EXISTENCE, COMPARISON, AND COMPACTNESS RESULTS FOR QUASILINEAR VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

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We consider quasilinear elliptic variational-hemivariational inequalities involving the indicator function of some closed convex set and a locally Lipschitz functional. We provide a generalization of the fundamental notion of sub- and supersolutions, on the basis of which we then develop the sub-supersolution method for variational-hemivariational inequalities, including existence, comparison, compactness, and extremality results.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, and let $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega)$, $1 < p < \infty$, denote the usual Sobolev spaces with their dual spaces V^* and V_0^*, respectively. In this paper, we deal with the following quasilinear variational-hemivariational inequality:

$$u \in K : \langle Au - f, v - u \rangle + \int_\Omega j^o(u; v - u)\,dx \geq 0 \quad \forall v \in K, \tag{1.1}$$

where $j^o(s;r)$ denotes the generalized directional derivative of the locally Lipschitz function $j : \mathbb{R} \to \mathbb{R}$ at s in the direction r given by

$$j^o(s;r) = \limsup_{y \to s, t \downarrow 0} \frac{j(y+tr) - j(y)}{t}, \tag{1.2}$$

(cf., e.g., [6, Chapter 2]), $f \in V_0^*$, and K is a closed and convex subset of V_0. The operator $A : V \to V_0^*$ is a second-order quasilinear differential operator in divergence form:

$$Au(x) = -\sum_{i=1}^N \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)) \quad \text{with } \nabla u = \left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_N}\right). \tag{1.3}$$

The main goal of this paper is to develop the sub-supersolution method for variational-hemivariational inequalities of form (1.1). Problem (1.1) includes various special cases.

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(i) For $K = V_0$ and $j : \mathbb{R} \to \mathbb{R}$ smooth, (1.1) is the weak formulation of the Dirichlet problem

$$u \in V_0 : Au + j'(u) = f \quad \text{in } V_0^*, \tag{1.4}$$

for which the sub-supersolution method is well known.

(ii) If $K = V_0$, and $j : \mathbb{R} \to \mathbb{R}$ not necessarily smooth, then (1.1) is a hemivariational inequality of the form

$$u \in V_0 : \langle Au - f, v - u \rangle + \int_\Omega j^o(u; v - u)\,dx \geq 0 \quad \forall v \in V_0, \tag{1.5}$$

for which an extension of the sub-supersolution method has been given recently in [3].

(iii) If $j = 0$, then (1.1) becomes a variational inequality for which a sub-supersolution method has been developed in [8, 9].

This paper continues the work on the extension of the sub-supersolution method started with the papers by Carl, Le, and Motreanu in [2, 3, 8, 9] to develop a strongly generalized and unified theory that includes all the above cited special cases.

2. Notation and hypotheses

We assume the following hypotheses of Leray-Lions type on the coefficient functions a_i, $i = 1, \ldots, N$, of the operator A.

(A1) Each $a_i : \Omega \times \mathbb{R}^N \to \mathbb{R}$ satisfies the Carathéodory conditions, that is, $a_i(x,\xi)$ is measurable in $x \in \Omega$ for all $\xi \in \mathbb{R}^N$ and continuous in ξ for almost all $x \in \Omega$. There exist a constant $c_0 > 0$ and a function $k_0 \in L^q(\Omega)$, $1/p + 1/q = 1$, such that

$$|a_i(x,\xi)| \leq k_0(x) + c_0|\xi|^{p-1} \tag{2.1}$$

for a.e. $x \in \Omega$ and for all $\xi \in \mathbb{R}^N$.

(A2)

$$\sum_{i=1}^{N} \left(a_i(x,\xi) - a_i(x,\xi')\right)\left(\xi_i - \xi_i'\right) > 0 \tag{2.2}$$

for a.e. $x \in \Omega$, and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

(A3)

$$\sum_{i=1}^{N} a_i(x,\xi)\xi_i \geq \nu|\xi|^p - k_1(x) \tag{2.3}$$

for a.e. $x \in \Omega$, and for all $\xi \in \mathbb{R}^N$ with some constant $\nu > 0$ and some function $k_1 \in L^1(\Omega)$.

As a consequence of (A1), (A2) the semilinear form a associated with the operator A by

$$\langle Au, \varphi \rangle := a(u, \varphi) = \int_\Omega \sum_{i=1}^{N} a_i(x, \nabla u) \frac{\partial \varphi}{\partial x_i} dx \quad \forall \varphi \in V_0 \tag{2.4}$$

is well defined for any $u \in V$, and the operator $A : V_0 \to V_0^*$ is continuous, bounded, and strictly monotone. For functions $w, z : \Omega \to \mathbb{R}$ and sets W and Z of functions defined on Ω we use the following notations: $w \wedge z = \min\{w, z\}$, $w \vee z = \max\{w, z\}$, $W \wedge Z = \{w \wedge z \mid w \in W,\ z \in Z\}$, $W \vee Z = \{w \vee z \mid w \in W,\ z \in Z\}$, and $w \wedge Z = \{w\} \wedge Z$, $w \vee Z = \{w\} \vee Z$. Next we introduce our basic notion of sub-supersolution.

Definition 2.1. A function $\underline{u} \in V$ is called a *subsolution* of (1.1) if the following holds:

(i) $\underline{u} \le 0$ on $\partial\Omega$,

(ii) $\langle A\underline{u} - f, v - \underline{u} \rangle + \int_\Omega j^o(\underline{u}; v - \underline{u}) dx \ge 0$, for all $v \in \underline{u} \wedge K$.

Definition 2.2. $\bar{u} \in V$ is a *supersolution* of (1.1) if the following holds:

(i) $\bar{u} \ge 0$ on $\partial\Omega$,

(ii) $\langle A\bar{u} - f, v - \bar{u} \rangle + \int_\Omega j^o(\bar{u}; v - \bar{u}) dx \ge 0$, for all $v \in \bar{u} \vee K$.

Note that the notion of sub-supersolution introduced here extends that for inclusions of hemivariational type introduced in [4, 5] and those for variational or hemivariational inequalities in [3, 8, 9].

Let $\partial j : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ denote Clarke's generalized gradient of j defined by

$$\partial j(s) := \{\zeta \in \mathbb{R} \mid j^o(s; r) \ge \zeta r,\ \forall r \in \mathbb{R}\}. \tag{2.5}$$

We assume the following hypothesis for j.

