STARLIKENESS ASSOCIATED WITH PARABOLIC REGIONS

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A parabolic starlike function f of order ρ in the unit disk is characterized by the fact that the quantity $zf'(z)/f(z)$ lies in a given parabolic region in the right half-plane. Denote the class of such functions by $\mathrm{PS}^*(\rho)$. This class is contained in the larger class of starlike functions of order ρ. Subordination results for $\mathrm{PS}^*(\rho)$ are established, which yield sharp growth, covering, and distortion theorems. Sharp bounds for the first four coefficients are also obtained. There exist different extremal functions for these coefficient problems. Additionally, we obtain a sharp estimate for the Fekete-Szegö coefficient functional and investigate convolution properties for $\mathrm{PS}^*(\rho)$.

1. Introduction

Let A denote the class of analytic functions f in the open unit disk $U = \{z : |z| < 1\}$ and let f be normalized so that $f(0) = f'(0) - 1 = 0$. In [4], Goodman introduced the class UCV of uniformly convex functions consisting of convex functions $f \in A$ with the property that for every circular arc γ contained in U, with center also in U, the image arc $f(\gamma)$ is a convex arc. He derived a two-variable characterization of functions in UCV, that is, $f \in A$ belongs to UCV if and only if for every pair $(z,\varsigma) \in U \times U$,

$$1 + \Re\left\{(z-\varsigma)\frac{f''(z)}{f'(z)}\right\} \geq 0. \tag{1.1}$$

Ma and Minda [6] and Rønning [10] independently developed a one-variable characterization that $f \in \mathrm{UCV}$ if and only if for every $z \in U$,

$$\left|\frac{zf''(z)}{f'(z)}\right| < \Re\left(1 + \frac{zf''(z)}{f'(z)}\right). \tag{1.2}$$

Rønning [10] also showed that $f \in \mathrm{UCV}$ if and only if the function $zf' \in \mathrm{PS}^*$, where PS^* is the class of functions $g \in A$ satisfying

$$\left|\frac{zg'(z)}{g(z)} - 1\right| < \Re\frac{zg'(z)}{g(z)}, \quad z \in U. \tag{1.3}$$

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Several authors have studied the classes above, amongst which the authors of [4, 6, 7, 8, 9, 10, 12].

In [9], the class PS^* was generalized by looking at functions $f \in A$ satisfying

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < \Re\frac{zf'(z)}{f(z)} - \alpha, \quad z \in U. \tag{1.4}$$

In this paper, we continue the investigation of this generalized class but under a slight modification of parameter. For $0 \le \rho < 1$, let Ω_ρ be the parabolic region in the right half-plane

$$\Omega_\rho = \{w = u + iv : v^2 < 4(1-\rho)(u-\rho)\} = \{w : |w-1| < 1 - 2\rho + \Re w\}. \tag{1.5}$$

The class of *parabolic starlike functions of order* ρ is the subclass $PS^*(\rho)$ of A consisting of functions f such that $zf'(z)/f(z) \in \Omega_\rho$, $z \in U$. Thus $f \in PS^*(\rho)$ if and only if for $z \in U$,

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - 2\rho + \Re\frac{zf'(z)}{f(z)}. \tag{1.6}$$

Similarly, a function $f \in A$ belongs to $UCV(\rho)$ if and only if for every pair (z, ς) in the polydisk $U \times U$,

$$1 + \Re\left\{(z-\varsigma)\frac{f''(z)}{f'(z)}\right\} > 2\rho - 1. \tag{1.7}$$

A function $f \in UCV(\rho)$ is called an *uniformly convex function of order* ρ. Thus the classes discussed earlier correspond to $UCV = UCV(1/2)$ and $PS^* = PS^*(1/2)$. In [5], Lee showed that

$$g \in UCV(\rho) \Longleftrightarrow f = zg' \in PS^*(\rho), \tag{1.8}$$

that is,

$$g \in UCV(\rho) \Longleftrightarrow \left|\frac{zg''(z)}{g'(z)}\right| < 2(1-\rho) + \Re\frac{zg''(z)}{g'(z)}. \tag{1.9}$$

In the present paper, we continue the study of $PS^*(\rho)$ realized by Ali and Singh [3], and more recently by Aghalary and Kulkarni [1]. We give examples of functions in the class $PS^*(\rho)$, and establish subordination results, which yield sharp growth, covering and distortion theorems. Sharp bounds on the first four coefficients are also obtained. There exist different extremal functions for these coefficient problems. Additionally, we obtain a sharp estimate for the Fekete-Szegö coefficient functional and examine convolution properties for $PS^*(\rho)$.

