end{aligned} \tag{3.8}$$

We employ the hyperbolic tangent method to obtain particular analytic solutions to the system (3.8). On balancing the highest-order derivative terms with the nonlinear terms, one can easily find that the solution will have the form as given by

$$\begin{aligned} F &= a_0 + a_1 \tanh \xi + a_2 \tanh^2 \xi, \\ G &= b_0 + b_1 \tanh \xi + b_2 \tanh^2 \xi, \end{aligned} \tag{3.9}$$

where  $a_0, a_1, a_2,$  and  $b_0, b_1, b_2$  are constants to be determined. Substituting these forms in (3.8) and performing the algebraic calculations on the relations obtained among various parameters, we get

$$\begin{aligned} F &= 12\sigma_1 \operatorname{sech}^2 \xi, \\ G &= -\frac{12\sigma_1}{k_3 b_1} + b_1 \tanh \xi \end{aligned} \tag{3.10}$$

with

$$b_1^2 = -\frac{24\sigma_1(1 - 4\sigma_1)}{k_3}, \quad \sigma_2 = \frac{1}{4}, \quad k_2 = \frac{1}{8\sigma_1}, \quad k_1 = d = 0. \tag{3.11}$$

Corresponding solution of system (1.1) can be expressed as follows:

$$\begin{aligned} u &= 12\sigma_1 \operatorname{sech}^2 \left[ \ln \left( \int \alpha_1(t) dt \right) - x \right] \left[ \int \alpha_1(t) dt \right]^{-1}, \\ v &= -\frac{12\sigma_1}{k_3 b_1} + b_1 \tanh \left[ \ln \left( \int \alpha_1(t) dt \right) - x \right]. \end{aligned} \tag{3.12}$$

Type (iv). For the case under consideration the reduced ODEs are

$$\begin{aligned} \sigma_1 F''' + k_3 \mu^{-1} GG' + FF' &= 0, \\ \sigma_2 G''' + k_2 FG' + k_1 GF' - \mu^{-1} G &= 0. \end{aligned} \tag{3.13}$$

We are only able to furnish here a linear solution for  $F$  and  $G$  of the type

$$\begin{aligned}
 F &= \frac{\mu^{-1}}{(k_1 + k_2)} \xi + a_2, \\
 G &= \pm \left( \frac{\mu^{-1}}{(k_1 + k_2)\sqrt{-k_3\mu^{-1}}} \xi + \frac{a_2}{\sqrt{-k_3\mu^{-1}}} \right).
 \end{aligned}
 \tag{3.14}$$

The solution of system (1.1) becomes

$$\begin{aligned}
 u &= \frac{\mu^{-1}}{(k_1 + k_2)} x + a_2, \\
 v &= \pm \left( \frac{\mu^{-1}}{(k_1 + k_2)\sqrt{-k_3\mu^{-1}}} x + \frac{a_2}{\sqrt{-k_3\mu^{-1}}} \right) \exp \left[ -\mu^{-1} \int \alpha_1(t) dt \right].
 \end{aligned}
 \tag{3.15}$$

Type (v). The system of reduced ODEs is given by

$$\begin{aligned}
 \sigma_1 F''' + \frac{k_3}{\lambda} GG' + FF' - \frac{1}{\lambda} F' &= 0, \\
 \sigma_2 G''' + k_2 FG' + k_1 GF' - \frac{1}{\lambda} G' - \frac{1}{\lambda} G &= 0.
 \end{aligned}
 \tag{3.16}$$

Herein, we exploit the idea used in [29] to construct the closed form traveling wave solutions for some nonlinear differential equations. Let us assume that the system (3.16) admits a solution in the form

$$\begin{aligned}
 F &= \sum_{i=0}^m a_i \phi^i + \sum_{i=1}^m b_i \psi \phi^{i-1}, \\
 G &= \sum_{j=0}^n c_j \phi^j + \sum_{j=1}^n d_j \psi \phi^{j-1},
 \end{aligned}
 \tag{3.17}$$

where  $\phi$  and  $\psi$  are solutions of a coupled Riccati system:

$$\phi' = -k\phi\psi, \quad \psi' = k(1 - \psi^2),
 \tag{3.18}$$

where  $k$  is a constant to be found later. The Riccati system possesses two types of general solutions

$$\begin{aligned}
 \phi &= \pm \operatorname{sech}(k\xi), & \psi &= \tanh(k\xi), \\
 \phi &= \pm \operatorname{csch}(k\xi), & \psi &= \operatorname{coth}(k\xi).
 \end{aligned}
 \tag{3.19}$$