(H) The function $j : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz and its Clarke's generalized gradient ∂j satisfies the following growth conditions:

(i) there exists a constant $c_1 \ge 0$ such that

$$\xi_1 \le \xi_2 + c_1 (s_2 - s_1)^{p-1} \tag{2.6}$$

for all $\xi_i \in \partial j(s_i)$, $i = 1, 2$, and for all s_1, s_2 with $s_1 < s_2$,

(ii) there is a constant $c_2 \ge 0$ such that

$$\xi \in \partial j(s) : |\xi| \le c_2 (1 + |s|^{p-1}) \quad \forall s \in \mathbb{R}. \tag{2.7}$$

Let $L^p(\Omega)$ be equipped with the natural partial ordering of functions defined by $u \le w$ if and only if $w - u$ belongs to the positive cone $L^p_+(\Omega)$ of all nonnegative elements of $L^p(\Omega)$. This induces a corresponding partial ordering also in the subspace V of $L^p(\Omega)$, and if $u, w \in V$ with $u \le w$, then

$$[u, w] = \{z \in V \mid u \le z \le w\} \tag{2.8}$$

denotes the ordered interval formed by u and w.

In the proofs of our main results we make use of the cut-off function $b:\Omega\times\mathbb{R}\to\mathbb{R}$ related to an ordered pair of functions $\underline{u}\le\bar{u}$, and given by

$$b(x,s)=\begin{cases}(s-\bar{u}(x))^{p-1} & \text{if } s>\bar{u}(x),\\ 0 & \text{if } \underline{u}(x)\le s\le\bar{u}(x),\\ -(\underline{u}(x)-s)^{p-1} & \text{if } s<\underline{u}(x).\end{cases} \tag{2.9}$$

One readily verifies that b is a Carathéodory function satisfying the growth condition

$$|b(x,s)|\le k(x)+c_3|s|^{p-1} \tag{2.10}$$

for a.e. $x\in\Omega$, for all $s\in\mathbb{R}$, with some function $k\in L^q_+(\Omega)$ and a constant $c_3\ge 0$. Moreover, one has the following estimate

$$\int_\Omega b(x,u(x))u(x)dx\ge c_4\|u\|^p_{L^p(\Omega)}-c_5\quad \forall u\in L^p(\Omega), \tag{2.11}$$

where c_4 and c_5 are some positive constants. In view of (2.10) the Nemytskij operator $B:L^p(\Omega)\to L^q(\Omega)$ defined by

$$Bu(x)=b(x,u(x)) \tag{2.12}$$

is continuous and bounded, and thus due to the compact embedding $V\subset L^p(\Omega)$ it follows that $B:V_0\to V_0^*$ is compact.

3. Preliminaries

In this section, we briefly recall a surjectivity result for multivalued mappings in reflexive Banach spaces (cf., e.g., [10, Theorem 2.12]) which among others will be used in the proof of our main result in this section.

Theorem 3.1. *Let X be a real reflexive Banach space with dual space X^*, $\Phi:X\to 2^{X^*}$ a maximal monotone operator, and $u_0\in\operatorname{dom}(\Phi)$. Let $A:X\to 2^{X^*}$ be a pseudomonotone operator, and assume that either A_{u_0} is quasibounded or Φ_{u_0} is strongly quasibounded. Assume further that $A:X\to 2^{X^*}$ is u_0-coercive, that is, there exists a real-valued function $c:\mathbb{R}_+\to\mathbb{R}$ with $c(r)\to+\infty$ as $r\to+\infty$ such that for all $(u,u^*)\in\operatorname{graph}(A)$, $\langle u^*,u-u_0\rangle\ge c(\|u\|_X)\|u\|_X$ holds. Then $A+\Phi$ is surjective, that is,* $\operatorname{range}(A+\Phi)=X^*$.

The operators A_{u_0} and Φ_{u_0} that appear in the theorem above are defined by $A_{u_0}(v):=A(u_0+v)$ and similarly for Φ_{u_0}. As for the notion of *quasibounded* and *strongly quasibounded*, we refer to [10, page 51]. In particular, one has that any bounded operator is quasibounded and strongly quasibounded as well. The following proposition provides sufficient conditions for an operator $A:X\to 2^{X^*}$ to be pseudomonotone, which is suitable for our purpose.

Proposition 3.2. *Let X be a real reflexive Banach space, and assume that $A:X\to 2^{X^*}$ satisfies the following conditions:*

(i) *for each $u\in X$, $A(u)$ is a nonempty, closed, and convex subset of X^*;*

(ii) $A : X \to 2^{X^*}$ *is bounded;*
(iii) *if* $u_n \rightharpoonup u$ *in* X *and* $u_n^* \rightharpoonup u^*$ *in* X^* *with* $u_n^* \in A(u_n)$ *and if* $\limsup \langle u_n^*, u_n - u \rangle \leq 0$, *then* $u^* \in A(u)$ *and* $\langle u_n^*, u_n \rangle \to \langle u^*, u \rangle$.

Then the operator $A : X \to 2^{X^*}$ *is pseudomonotone.*

As for the proof of Proposition 3.2 we refer, for example, to [10, Chapter 2].

4. Existence and comparison result

The main result of this section is given by the following theorem which provides an existence and comparison result for the variational-hemivariational inequality (1.1).

Theorem 4.1. *Let* $\bar{u}$ *and* $\underline{u}$ *be super- and subsolutions of (1.1), respectively, satisfying* $\underline{u} \leq \bar{u}$, *and assume* $\bar{u} \wedge K \subset K$ *and* $\underline{u} \vee K \subset K$. *Then under hypotheses (A1)–(A3) and (H), there exist solutions of (1.1) within the ordered interval* $[\underline{u}, \bar{u}]$.