2. Preliminary results

From its definition, it is clear that the class $PS^*(\rho)$ is contained in the class $S^*(\rho)$ of starlike functions of order ρ, that is, $\Re(zf'(z)/f(z)) > \rho$, $z \in U$. It is also fairly immediate that $\mathrm{PS}^*(\rho)$ is related to the class of strongly starlike functions, where a function $f \in A$ is said to be strongly starlike of order α, $0 < \alpha \le 1$, if f satisfies $|\operatorname{Arg} zf'(z)/f(z)| < \pi\alpha/2$, $z \in U$. We state the relation in the theorem below.

Theorem 2.1. *If $f \in \mathrm{PS}^*(\rho)$, then f is strongly starlike of order γ, where $(\pi/2)\gamma = \tan^{-1}\sqrt{(1-\rho)/\rho}$. In other words, for $z \in U$,*

$$\left|\operatorname{Arg}\frac{zf'(z)}{f(z)}\right| \le \frac{\pi\gamma}{2}. \tag{2.1}$$

A sufficient condition for a function f to be parabolic starlike of order ρ is given by the following theorem.

Theorem 2.2. *If $f \in A$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \rho, \tag{2.2}$$

then $f \in \mathrm{PS}^*(\rho)$.

Proof. The given condition implies that

$$\Re\frac{zf'(z)}{f(z)} - \left|\frac{zf'(z)}{f(z)} - 1\right| + 1 - 2\rho \ge 2(1-\rho) - 2\left|\frac{zf'(z)}{f(z)} - 1\right| > 0. \tag{2.3}$$

□

The following two examples are now easily established from Theorem 2.2.

Example 2.3. The function $f(z) = z + \alpha z^n \in \mathrm{PS}^*(\rho)$ if and only if $|\alpha| \le (1-\rho)/(n-\rho)$.

Example 2.4. The generalized hypergeometric function is defined by

$$F(a_1,\dots,a_p;b_1,\dots,b_q;z) = 1 + \sum_{n=1}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}, \quad b_j \ne 0,-1,\dots, \tag{2.4}$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \begin{cases} 1, & n = 0, \\ \lambda(\lambda+1)(\lambda+2)\cdots(\lambda+n-1), & n = 1,2,\dots. \end{cases} \tag{2.5}$$

If $|zF'(z)/F(z)| < 1 - \rho$, then $zF \in \mathrm{PS}^*(\rho)$.

Ali and Singh [3] showed that the normalized Riemann mapping function q_ρ from U onto Ω_ρ is given by

$$q_\rho(z) = 1 + \frac{4(1-\rho)}{\pi^2}\left[\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right]^2 = 1 + \sum_{n=1}^{\infty} B_n z^n. \tag{2.6}$$

Here

$$B_1 = \frac{16(1-\rho)}{\pi^2}, \qquad B_n = \frac{16(1-\rho)}{n\pi^2}\sum_{k=0}^{n-1}\frac{1}{2k+1}, \quad n = 2,3,\dots. \tag{2.7}$$

Since the latter sum is bounded above by $1+(1/2)\log(2n-1)$ (see [6]) an upper bound for each coefficient is given by

$$B_n < \frac{16(1-\rho)}{n\pi^2}\left(1+\frac{1}{2}\log(2n-1)\right). \tag{2.8}$$

However these bounds do not yield sharp coefficient estimates for the class $\mathrm{PS}^*(\rho)$. We will return to the coefficient problem in the next section.

Let $k \in \mathrm{PS}^*(\rho)$ be defined by $k(0) = k'(0) - 1 = 0$ and

$$\frac{zk'(z)}{k(z)} = q_\rho(z). \tag{2.9}$$

In [8], Ma and Minda established a general result that leads to the following result.