These two types of solutions satisfy the relations  $\psi^2 = 1 - \phi^2$  and  $\psi^2 = 1 + \phi^2$ . We will consider only the first solution for our system. Balancing the highest-order derivative terms with the nonlinear terms in system (3.16) we obtain  $m, n = 2$ . This suggests the following ansatz:

$$\begin{aligned}
 F &= a_0 + a_1 \phi + a_2 \phi^2 + b_1 \psi + b_2 \phi \psi, \\
 G &= c_0 + c_1 \phi + c_2 \phi^2 + d_1 \psi + d_2 \phi \psi.
 \end{aligned}
 \tag{3.20}$$

Using these in ODEs (3.16), we arrive at a set of algebraic equations which can be solved to get eventually

$$\begin{aligned}
 F &= \frac{1}{\lambda} + \frac{3}{\lambda(1-12k)} \operatorname{sech}^2(k\xi) + \frac{6}{\lambda(1-12k)} \tanh(k\xi), \\
 G &= \pm \left[ \frac{45}{\lambda k_3(1-12k)^2} \right]^{1/2} \operatorname{sech}^2(k\xi),
 \end{aligned}
 \tag{3.21}$$

subject to the following restrictions on the parameters:

$$\sigma_1 = \frac{3}{2k^2(1-12k)\lambda}, \quad \sigma_2 = \frac{1}{48k^3\lambda}, \quad k_2 = \frac{12k-1}{12k}, \quad k_1 = \frac{1-12k}{6k}.
 \tag{3.22}$$

Thus, we obtain a very elegant form of an exact analytic periodic solution of the KdV system (1.1)

$$\begin{aligned}
 u &= \frac{1}{\lambda} + \frac{3}{\lambda(1-12k)} \operatorname{sech}^2 \left[ k \left( x - \frac{1}{\lambda} \int \alpha_1(t) dt \right) \right] + \frac{6}{\lambda(1-12k)} \tanh \left[ k \left( x - \frac{1}{\lambda} \int \alpha_1(t) dt \right) \right], \\
 v &= \pm \left[ \frac{45}{\lambda k_3(1-12k)^2} \right]^{1/2} \operatorname{sech}^2 \left[ k \left( x - \frac{1}{\lambda} \int \alpha_1(t) dt \right) \right] \exp \left[ -\frac{1}{\lambda} \int \alpha_1(t) dt \right].
 \end{aligned}
 \tag{3.23}$$

*Type (vi).* The Jacobi elliptic function method is used to find an exact solution for the system

$$\begin{aligned}
 \sigma_1 F''' + k_3 G G' + F F' - F' &= 0, \\
 \sigma_2 G''' + k_2 F G' + k_1 G F' - G' &= 0.
 \end{aligned}
 \tag{3.24}$$

Following the procedure applied in [15], we take the solution of (3.24) in the form

$$\begin{aligned}
 F &= \sum_i^m a_i sn^i(\xi), \\
 G &= \sum_i^n a_i sn^i(\xi),
 \end{aligned}
 \tag{3.25}$$

where  $sn(\xi)$  is the Jacobi elliptic sine function.