Proof. Let $I_K : V_0 \to \mathbb{R} \cup \{+\infty\}$ denote the indicator function related to the given closed convex set $K \neq \emptyset$ and defined by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K, \end{cases} \tag{4.1}$$

which is proper, convex, and lower semicontinuous. By means of the indicator function the variational-hemivariational inequality (1.1) can be rewritten in the following form. Find $u \in K$ such that

$$\langle Au - f, v - u \rangle + I_K(v) - I_K(u) + \int_\Omega j^o(u; v - u)\,dx \geq 0 \quad \forall v \in V_0. \tag{4.2}$$

Since we are looking for solutions of (4.2) within $[\underline{u}, \bar{u}]$, we consider the following auxiliary problem: Find $u \in K$ such that

$$\langle Au - f + \lambda B(u), v - u \rangle + I_K(v) - I_K(u) + \int_\Omega j^o(u; v - u)\,dx \geq 0 \quad \forall v \in V_0, \tag{4.3}$$

where B is the cut-off operator introduced in Section 2, and $\lambda \geq 0$ is some parameter to be specified later. As will be seen in the course of the proof, the role of λB is twofold. First it provides a coercivity generating term, and second, it allows for comparison. The proof of the theorem will be done in two steps. In Step 1 we prove the existence of solutions of auxiliary problem (4.3), and in Step 2 we are going to show that any solution of (4.3) belongs to the interval $[\underline{u}, \bar{u}]$, which completes the proof, since then $B(u) = 0$ and (4.2) holds.

Step 1 (existence for (4.3)). We introduce the functional $J : L^p(\Omega) \to \mathbb{R}$ defined by

$$J(v) = \int_\Omega j(v(x))\,dx \quad \forall v \in L^p(\Omega), \tag{4.4}$$

which by hypothesis (H) is locally Lipschitz, and moreover, by Aubin-Clarke theorem (see [6, page 83]) for each $u \in L^p(\Omega)$ we have

$$\xi \in \partial J(u) \Longrightarrow \xi \in L^q(\Omega) \quad \text{with } \xi(x) \in \partial j(u(x)) \text{ for a.e. } x \in \Omega. \tag{4.5}$$

Consider now the multivalued operator

$$A + \lambda B + \partial(J|_{V_0}) + \partial I_K : V_0 \longrightarrow 2^{V_0^*}, \tag{4.6}$$

where $J|_{V_0}$ denotes the restriction of J to V_0 and ∂I_K is the subdifferential of I_K in the sense of convex analysis. It is well known that $\Phi := \partial I_K : V_0 \to 2^{V_0^*}$ is a maximal monotone operator (cf., e.g., [11]). Since $A : V_0 \to V_0^*$ is strictly monotone, bounded, and continuous, and $\lambda B : V_0 \to V_0^*$ is bounded, continuous, and compact, it follows that $A + \lambda B : V_0 \to V_0^*$ is a (single-valued) pseudomonotone, continuous, and bounded operator. In [3], it has been shown that $\partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is a (multivalued) pseudomonotone operator, which, due to (H), is bounded. Thus $A_0 := A + \lambda B + \partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is a pseudomonotone and bounded operator. Hence, it follows by Theorem 3.1 that range$(A_0 + \Phi) = V_0^*$ provided A_0 is u_0-coercive for some $u_0 \in K$, which can readily be seen as follows. For any $v \in V_0$ and any $w \in \partial(J|_{V_0})(v)$, we obtain by applying (A3), (H)(ii), and (2.11) the estimate

$$\begin{aligned}
&\langle Av + \lambda B(v) + w, v - u_0 \rangle \\
&\quad = \int_\Omega \sum_{i=1}^N a_i(x, \nabla v) \frac{\partial v}{\partial x_i} dx + \lambda \langle B(v), v \rangle + \int_\Omega wv\, dx - \langle Av + \lambda B(v) + w, u_0 \rangle \\
&\quad \geq \nu \int_\Omega |\nabla v|^p dx - ||k_1||_{L^1(\Omega)} + c_4 \lambda \|v\|^p_{L^p(\Omega)} - c_5 \lambda \\
&\qquad - c_2 \int_\Omega (1 + |v|^{p-1}) |v| dx - |\langle Av + \lambda B(v) + w, u_0 \rangle| \\
&\quad \geq \nu \|v\|^p_{V_0} - C\Big(1 + ||v||^{p-1}_{V_0}\Big)
\end{aligned} \tag{4.7}$$

for some constant $C > 0$, by choosing the constant λ in such a way that $c_4 \lambda > c_2$. Since $p > 1$, the coercivity of A_0 follows from (4.7). In view of the surjectivity of the operator $A_0 + \Phi$, there exists a $u \in K$ such that $f \in A_0(u) + \Phi(u)$, that is, there is an $\xi \in \partial(J|_{V_0})(u)$ with $\xi \in L^q(\Omega)$ and $\xi(x) \in \partial j(u(x))$ for a.e. $x \in \Omega$, and an $\eta \in \Phi(u)$ such that

$$Au - f + \lambda B(u) + \xi + \eta = 0 \quad \text{in } V_0^*, \tag{4.8}$$

where

$$\langle \xi, \varphi \rangle = \int_\Omega \xi(x) \varphi(x) dx \quad \forall \varphi \in V_0, \tag{4.9}$$

$$I_K(v) \geq I_K(u) + \langle \eta, v - u \rangle \quad \forall v \in V_0. \tag{4.10}$$

By definition of Clarke's generalized gradient ∂j from (4.9) we get

$$\langle \xi, \varphi \rangle = \int_\Omega \xi(x) \varphi(x) dx \leq \int_\Omega j^o(u(x); \varphi(x)) dx \quad \forall \varphi \in V_0. \tag{4.11}$$

Thus from (4.8), (4.9), (4.10), and (4.11) with φ replaced by $v-u$ we obtain (4.3), which proves the existence of solutions of problem (4.3).

Step 2 ($\underline{u} \le u \le \bar{u}$ for any solution u of (4.3)). We first show $u \le \bar{u}$. By definition, the supersolution $\bar{u}$ satisfies $\bar{u} \ge 0$ on $\partial\Omega$, and

$$\langle A\bar{u} - f, v - \bar{u}\rangle + \int_\Omega j^o(\bar{u}; v - \bar{u})dx \ge 0 \quad \forall v \in \bar{u} \vee K. \tag{4.12}$$

Let u be any solution of (4.3) which is equivalent to the following variational-hemi-variational inequality:

$$u \in K : \langle Au - f, v - u\rangle + \langle \lambda B(u), v - u\rangle + \int_\Omega j^o(u; v - u)dx \ge 0 \quad \forall v \in K. \tag{4.13}$$