Theorem 2.5 [8]. *If $f \in \mathrm{PS}^*(\rho)$, then*

(a) *$zf'(z)/f(z) \prec zk'(z)/k(z)$ and $f(z)/z \prec k(z)/z$,*
(b) $-k(-r) \le |f(z)| \le k(r)$, $|z| \le r < 1$,
(c) $|\mathrm{Arg}(f(z)/z)| \le \max_{|z|=r}|\mathrm{Arg}(k(z)/z)|$, $|z| \le r < 1$,
(d) $k'(-r) \le |f'(z)| \le k'(r)$, $|z| \le r < 1$.

Equality in (b), (c), and (d) holds for some $z \neq 0$ if and only if f is a rotation of k.

Since the function k is continuous in $\overline{U}$, $-k(-1) = \lim_{r\to 1} -k(-r)$ and $k(1) = \lim_{r\to 1} k(r)$ exist. Rønning [9] established the following corollary.

Corollary 2.6 [9]. *(a) Let $f \in \mathrm{PS}^*(\rho)$. Then either f is a rotation of k or $f(U) \supset \{w : |w| \le -k(-1)\}$, where the Koebe constant is $-k(-1) = e^{-(1-\rho)(1.25475)}$.*

(b) The functions in $\mathrm{PS}^*(\rho)$ are uniformly bounded by the sharp constant $k(1) = e^{3.41023(1-\rho)}$.

3. Coefficient bounds

We first give another sufficient condition for a function f to belong to $\mathrm{PS}^*(\rho)$.

Theorem 3.1. *If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfies $\sum_{n=2}^{\infty}(n-1)|a_n| \le (1-\rho)/(2-\rho)$, then $f \in \mathrm{PS}^*(\rho)$. The constant $(1-\rho)/(2-\rho)$ cannot be replaced by a larger number.*

Proof. Let $g(z) = \int_0^z (f(\xi)/\xi)d\xi = z + \sum_{n=2}^{\infty}(a_n/n)z^n$. In view of (1.8), it suffices to show that $g \in \mathrm{UCV}(\rho)$. Since

$$\sum_{n=2}^{\infty}|a_n| \le \frac{1-\rho}{2-\rho}, \tag{3.1}$$

it follows that

$$1+\Re(z-\varsigma)\frac{g''(z)}{g'(z)} \geq 1-\frac{\sum_{n=2}^{\infty}(n-1)\,|\,a_n\,|\,|z|^{n-2}}{1-\sum_{n=2}^{\infty}\,|\,a_n\,|\,|z|^{n-1}}|z-\varsigma| \geq 2\rho-1. \tag{3.2}$$

Thus $g\in \mathrm{UCV}(\rho)$. The function $f(z)=z+((1-\rho)/(2-\rho))z^2$ in Example 2.3 shows that the constant $(1-\rho)/(2-\rho)$ is the best possible. □

We next consider the problem of finding

$$A_n=\max_{f\in \mathrm{PS}^*(\rho)}\,|\,a_n\,|\,. \tag{3.3}$$

If $f(z)=z+a_2z^2+a_3z^3+\cdots\in \mathrm{PS}^*(\rho)$ and $h(z)=zf'(z)/f(z)$, then there exists a Schwarz function w defined in U with $w(0)=0$, $|w(z)|<1$, and satisfying

$$h(z)=\frac{zf'(z)}{f(z)}=q_\rho(w(z)). \tag{3.4}$$

If $h(z)=1+b_1z+b_2z^2+\cdots$, the first equality in (3.4) implies that

$$(n-1)a_n=\sum_{k=1}^{n-1}a_kb_{n-k}. \tag{3.5}$$

Since q_ρ is univalent in U and $h\prec q_\rho$, the function

$$p(z)=\frac{1+q_\rho^{-1}(h(z))}{1-q_\rho^{-1}(h(z))}=1+c_1z+c_2z^2+\cdots \tag{3.6}$$

belongs to the class P consisting of analytic functions p in the unit disk U with positive real part such that $p(0)=1$ and $\Re p(z)>0$, $z\in U$. In other words,

$$h(z)=q_\rho\left(\frac{p(z)-1}{p(z)+1}\right). \tag{3.7}$$

While (3.5) gives a_n in terms of the coefficients b_k, (3.7) expresses the b_k's in terms of the coefficients c_m's and B_m's. It is now easily established that

$$\begin{aligned}
a_2&=\frac{8(1-\rho)}{\pi^2}c_1,\\
a_3&=\frac{8(1-\rho)}{2\pi^2}\left[c_2-\left(\frac{1}{6}-\frac{8(1-\rho)}{\pi^2}\right)c_1^2\right],\\
a_4&=\frac{8(1-\rho)}{3\pi^2}\left[c_3-\left(\frac{1}{3}-\frac{12(1-\rho)}{\pi^2}\right)c_1c_2+\left(\frac{2}{45}-\frac{2(1-\rho)}{\pi^2}+\frac{32(1-\rho)^2}{\pi^4}\right)c_1^3\right].
\end{aligned} \tag{3.8}$$