We recall first that the Jacobi elliptic cosine function  $cn(\xi)$ , and the Jacobi elliptic function of third kind  $dn(\xi)$ , satisfy the following:

$$\begin{aligned}
 cn^2(\xi) &= 1 - sn^2(\xi), & dn^2(\xi) &= 1 - m^2 sn^2(\xi), \\
 \frac{d}{d\xi} sn(\xi) &= cn(\xi) dn(\xi), & \frac{d}{d\xi} cn(\xi) &= -sn(\xi) dn(\xi), \\
 \frac{d}{d\xi} dn(\xi) &= -m^2 cn(\xi) sn(\xi), & & \text{with modulus } m \ (0 < m < 1).
 \end{aligned}
 \tag{3.26}$$

We select the values of  $m, n = 2$  by balancing the highest-order derivative with the non-linear term, and this brings in the following ansatz:

$$\begin{aligned} F &= a_0 + a_1 sn(\xi) + a_2 sn^2(\xi), \\ G &= b_0 + b_1 sn(\xi) + b_2 sn^2(\xi). \end{aligned} \tag{3.27}$$

Combining (3.24) and (3.27), and performing algebraic computations, we deduce an interesting form of another periodic solution for (1.1) as follows:

$$\begin{aligned} u = F(\xi) &= \frac{(k_1 - k_3\gamma^2) + 4(1 + m^2)(k_1\sigma_1 - k_3\sigma_2\gamma^2)}{(k_1 - k_2k_3\gamma^2)} - \frac{12\sigma_2m^2}{(k_1 + k_2)} sn^2(\xi), \\ v = G(\xi) &= \pm \left( \frac{(k_2 - 1) + 4(1 + m^2)(k_2\sigma_1 - \sigma_2)}{(k_1/\gamma - k_2k_3\gamma)} + \frac{12\sigma_2m^2\gamma}{(k_1 + k_2)} sn^2(\xi) \right), \end{aligned} \tag{3.28}$$

where  $\gamma = [((k_1 + k_2)\sigma_1 - \sigma_2)/\sigma_2k_3]^{1/2}$  and  $\xi = x - \int \alpha_1(t) dt$ .

It is worth mentioning here that the class of solutions (3.28) gives rise, in particular, to a completely new exact periodic solution for the system analyzed in [32] corresponding to the parameter value  $k_1 = 0$ .

*Type (vii).* In this, we follow the procedure in a manner similar to the preceding case to solve the equations

$$\begin{aligned} \sigma_1 F''' + k_3 GG' + k_0 FF' &= 0, \\ \sigma_2 G''' + k_2 FG' + k_1 GF' &= 0, \end{aligned} \tag{3.29}$$

and the following form of the solution is produced:

$$\begin{aligned} F &= \frac{4(1 + m^2)(k_1\sigma_1 - k_3\sigma_2\gamma^2)}{(k_0k_1 - k_2k_3\gamma^2)} - \frac{12\sigma_2m^2}{(k_1 + k_2)} sn^2(\xi), \\ G &= \pm \left( \frac{4(1 + m^2)(k_2\sigma_1 - k_0\sigma_2)}{(k_0k_1/\gamma - k_2k_3\gamma)} + \frac{12\sigma_2m^2\gamma}{(k_1 + k_2)} sn^2(\xi) \right), \end{aligned} \tag{3.30}$$

where  $\gamma = [((k_1 + k_2)\sigma_1 - k_0\sigma_2)/\sigma_2k_3]^{1/2}$ . The solution to (1.1) can be expressed as

$$u = F(\xi), \quad v = G(\xi), \quad \text{with } \xi = x. \tag{3.31}$$

*Type (viii).* The reduced ODEs, in this case are

$$F' = 0, \quad G' = 0. \tag{3.32}$$

The obvious solution of (1.1) is

$$u = \text{constant}, \quad v = \text{constant}. \tag{3.33}$$

#### 4. Discussion and concluding remarks

We have investigated the symmetries and invariant solutions of a variable coefficient KdV system. The generalized symmetry method is utilized for the purpose of obtaining the

group infinitesimals. Using the adjoint action of the symmetry group an optimal system is identified. The basic fields of the optimal system lead to reductions that are inequivalent with respect to the symmetry transformations. For each element in the optimal system, some exact solutions are attempted for the KdV system. It is important to note that, besides deriving the exact solutions for the system under investigation, new exact solutions have also been deduced in particular, for a system examined in [32]. Moreover, in one instance (refer to (3.28)), the solution furnished brings forth a new class of exact periodic solutions of the same system [32]. In almost all the cases, one can choose the arbitrary function  $\alpha_1(t)$ , along with various other arbitrary parameters, in a suitable manner, and this provides enough freedom to build solutions that may correspond to a particular physical situation. It is to be mentioned that, with the aid of Maple, the exact solutions reported here in this paper are found to indeed satisfy the KdV system. Finally, it is being pointed out that the KdV system examined in the present study will also admit some additional symmetries for the special case  $\alpha_1(t) = \beta_2(t)$ . The work is in progress and will be communicated later upon completion.