We apply the special test function $v = \bar{u} \vee u = \bar{u} + (u-\bar{u})^+ (\in \bar{u} \vee K)$ in (4.12) and $v = \bar{u} \wedge u = u - (u-\bar{u})^+ (\in K)$ in (4.13), and get by adding the resulting inequalities the following one:

$$\begin{aligned} &\langle A\bar{u} - Au, (u-\bar{u})^+\rangle + \lambda\langle B(u), -(u-\bar{u})^+\rangle \\ &\quad + \int_\Omega \left(j^o(\bar{u}; (u-\bar{u})^+) + j^o(u; -(u-\bar{u})^+)\right)dx \ge 0, \end{aligned} \tag{4.14}$$

which yields due to

$$\langle Au - A\bar{u}, (u-\bar{u})^+\rangle \ge 0, \tag{4.15}$$

the inequality

$$\lambda\langle B(u), (u-\bar{u})^+\rangle \le \int_\Omega \left(j^o(\bar{u}; (u-\bar{u})^+) + j^o(u; -(u-\bar{u})^+)\right)dx. \tag{4.16}$$

By using (H) and the properties on j^o and ∂j we get for certain $\bar{\xi}(x) \in \partial j(\bar{u}(x))$ and $\xi(x) \in \partial j(u(x))$ the following estimate of the right-hand side of (4.16):

$$\begin{aligned} &\int_\Omega \left(j^o(\bar{u}; (u-\bar{u})^+) + j^o(u; -(u-\bar{u})^+)\right)dx \\ &\quad = \int_{\{u>\bar{u}\}} \left(j^o(\bar{u}; u-\bar{u}) + j^o(u; -(u-\bar{u}))\right)dx \\ &\quad = \int_{\{u>\bar{u}\}} \left(\bar{\xi}(x)(u(x)-\bar{u}(x)) + \xi(x)\left(-(u(x)-\bar{u}(x))\right)\right)dx \\ &\quad = \int_{\{u>\bar{u}\}} \left(\bar{\xi}(x) - \xi(x)\right)(u(x)-\bar{u}(x))dx \\ &\quad \le \int_{\{u>\bar{u}\}} c_1 (u(x)-\bar{u}(x))^p dx. \end{aligned} \tag{4.17}$$

Since

$$\langle B(u), (u-\bar{u})^+\rangle = \int_{\{u>\bar{u}\}} (u-\bar{u})^p dx, \tag{4.18}$$

we get from (4.16) and (4.17) the estimate

$$(\lambda - c_1)\int_{\{u>\bar{u}\}} (u-\bar{u})^p\,dx \le 0. \tag{4.19}$$

Selecting the parameter λ, in addition, such that $\lambda - c_1 > 0$, then (4.19) yields

$$\int_\Omega \left((u-\bar{u})^+\right)^p dx \le 0, \tag{4.20}$$

which implies $(u-\bar{u})^+ = 0$ and thus $u \le \bar{u}$. The proof for the inequality $\underline{u} \le u$ can be carried out in a similar way which completes the proof of the theorem. □

5. Compactness and existence of extremal solutions

Let $\mathcal{S}$ denote the set of all solutions of (1.1) within the interval $[\underline{u}, \bar{u}]$ of an ordered pair of sub- and supersolutions. In this section, we are going to show that the solution set $\mathcal{S}$ is compact, and under certain lattice conditions on K, $\mathcal{S}$ possesses the smallest and greatest elements with respect to the given partial ordering. The smallest and greatest elements of $\mathcal{S}$ are called the *extremal solutions* of (1.1) within $[\underline{u}, \bar{u}]$.

Theorem 5.1. *Under the hypotheses of Theorem 4.1 the solution set $\mathcal{S}$ is compact in V_0.*

Proof. First we prove that $\mathcal{S}$ is bounded in V_0. Since any $u \in \mathcal{S}$ belongs to the interval $[\underline{u}, \bar{u}]$, it follows that $\mathcal{S}$ is bounded in $L^p(\Omega)$. Moreover, any $u \in \mathcal{S}$ solves (1.1), that is, we have $u \in K : \langle Au - f, v - u\rangle + \int_\Omega j^o(u; v-u)\,dx \ge 0$, for all $v \in K$. Let u_0 be any (fixed) element of K. By taking $v = u_0$ in the above inequality we get

$$\langle Au, u\rangle \le \langle Au, u_0\rangle + \langle f, u - u_0\rangle + \int_\Omega j^o(u; u_0 - u)\,dx. \tag{5.1}$$

This yields, by applying (A3), (H)(ii), and Young's inequality, the following estimate:

$$\nu\|\nabla u\|^p_{L^p(\Omega)} \le \|k_1\|_{L^1(\Omega)} + c(\varepsilon)\left(\|f\|^q_{V_0^*} + 1\right) + \varepsilon\|u\|^p_{V_0} + \tilde{\alpha}\left(\|u\|_{L^p(\Omega)} + \|u\|^p_{L^p(\Omega)} + 1\right) \tag{5.2}$$

for any $\varepsilon > 0$. Hence, the boundedness of $\mathcal{S}$ in V_0 follows by choosing ε sufficiently small and by taking into account that $\mathcal{S}$ is bounded in $L^p(\Omega)$.

Let $(u_n) \subset \mathcal{S}$. From the above boundedness of $\mathcal{S}$ in V_0, we can choose a subsequence (u_k) of (u_n) such that

$$u_k \rightharpoonup u \quad \text{in } V_0, \qquad u_k \longrightarrow u \quad \text{in } L^p(\Omega), \qquad u_k(x) \longrightarrow u(x) \quad \text{a.e. in } \Omega. \tag{5.3}$$

Obviously $u \in [\underline{u}, \bar{u}]$. On the other hand, because K is closed and convex in V_0, it is weakly closed. As $u_k \in K$ for all k, we see that u is also in K. Since u_k solve (1.1), we can put $v = u \in K$ in (1.1) (with u_k instead of u) and get

$$\langle Au_k - f, u - u_k\rangle + \int_\Omega j^o(u_k; u - u_k)\,dx \ge 0, \tag{5.4}$$

and thus

$$\langle Au_k, u_k - u\rangle \le \langle f, u_k - u\rangle + \int_\Omega j^o(u_k; u - u_k)\,dx. \tag{5.5}$$

Due to (5.3) and due to the fact that $(s,r) \mapsto j^o(s;r)$ is upper semicontinuous, we get by applying Fatou's lemma

$$\limsup_k \int_\Omega j^o(u_k; u - u_k)\,dx \le \int_\Omega \limsup_k j^o(u_k; u - u_k)\,dx = 0. \tag{5.6}$$

In view of (5.6) we thus obtain from (5.3) and (5.5)

$$\limsup_k \langle Au_k, u_k - u\rangle \le 0. \tag{5.7}$$

Since the operator A has the (S_+)-property (we refer, e.g., to [1] for the definition of the (S_+)-property being used here), the weak convergence of (u_k) in V_0 along with (5.7) imply the strong convergence $u_k \to u$ in V_0, see, for example, [1, Theorem D.2.1]. Moreover, the limit u belongs to $\mathcal{S}$ as can be seen by passing to the lim sup on the left-hand side of the following inequality:

$$\langle Au_k - f, v - u_k\rangle + \int_\Omega j^o(u_k; v - u_k)\,dx \ge 0, \tag{5.8}$$

where we have used Fatou's lemma and the strong convergence of (u_k) in V_0. This completes the proof. □

As for the existence of extremal solutions in $\mathcal{S}$, we introduce the following notion.