Thus the coefficient estimates for $\mathrm{PS}^*(\rho)$ may be viewed in terms of nonlinear coefficient problems for the class P.

We now introduce the following functions in $\mathrm{PS}^*(\rho)$. Define $k_n, G, H \in A$, respectively, by

$$\frac{zk_n'(z)}{k_n(z)} = q_\rho(z^{n-1}), \quad \frac{zH'(z)}{H(z)} = q_\rho\left(\frac{z(z-r)}{1-rz}\right), \quad \frac{zG'(z)}{G(z)} = q_\rho\left(-\frac{z(z-r)}{1-rz}\right), \quad 0 \le r \le 1. \tag{3.9}$$

It is clear from (3.4) that $k_n, G, H \in \mathrm{PS}^*(\rho)$, and that $k_2(z) = k(z)$. Since

$$k_n(z) = z + \frac{16(1-\rho)}{(n-1)\pi^2} z^n + \cdots, \tag{3.10}$$

we find that

$$A_n \ge \frac{16(1-\rho)}{(n-1)\pi^2}. \tag{3.11}$$

On the other hand, Ali and Singh [3] proved that

$$(n-1)A_n \le 2\sqrt{2}(1-\rho)e^{4(1-\rho)^2}, \tag{3.12}$$

which also yields the sharp order of growth $|a_n| = O(1/n)$.

From a result of Ma and Minda [8], we can also deduce the following solution to the Fekete-Szegö coefficient functional over the class $\mathrm{PS}^*(\rho)$. We will omit the details.

Theorem 3.2. *Let $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathrm{PS}^*(\rho)$. Then*

$$\left|a_3 - ta_2^2\right| \le \begin{cases} \dfrac{16(1-\rho)}{3\pi^4}\left[24(1-\rho)(1-2t)+\pi^2\right], & t \le \dfrac{1}{2} - \dfrac{\pi^2}{96(1-\rho)}, \\ \dfrac{8(1-\rho)}{\pi^2}, & \dfrac{1}{2} - \dfrac{\pi^2}{96(1-\rho)} \le t \le \dfrac{1}{2} + \dfrac{5\pi^2}{96(1-\rho)}, \\ \dfrac{16(1-\rho)}{3\pi^4}\left[24(1-\rho)(2t-1)-\pi^2\right], & t \ge \dfrac{1}{2} + \dfrac{5\pi^2}{96(1-\rho)}. \end{cases} \tag{3.13}$$

If $1/2 - \pi^2/96(1-\rho) < t < 1/2 + 5\pi^2/96(1-\rho)$, equality holds if and only if $f = k_3$ or one of its rotations. If $t < 1/2 - \pi^2/96(1-\rho)$ or $t > 1/2 + 5\pi^2/96(1-\rho)$, equality holds if and only if $f = k_2$ or one of its rotations. If $t = 1/2 - \pi^2/96(1-\rho)$, equality holds if and only if $f = H$ or one of its rotations, while if $t = 1/2 + 5\pi^2/96(1-\rho)$, then equality holds if and only if $f = G$ or one of its rotations.

The above estimates can be used to determine sharp upper bounds on the second and third coefficients, respectively, which we will state below. In addition, the sharp bound on the fourth coefficient A_4 is determined with the aid of the following lemma.