The main results of Sections 2 and 3 are summarized as follows.

**THEOREM 4.1.** *For a given smooth function  $\alpha_1(t)$ , the KdV system (1.1) remains invariant under a 5-parameter Lie algebra spanned by the following basis elements:*

$$\begin{aligned} V_1 &= \left[ \frac{1}{\alpha_1(t)} \int \alpha_1(t) dt \right] \frac{\partial}{\partial t} + x \frac{\partial}{\partial x}, \\ V_2 &= \left[ \frac{1}{\alpha_1(t)} \int \alpha_1(t) dt \right] \frac{\partial}{\partial t} - u \frac{\partial}{\partial u}, \\ V_3 &= -v \frac{\partial}{\partial v}, \quad V_4 = \frac{1}{\alpha_1(t)} \frac{\partial}{\partial t}, \quad V_5 = \frac{\partial}{\partial x}. \end{aligned} \tag{4.1}$$

The admissible forms of the coefficient functions satisfy the following equations:

$$\begin{aligned} \frac{d}{dt} \left[ \frac{\beta_1(t)}{\alpha_1(t)} \left\{ (a_1 + a_2) \int \alpha_1(t) dt + a_4 \right\} \right] + (a_2 - 2a_3 - a_1) \beta_1(t) &= 0, \\ \frac{d}{dt} \left[ \frac{\gamma_i(t)}{\alpha_1(t)} \left\{ (a_1 + a_2) \int \alpha_1(t) dt + a_4 \right\} \right] - 3a_1 \gamma_i(t) &= 0, \quad i = 1, 2, \\ \alpha_2(t) &= k_1 \alpha_1(t), \quad \beta_2(t) = k_2 \alpha_1(t), \end{aligned} \tag{4.2}$$

where  $a_1, a_2, a_3, a_4, k_1,$  and  $k_2$  are arbitrary constants.

**THEOREM 4.2 (periodic solution).** *Suppose that the coefficient functions are as given in row (v) of Table 2.1. Then*

$$\begin{aligned} u &= \frac{1}{\lambda} + \frac{3}{\lambda(1-12k)} \operatorname{sech}^2 \left[ k \left( x - \frac{1}{\lambda} \int \alpha_1(t) dt \right) \right] + \frac{6}{\lambda(1-12k)} \tanh \left[ k \left( x - \frac{1}{\lambda} \int \alpha_1(t) dt \right) \right], \\ v &= \pm \left[ \frac{45}{\lambda k_3 (1-12k)^2} \right]^{1/2} \operatorname{sech}^2 \left[ k \left( x - \frac{1}{\lambda} \int \alpha_1(t) dt \right) \right] \exp \left[ -\frac{1}{\lambda} \int \alpha_1(t) dt \right] \end{aligned} \tag{4.3}$$

form a solution to the KdV equations (1.1), when

$$\sigma_1 = \frac{3}{2k^2(1-12k)\lambda}, \quad \sigma_2 = \frac{1}{48k^3\lambda}, \quad k_2 = \frac{12k-1}{12k}, \quad k_1 = \frac{1-12k}{6k}; \quad (4.4)$$

$\lambda \neq 0, k_3 \neq 0, k \neq 1/12$  are arbitrary constants; and  $\alpha_1(t)$  is an arbitrary smooth function.