Definition 5.2. Let $(\mathcal{P}, \le)$ be a partially ordered set. A subset $\mathcal{C}$ of $\mathcal{P}$ is said to be *upward-directed* if for each pair $x, y \in \mathcal{C}$, there is a $z \in \mathcal{C}$ such that $x \le z$ and $y \le z$, and $\mathcal{C}$ is *downward-directed* if for each pair $x, y \in \mathcal{C}$, there is a $w \in \mathcal{C}$ such that $w \le x$ and $w \le y$. If $\mathcal{C}$ is both upward and downward directed it is called *directed*.

We are now ready to prove our extremality result.

Theorem 5.3. *Let the hypotheses of Theorem 4.1 be satisfied, and assume, moreover,*

$$K \wedge K \subset K, \qquad K \vee K \subset K. \tag{5.9}$$

Then, the solution set $\mathcal{S}$ possesses extremal elements.

Proof. The proof of Theorem 5.3 is divided into two steps. In Step 1, we show that the solution set $\mathcal{S}$ is directed, and the existence of extremal elements of $\mathcal{S}$ is proved in Step 2.

Step 1 ($\mathcal{S}$ is a directed set). As a consequence of Theorem 4.1, we have $\mathcal{S} \ne \emptyset$. Given $u_1, u_2 \in \mathcal{S}$, we show that there is a $u \in \mathcal{S}$ such that $u_k \le u$, $k = 1, 2$, which means $\mathcal{S}$ is upward-directed. To this end we consider the following auxiliary variational-hemivariational inequality

$$u \in K : \langle Au - f + \lambda B(u), v - u\rangle + \int_\Omega j^o(u; v - u)\,dx \ge 0 \quad \forall v \in K, \tag{5.10}$$

where $\lambda \geq 0$ is a free parameter to be chosen later. Unlike in the proof of Theorem 4.1, the operator B is now given by the following cut-off function $b : \Omega \times \mathbb{R} \to \mathbb{R}$:

$$b(x,s) = \begin{cases} (s - \bar{u}(x))^{p-1} & \text{if } s > \bar{u}(x), \\ 0 & \text{if } u_0(x) \leq s \leq \bar{u}(x), \\ -(u_0(x) - s)^{p-1} & \text{if } s < u_0(x), \end{cases} \tag{5.11}$$

where $u_0 = \max\{u_1, u_2\}$. By arguments similar to those in the proof of Theorem 4.1 we get the existence of solutions of (5.10). The set $\mathcal{S}$ is shown to be upward-directed provided that any solution u of (5.10) satisfies $u_k \leq u \leq \bar{u}$, $k = 1,2$, because then $Bu = 0$ and thus $u \in \mathcal{S}$ exceeding u_k.

For $k = 1,2$, because $u_k \in \mathcal{S}$, we have $u_k \in K \cap [\underline{u}, \bar{u}]$ and

$$\langle Au_k - f, v - u_k \rangle + \int_\Omega j^o(u_k; v - u_k)\,dx \geq 0 \quad \forall v \in K. \tag{5.12}$$

Note that since $u, u_1, u_2 \in K$, (5.9) implies that

$$u + (u_k - u)^+ = u \vee u_k \in K, \qquad u_k - (u_k - u)^+ = u \wedge u_k \in K. \tag{5.13}$$

Therefore, one can take as special functions $v = u + (u_k - u)^+$ in (5.10) and $v = u_k - (u_k - u)^+$ in (5.12). Adding the resulting inequalities we obtain

$$\begin{aligned} &\left\langle Au_k - Au, (u_k - u)^+ \right\rangle - \lambda \left\langle B(u), (u_k - u)^+ \right\rangle \\ &\quad \leq \int_\Omega \left(j^o\left(u; (u_k - u)^+\right) + j^o\left(u_k; -(u_k - u)^+\right)\right) dx. \end{aligned} \tag{5.14}$$

Arguing as in (4.17), we have for the right-hand side of (5.14) the estimate

$$\int_\Omega \left(j^o\left(u; (u_k - u)^+\right) + j^o\left(u_k; -(u_k - u)^+\right)\right) dx \leq \int_{\{u_k > u\}} c_1 (u_k(x) - u(x))^p\,dx. \tag{5.15}$$

For the terms on the left-hand side we have

$$\left\langle Au_k - Au, (u_k - u)^+ \right\rangle \geq 0, \tag{5.16}$$

and (5.11) yields

$$\begin{aligned} \left\langle B(u), (u_k - u)^+ \right\rangle &= -\int_{\{u_k > u\}} (u_0(x) - u(x))^{p-1} (u_k(x) - u(x))\,dx \\ &\leq -\int_{\{u_k > u\}} (u_k(x) - u(x))^p\,dx. \end{aligned} \tag{5.17}$$

By means of (5.15), (5.16), (5.17) we get from (5.14) the inequality

$$(\lambda - c_1) \int_{\{u_k > u\}} (u_k(x) - u(x))^p\,dx \leq 0. \tag{5.18}$$

Selecting λ such that $\lambda > c_1$ from (5.18) it follows $u_k \le u$. The proof for $u \le \bar{u}$ follows similar arguments, and thus $\mathcal{S}$ is upward-directed. By obvious modifications of the auxiliary problem, one can show analogously that $\mathcal{S}$ is also downward-directed.