Lemma 3.3 [2]. *Let $p(z) = 1 + \sum_{k=1}^\infty c_k z^k \in P$. If $0 \le \beta \le 1$ and $\beta(2\beta - 1) \le \delta \le \beta$, then*

$$\left|c_3 - 2\beta c_1c_2 + \delta c_1^3\right| \le 2. \tag{3.14}$$

In particular,

$$|c_3 - 2\beta c_1 c_2 + \beta c_1^3| \leq 2. \tag{3.15}$$

When $\beta = 0$, *equality holds if and only if*

$$p(z) := p_3(z) = \sum_{k=1}^{3} \lambda_k \frac{1 + \epsilon e^{-2\pi i k/3} z}{1 - \epsilon e^{-2\pi i k/3} z}, \quad |\epsilon| = 1,\ \lambda_k \geq 0, \tag{3.16}$$

with $\lambda_1 + \lambda_2 + \lambda_3 = 1$. *If* $\beta = 1$, *equality holds if and only if* p *is the reciprocal of* p_3. *If* $0 < \beta < 1$, *equality holds if and only if*

$$p(z) = \frac{1 + \epsilon z}{1 - \epsilon z}, \quad |\epsilon| = 1 \qquad \text{or} \qquad p(z) = \frac{1 + \epsilon z^3}{1 - \epsilon z^3}, \quad |\epsilon| = 1. \tag{3.17}$$

Theorem 3.4. *Let* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathrm{PS}^*(\rho)$. *Then*

$$|a_2| \leq \frac{16(1-\rho)}{\pi^2}, \tag{3.18}$$

with equality if and only if $f = k$ *or its rotations. Further*

$$|a_3| \leq \begin{cases} \dfrac{8(1-\rho)}{\pi^2}\left(\dfrac{2}{3} + \dfrac{16(1-\rho)}{\pi^2}\right), & 0 \leq \rho \leq 1 - \dfrac{\pi^2}{48}, \\[2ex] \dfrac{8(1-\rho)}{\pi^2}, & 1 - \dfrac{\pi^2}{48} \leq \rho < 1. \end{cases} \tag{3.19}$$

For $0 \leq \rho < 1 - \pi^2/48$, *equality holds if and only if* $f = k$ *or its rotations. For* $1 - \pi^2/48 < \rho < 1$, *equality holds if and only if* $f = k_3$ *or its rotations. If* $\rho = 1 - \pi^2/48$, *equality holds if and only if* $f = H$ *or its rotations. Additionally,*

$$|a_4| \leq \begin{cases} \dfrac{16(1-\rho)}{3\pi^2}\left[\dfrac{128(1-\rho)^2}{\pi^4} + \dfrac{16(1-\rho)}{\pi^2} + \dfrac{23}{45}\right], & 0 \leq \rho \leq 1 + \dfrac{\pi^2}{16}\left(1 - \sqrt{\dfrac{89}{45}}\right), \\[2ex] \dfrac{16(1-\rho)}{3\pi^2}, & 1 + \dfrac{\pi^2}{16}\left(1 - \sqrt{\dfrac{89}{45}}\right) \leq \rho < 1. \end{cases} \tag{3.20}$$

Equality holds in the upper expression of the right inequality if and only if $f = k$ *or its rotations, while equality holds in the lower expression of the right inequality if and only if* $f = k_4$ *or its rotations.*

Proof. In the light of Theorem 3.2, we are left to finding an estimate on the fourth coefficient. The relation (3.8) gives

$$\begin{aligned} a_4 &= \frac{8(1-\rho)}{3\pi^2}\left[c_3 - \left(\frac{1}{3} - \frac{12(1-\rho)}{\pi^2}\right)c_1 c_2 + \left(\frac{2}{45} - \frac{2(1-\rho)}{\pi^2} + \frac{32(1-\rho)^2}{\pi^4}\right)c_1^3\right] \\ &:= \frac{8(1-\rho)}{3\pi^2} E. \end{aligned} \tag{3.21}$$

We will apply Lemma 3.3 with

$$2\beta = \frac{1}{3} - \frac{12(1-\rho)}{\pi^2}, \qquad \delta = \frac{2}{45} - \frac{2(1-\rho)}{\pi^2} + \frac{32(1-\rho)^2}{\pi^4}. \tag{3.22}$$

The conditions on β and δ are satisfied if

$$1 + \frac{\pi^2}{16}\left(1 - \sqrt{\frac{89}{45}}\right) \leq \rho < 1. \tag{3.23}$$

Thus $|a_4| \leq 16(1-\rho)/3\pi^2$, with equality if and only if the function p in (3.7) is given by $p(z) = (1+\epsilon z^3)/(1-\epsilon z^3)$. This implies that $f = k_4$.