**THEOREM 4.3** (periodic solution). *Suppose that the coefficient functions are as given in row (vi) of Table 2.1. Then*

$$\begin{aligned} u = F(\xi) &= \frac{(k_1 - k_3\gamma^2) + 4(1 + m^2)(k_1\sigma_1 - k_3\sigma_2\gamma^2)}{(k_1 - k_2k_3\gamma^2)} - \frac{12\sigma_2m^2}{(k_1 + k_2)}\text{sn}^2(\xi), \\ v = G(\xi) &= \pm \left( \frac{(k_2 - 1) + 4(1 + m^2)(k_2\sigma_1 - \sigma_2)}{(k_1/\gamma - k_2k_3\gamma)} + \frac{12\sigma_2m^2\gamma}{(k_1 + k_2)}\text{sn}^2(\xi) \right), \end{aligned} \quad (4.5)$$

where  $\gamma = [((k_1 + k_2)\sigma_1 - \sigma_2)/\sigma_2k_3]^{1/2} \neq 0, k_1 \neq k_2k_3\gamma^2, 0 < m < 1, \sigma_2 \neq 0, k_3 \neq 0,$  and  $k_1 \neq -k_2$  are all arbitrary constants and  $\xi = x - \int \alpha_1(t)dt,$  with  $\alpha_1(t)$  as a smooth arbitrary function, is a solution of the KdV system (1.1).

**THEOREM 4.4.** *Suppose that the coefficient functions are as given in row (i) of Table 2.1. Then, for an arbitrary smooth function  $\alpha_1(t),$  the KdV system (1.1) has a solution of the form*

$$\begin{aligned} u &= A_1x^{-2} \left[ \int \alpha_1(t)dt \right]^{(2-a)/(1+a)} + B_1x \left[ \int \alpha_1(t)dt \right]^{-1}, \\ v &= A_2x^{-2} \left[ \int \alpha_1(t)dt \right]^{(2-b)/(1+a)} + B_2x \left[ \int \alpha_1(t)dt \right]^{-(1+b)/(1+a)}, \end{aligned} \quad (4.6)$$

where

$$\begin{aligned} B_1 &= \frac{2(1+a)(1+b) - 3(2-a)}{(2b-1)(1+a) + 3k_2(2a-1)(1+a)}, \\ B_2 &= \pm \left[ \frac{B_1 - B_1^2}{k_3} \right]^{1/2}, \quad A_1 = -\frac{12\sigma_2}{(k_1 + k_2)}, \quad A_2 = \frac{[(2-a) - (1+a)B_1]}{k_3(1+a)B_2}A_1, \\ k_1 &= \frac{(1+b)}{(1+a)B_1} - k_2, \quad \sigma_1 = -\frac{(A_1^2 + k_3A_2^2)}{12A_1} \end{aligned} \quad (4.7)$$

with  $a, b, k_2, k_3,$  and  $\sigma_2$  as arbitrary constants.

**THEOREM 4.5.** *Suppose that the coefficient functions are as given in row (iii) of Table 2.1. Then, for an arbitrary smooth function  $\alpha_1(t),$  the KdV system (1.1) has a solution of the form*

$$\begin{aligned} u &= 12\sigma_1 \text{sech}^2 \left[ \ln \left( \int \alpha_1(t)dt \right) - x \right] \left[ \int \alpha_1(t)dt \right]^{-1}, \\ v &= -\frac{12\sigma_1}{k_3b_1} + b_1 \tanh \left[ \ln \left( \int \alpha_1(t)dt \right) - x \right], \end{aligned} \quad (4.8)$$

where

$$b_1^2 = -\frac{24\sigma_1(1-4\sigma_1)}{k_3}, \quad \sigma_2 = \frac{1}{4}, \quad k_2 = \frac{1}{8\sigma_1}, \quad k_1 = d = 0 \quad (4.9)$$

with  $\sigma_1, k_3$  as arbitrary constants.

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K. Singh: Department of Mathematics, Jaypee University of Information Technology, Wagnaghat, P.O. Dumehar Bani, Kandaghat, Distt. Solan, Pin-173215 (H.P.), India  
*E-mail addresses:* karan\_jeet@yahoo.com ; ksingh\_juit@hotmail.com

R. K. Gupta: Department of Mathematics, Jaypee University of Information Technology, Wagnaghat, P.O. Dumehar Bani, Kandaghat, Distt. Solan, Pin-173215 (H.P.), India  
*E-mail address:* rajeshateli@rediffmail.com