Step 2 (existence of extremal solutions). We show the existence of the greatest element of $\mathcal{S}$. Since V_0 is separable, we have that $\mathcal{S} \subset V_0$ is separable too, so there exists a countable, dense subset $Z = \{z_n \mid n \in \mathbb{N}\}$ of $\mathcal{S}$. From Step 1, $\mathcal{S}$ is upward-directed, so we can construct an increasing sequence $(u_n) \subset \mathcal{S}$ as follows. Let $u_1 = z_1$. Select $u_{n+1} \in \mathcal{S}$ such that

$$\max\{z_n, u_n\} \le u_{n+1} \le \bar{u}. \tag{5.19}$$

The existence of u_{n+1} is established in Step 1. From the compactness of $\mathcal{S}$ according to Theorem 5.1, we can choose a subsequence of (u_n), denoted again (u_n), and an element $u \in \mathcal{S}$ such that $u_n \to u$ in V_0, and $u_n(x) \to u(x)$ a.e. in Ω. This last property of (u_n) combined with its increasing monotonicity implies that the entire sequence is convergent in V_0 and, moreover, $u = \sup_n u_n$. By construction, we see that

$$\max\{z_1, z_2, \dots, z_n\} \le u_{n+1} \le u \quad \forall n, \tag{5.20}$$

thus $Z \subset [\underline{u}, u]$. Since the interval $[\underline{u}, u]$ is closed in V_0, we infer

$$\mathcal{S} \subset \overline{Z} \subset \overline{[\underline{u}, u]} = [\underline{u}, u], \tag{5.21}$$

which in conjunction with $u \in \mathcal{S}$ ensures that u is the greatest solution of (1.1).

The existence of the least solution of (1.1) can be proved in a similar way. □

Remark 5.4. From the proof of Theorem 5.3 it can be seen that instead of lattice condition (5.9), it is enough to assume the following weaker condition:

$$K \wedge (K \cap [\underline{u}, \bar{u}]) \subset K, \qquad K \vee (K \cap [\underline{u}, \bar{u}]) \subset K. \tag{5.22}$$

6. Example and generalization

6.1. Example. We consider (1.1) with $f \in L^{p^{*\prime}}(\Omega)$, where $p^{*\prime}$ is the Hölder conjugate of the critical Sobolev exponent p^*, and K representing the following obstacle problem:

$$K = \{v \in V_0 \mid v(x) \le \psi(x) \text{ for a.e. } x \in \Omega\} \tag{6.1}$$

with $\psi : \Omega \to \mathbb{R}$ measurable. We are going to provide sufficient conditions for the existence of an ordered pair of constant sub-and supersolutions α and β, respectively.

Proposition 6.1. *Let $K \neq \emptyset$ be given by (6.1) and assume f and ψ as given above, and let $a_i(x,0) = 0$, $i = 1, \dots, N$. Then*

(a) *the constant function $\underline{u}(x) \equiv \alpha \le 0$ is a subsolution of (1.1) if*

$$f(x) \ge -j^o(\alpha; -1) \quad \text{for a.e. } x \in \Omega, \tag{6.2}$$

(b) *the constant function $\bar{u}(x) \equiv \beta \ge 0$ is a supersolution of (1.1) if*

$$f(x) \le j^o(\beta; 1) \quad \text{for a.e. } x \in \Omega, \tag{6.3}$$

(c) *if* $f \in L^\infty(\Omega)$ *and* $\alpha, \beta \in \mathbb{R}$ *satisfy* $\alpha \le 0 \le \beta$ *and*

$$-j^o(\alpha;-1) \le f(x) \le j^o(\beta;1) \quad \text{for a.e. } x \in \Omega, \tag{6.4}$$

then α *and* β *is an ordered pair of sub- and supersolutions.*

Proof. Let $\alpha \le 0$ satisfy (6.2). According to Definition 2.1, we only need to verify that α satisfies Definition 2.1(ii). To this end let $v \in \alpha \wedge K$ be given. Then $v - \alpha \le 0$ in Ω and in view of (6.2) we get

$$\begin{aligned}
&\langle A\alpha - f, v - \alpha\rangle + \int_\Omega j^o(\alpha; v(x) - \alpha)\,dx \\
&\quad = \int_\Omega \big(j^o(\alpha; v(x) - \alpha) - f(x)(v(x) - \alpha)\big)\,dx \\
&\quad = \int_\Omega \big(j^o(\alpha;-1) + f(x)\big)(\alpha - v(x))\,dx \ge 0 \quad \forall v \in \alpha \wedge K,
\end{aligned} \tag{6.5}$$

which proves that α is a subsolution. In a similar way one can show that under (6.3), the constant $\beta \ge 0$ is a supersolution. Finally, (c) follows immediately from (a) and (b). □

In order to apply Theorem 4.1 to our example, we only need to make sure that, in addition, $\beta \wedge K \subset K$ and $\alpha \vee K \subset K$ is satisfied. For the obstacle problem $\beta \wedge K \subset K$ is trivially satisfied and $\alpha \vee K \subset K$ holds provided $\alpha \le \psi(x)$ for a.e. $x \in \Omega$.

Moreover, straightforward calculations show that both lattice conditions in (5.9) are satisfied for our convex set K here. Thus, Theorem 5.3 also holds in the present example if $\alpha \le \psi(x)$ for a.e. $x \in \Omega$.

Remark 6.2. Our main goal is a general sub-supersolution approach for variational-hemivariational inequalities and the example given here illustrates the above results in a simple circumstance. Calculations of nonconstant sub-supersolutions in inclusions and variational inequalities were presented, for example, in [3, 4, 7].

Applications of the sub-supersolution method presented above to some variational-hemivariational inequalities in material science (in which nonconstant sub-supersolutions are constructed) will be studied in a forthcoming project.

6.2. Generalization. Our discussions above could be extended to the case where the principal operator A is perturbed by a lower-order term G. The inequality (1.1) is extended to

$$u \in K : \langle Au + Gu - f, v - u\rangle + \int_\Omega j^o(u; v - u)\,dx \ge 0 \quad \forall v \in K, \tag{6.6}$$

where G is the Nemytskij operator associated with a Carathéodory function $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$:

$$\langle Gu, v\rangle = \int_\Omega g(\cdot, u, \nabla u)v\,dx \quad \forall u, v \in V. \tag{6.7}$$

For the integral in (6.7) to be defined, we need some growth condition on g, which will be specified later. Note that the operator $A+G$ is not coercive in general. The definition of supersolutions of (6.6) now becomes as follows.

Definition 6.3. A function $\bar{u} \in V$ is called a supersolution of (6.6) if the following holds:
(i) $\bar{u} \geq 0$ on $\partial\Omega$,
(ii) $G\bar{u} \in L^q(\Omega)$,
(iii) $\langle A\bar{u} + G\bar{u} - f, v - \bar{u}\rangle + \int_\Omega j^o(\bar{u}; v - \bar{u})dx \geq 0$, for all $v \in \bar{u} \vee K$.

We have a similar definition for subsolutions of (6.6). Combining this notion of sub-supersolutions with appropriate modifications of the arguments in Section 5, we can prove the following existence and extremality result for (6.6).

Theorem 6.4. (a) *Assume the hypotheses (A1)–(A3), (H), and that (6.6) has subsolutions $\underline{u}_1, \dots, \underline{u}_k$ and supersolutions $\bar{u}_1, \dots, \bar{u}_m$ such that*

$$\underline{u} := \max\{\underline{u}_1, \dots, \underline{u}_k\} \leq \bar{u} := \min\{\bar{u}_1, \dots, \bar{u}_m\}, \tag{6.8}$$

and $\bar{u}_i \wedge K \subset K$, $\underline{u}_j \vee K \subset K$ for all $1 \leq i \leq m$, $1 \leq j \leq k$. Suppose furthermore g has the growth condition

$$|g(x,u,\xi)| \leq k_2(x) + c_6|\xi|^{p-1} \tag{6.9}$$

for a.e. $x \in \Omega$, all $\xi \in \mathbb{R}^N$, and all $u \in \mathbb{R}$ such that

$$\min\{\underline{u}_1(x), \dots, \underline{u}_k(x)\} \leq u \leq \max\{\bar{u}_1(x), \dots, \bar{u}_m(x)\}, \tag{6.10}$$

where $k_2 \in L^q(\Omega)$, $c_6 > 0$. Then there exists a solution u of (6.6) such that

$$\underline{u} \leq u \leq \bar{u}. \tag{6.11}$$

(b) *Furthermore, if K satisfies (5.9), then under the assumptions in (a), (6.6) possesses extremal solutions within $[\underline{u}, \bar{u}]$.*

Proof. To prove the assertion in part (a), we follow the idea of the proof of Theorem 4.1. We first note that variational-hemivariational inequality (6.6) is equivalent to the following. Find $u \in V_0$ such that

$$\langle Au + Gu - f, v - u\rangle + I_K(v) - I_K(u) + \int_\Omega j^o(u; v-u)dx \geq 0 \quad \forall v \in V_0, \tag{6.12}$$

where I_K denotes the indicator function related to K. However, unlike in Theorem 4.1 the functions $\underline{u}$ and $\bar{u}$ defined in (6.8) are no longer sub- and supersolutions, respectively. Therefore our existence proof will be based on the following modified auxiliary truncated problem: find $u \in V_0$ such that

$$\langle Au - f + \lambda B(u) + Pu, v - u\rangle + I_K(v) - I_K(u) + \int_\Omega j^o(u; v-u)dx \geq 0 \quad \forall v \in V_0, \tag{6.13}$$

where B is the cut-off operator as given by (2.9) and $\lambda \geq 0$ is some free parameter to be specified later. The operator $P : V_0 \to V_0^*$ is defined by

$$Pu := G \circ Tu + \sum_{i=1}^{m} \left| G \circ T^i u - G \circ Tu \right| - \sum_{j=1}^{k} \left| G \circ T_j u - G \circ Tu \right|, \tag{6.14}$$

where the truncation operators $T_j, T^i, T : V \to [\underline{u}, \bar{u}] \subset V$ are defined as follows:

$$Tu(x) = \begin{cases} \underline{u}(x) & \text{if } u(x) < \underline{u}(x), \\ u(x) & \text{if } \underline{u}(x) \leq u(x) \leq \bar{u}(x), \\ \bar{u}(x) & \text{if } u(x) > \bar{u}(x), \end{cases}$$
$$T_j u(x) = \begin{cases} \underline{u}_j(x) & \text{if } u(x) < \underline{u}_j(x), \\ u(x) & \text{if } \underline{u}_j(x) \leq u(x) \leq \bar{u}(x), \\ \bar{u}(x) & \text{if } u(x) > \bar{u}(x), \end{cases} \tag{6.15}$$
$$T^i u(x) = \begin{cases} \underline{u}(x) & \text{if } u(x) < \underline{u}(x), \\ u(x) & \text{if } \underline{u}(x) \leq u(x) \leq \bar{u}_i(x), \\ \bar{u}_i(x) & \text{if } u(x) > \bar{u}_i(x), \end{cases}$$

for $1 \leq i \leq m$, $1 \leq j \leq k$, $x \in \Omega$. The operators $G \circ T$, $G \circ T_j$, $G \circ T^i$ stand for the compositions of the Nemytskij operator G and the truncation operators T, T_j, T^i, respectively, and we have

$$\langle \left| G \circ T_j u - G \circ Tu \right|, v \rangle = \int_\Omega \left| g(\cdot, T_j u, \nabla T_j u) - g(\cdot, Tu, \nabla Tu) \right| v \, dx \tag{6.16}$$

for all $u, v \in V_0$. Since $T_j, T^i, T : V_0 \to V_0$ are bounded and continuous, it follows in view of the growth condition imposed on g that $P : V_0 \to L^q(\Omega) \subset V_0^*$ is bounded and continuous as well. Moreover, by applying [1, Theorem D.2.1] one sees that $A + \lambda B + P : V_0 \to V_0^*$ is continuous, bounded, and pseudomonotone. Introducing the same functional J as in the proof of Theorem 4.1, we can show that the multivalued operator $A + \lambda B + P + \partial(J|_{V_0}) : V_0 \to 2^{V_0^*}$ is pseudomonotone, bounded, and due to the growth condition on g as well as the mapping properties of the truncation operators, it is also coercive for λ chosen sufficiently large. Hence, by similar arguments as in the proof of Theorem 4.1, we infer that (6.13) has a solution u. The proof of the existence result of part (a) is accomplished provided any solution u of (6.13) can be shown to satisfy

$$\underline{u}_j \leq u \leq \bar{u}_i, \quad 1 \leq i \leq m, \ 1 \leq j \leq k. \tag{6.17}$$

This is because then u satisfies also $\underline{u} \leq u \leq \bar{u}$ which finally results in $Tu = u, T_j u = u$, $T^i u = u$, and thus $Pu = Gu$ as well as $Bu = 0$ showing that u is a solution of (6.12) (i.e., of (6.6)) within $[\underline{u}, \bar{u}]$.

We first show that $u \le \bar{u}_l$ for $l \in \{1,\dots,m\}$ fixed. By Definition 6.3 we have $\bar{u}_l \ge 0$ on $\partial\Omega$, and

$$\langle A\bar{u}_l + G\bar{u}_l - f, v - \bar{u}_l\rangle + \int_\Omega j^o(\bar{u}_l; v - \bar{u}_l)\,dx \ge 0 \quad \forall v \in \bar{u}_l \vee K, \tag{6.18}$$

and u is a solution of auxiliary problem (6.13) which is equivalent to the following. Find $u \in K$ such that

$$\langle Au - f + \lambda B(u) + Pu, v - u\rangle + \int_\Omega j^o(u; v - u)\,dx \ge 0 \quad \forall v \in K. \tag{6.19}$$

We apply the special test function $v = \bar{u}_l \vee u = \bar{u}_l + (u - \bar{u}_l)^+$ in (6.18) and $v = \bar{u}_l \wedge u = u - (u - \bar{u}_l)^+ (\in K)$ in (6.19), and get by adding the resulting inequalities the following one:

$$\begin{aligned} &\Big\langle A\bar{u}_l - Au, (u - \bar{u}_l)^+\Big\rangle + \Big\langle \lambda B(u) + Pu - G\bar{u}_l, -(u - \bar{u}_l)^+\Big\rangle \\ &\qquad + \int_\Omega \Big(j^o\Big(\bar{u}_l; (u - \bar{u}_l)^+\Big) + j^o\Big(u; -(u - \bar{u}_l)^+\Big)\Big)\,dx \ge 0, \end{aligned} \tag{6.20}$$

which yields due to

$$\Big\langle Au - A\bar{u}_l, (u - \bar{u}_l)^+\Big\rangle \ge 0, \tag{6.21}$$

the inequality

$$\Big\langle \lambda B(u) + Pu - G\bar{u}_l, (u - \bar{u}_l)^+\Big\rangle \le \int_\Omega \Big(j^o\Big(\bar{u}_l; (u - \bar{u}_l)^+\Big) + j^o\Big(u; -(u - \bar{u}_l)^+\Big)\Big)\,dx. \tag{6.22}$$

As in (4.17), for the right-hand side of (6.22) we get the estimate

$$\int_\Omega \Big(j^o\Big(\bar{u}_l; (u - \bar{u}_l)^+\Big) + j^o\Big(u; -(u - \bar{u}_l)^+\Big)\Big)\,dx \le \int_{\{u > \bar{u}_l\}} c_1\big(u(x) - \bar{u}_l(x)\big)^p\,dx. \tag{6.23}$$

As for the estimates of the terms on the left-hand side of (6.22) we note that $\bar{u}_l \ge \bar{u} \ge \underline{u} \ge \underline{u}_j$ which by taking into account the definition of the truncation operators yields

$$\int_{\{u > \bar{u}_l\}} \sum_{j=1}^{k} \big| G \circ T_j u - G \circ T u \big| \,(u - \bar{u}_l)\,dx = 0, \tag{6.24}$$

and the following estimates

$$
\begin{aligned}
\left\langle B(u),(u-\bar{u}_l)^+\right\rangle &= \int_{\{u>\bar{u}_l\}}(u-\bar{u})^p\,dx \geq \int_{\{u>\bar{u}_l\}}(u-\bar{u}_l)^p\,dx,\\
\left\langle Pu-G\bar{u}_l,(u-\bar{u}_l)^+\right\rangle &\\
&= \int_{\{u>\bar{u}_l\}}(Pu-G\bar{u}_l)(u-\bar{u}_l)\,dx\\
&= \int_{\{u>\bar{u}_l\}}\left[(G\circ Tu-G\bar{u}_l)(u-\bar{u}_l)+\sum_{i=1}^{m}\left|G\circ T^iu-G\circ Tu\right|(u-\bar{u}_l)\right]dx\\
&= \int_{\{u>\bar{u}_l\}}\left((G\circ Tu-G\bar{u}_l)+\left|G\bar{u}_l-G\circ Tu\right|\right)(u-\bar{u}_l)\,dx\\
&\quad+\int_{\{u>\bar{u}_l\}}\sum_{i\neq l}\left|G\circ T^iu-G\circ Tu\right|(u-\bar{u}_l)\,dx \geq 0.
\end{aligned}
\tag{6.25}
$$

Thus from (6.22) we get by means of (6.23), and (6.25),

$$
(\lambda-c_1)\int_{\{u>\bar{u}_l\}}(u-\bar{u}_l)^p\,dx \leq 0. \tag{6.26}
$$

By selecting λ in addition large enough such that $\lambda-c_1>0$, from (6.26) we obtain $u\leq\bar{u}_l$. In a similar way one can prove that for any $l\in\{1,\dots,k\}$ one has also $u\geq\underline{u}_l$ which completes the proof of part (a) of the theorem.

In order to prove (b), that is, the existence of extremal solutions in $[\underline{u},\bar{u}]$, we denote again by $\mathcal{S}$ the set of all solutions of (6.6) within $[\underline{u},\bar{u}]$. Following the line in the proof of Theorem 5.1, one readily verifies the compactness of $\mathcal{S}$ in V_0. Due to lattice condition (5.9) assumed in (b), one observes that any solution $u\in\mathcal{S}$ is, in particular, a subsolution and a supersolution of (6.6). Therefore, the statement of part (a) implies that $\mathcal{S}$ is a directed set. In just the same way as in Step 2 of the proof of Theorem 5.3, the compactness and directedness of $\mathcal{S}$ yield the existence of extremal elements of $\mathcal{S}$, which completes the proof of the theorem. □

Remark 6.5. The results and methods in this paper can be extended to variational-hemivariational inequalities involving more general quasilinear elliptic operators of Leray-Lions type and functions $j:\Omega\times\mathbb{R}\to\mathbb{R}$ depending also on the space variable x, which, however, has been omitted in order to avoid too many technicalities and in order to emphasize the main ideas.

We could also extend the above results to more general cases where the operator A satisfies a monotonicity condition such as

$$
\left\langle Au_1-Au_2,(u_1-u_2)^+\right\rangle \geq 0 \tag{6.27}
$$

for u_1, u_2 in some appropriate function space (such as V_0 or its analogue). This extension would allow us to study problems with weighted or degenerate operators.

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