In view of the fact that $0 < \delta < 1$, and that $\delta - \beta \geq 0$ provided

$$1 + \frac{\pi^2}{16}\left(1 - \sqrt{\frac{89}{45}}\right) \geq \rho, \tag{3.24}$$

Lemma 3.3 yields

$$\begin{aligned} |E| &\leq \left| c_3 - 2\delta c_1 c_2 + \delta c_1^3 \right| + 2(\delta - \beta)\left| c_1 c_2 \right| \\ &\leq 2 + 8\left(\frac{32(1-\rho)^2}{\pi^4} + \frac{4(1-\rho)}{\pi^2} - \frac{11}{90}\right) \\ &= 2\left(\frac{128(1-\rho)^2}{\pi^4} + \frac{16(1-\rho)}{\pi^2} + \frac{23}{45}\right). \end{aligned} \tag{3.25}$$

Equality holds if and only if the function p in (3.7) is given by $p(z) = (1+\epsilon z)/(1-\epsilon z)$, that is, $f = k$. This completes the proof. □

Theorem 3.5. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathrm{PS}^*(\rho)$. For $\mu \in C$ and*

$$\lambda(\mu) = \frac{1}{3} + \frac{16(1-\rho)}{\pi^2}(2\mu - 1),$$

$$\left| a_3 - \mu a_2^2 \right| \leq \begin{cases} \dfrac{16(1-\rho)}{3\pi^4}\left| 24(1-\rho)(1-2\mu) + \pi^2 \right|, & \left| \lambda(\mu) - 1 \right| \geq 1, \\ \dfrac{8(1-\rho)}{\pi^2}, & \left| \lambda(\mu) - 1 \right| \leq 1. \end{cases} \tag{3.26}$$

Equality holds in the upper expression of the right inequality if $f = k$ or its rotations, while equality holds in the lower expression of the right inequality if $f = k_3$ or its rotations.

Proof. From the relation (3.8), we get

$$a_3 - \mu a_2^2 = \frac{4(1-\rho)}{\pi^2}\left[c_2 - \frac{\lambda(\mu)}{2}c_1^2\right]. \tag{3.27}$$

The well-known estimate

$$\left| c_2 - \frac{1}{2}c_1^2 \right| \leq 2 - \frac{1}{2}\left| c_1 \right|^2 \tag{3.28}$$

leads to

$$\left|c_2 - \frac{\lambda(\mu)}{2}c_1^2\right| \leq \left|c_2 - \frac{1}{2}c_1^2\right| + \frac{|1-\lambda(\mu)|}{2}|c_1|^2 \leq 2 + \frac{|\lambda(\mu)-1|-1}{2}|c_1|^2, \tag{3.29}$$

which yields the desired result. □

4. Convolution properties

The convolution of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined to be the function $(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. For $\alpha < 1$, denote by R_α the class of prestarlike functions of order α consisting of $f \in A$ such that $f * ((z)/(1-z)^{2-2\alpha}) \in S^*(\alpha)$. Here $S^*(\alpha)$ is the class of starlike functions of order α. An important result in convolution is contained in the following lemma of Ruscheweyh.

Lemma 4.1 [11, page 54]. *If $f \in R_\alpha$, $g \in S^*(\alpha)$, and H is an analytic function in U, then*

$$\frac{f * gH}{f * g}(U) \subset \overline{\mathrm{co}}H(U), \tag{4.1}$$

where $\overline{\mathrm{co}}H(U)$ is the closed convex hull of $H(U)$.

Theorem 4.2. *If $f \in R_\rho$ and $g \in \mathrm{PS}^*(\rho)$, then $f * g \in \mathrm{PS}^*(\rho)$.*

Proof. Since g also belongs to $S^*(\rho)$ and $H(z) = zg'(z)/g(z) \prec q_\rho(z)$, Lemma 4.1 yields

$$\frac{z(f*g)'}{f*g}(U) = \frac{f * zg'}{f*g}(U) = \frac{f*g(zg'/g)}{f*g}(U) \subset \overline{\mathrm{co}}\frac{zg'}{g}(U) \subset \overline{\Omega}_\rho, \tag{4.2}$$

and hence, $f * g \in \mathrm{PS}^*(\rho)$. □

Since $R_{1/2} = S^*(1/2)$ (see [11]), and $R_0 = C$, where C is the class of convex functions in A, a similar proof also yields the following result.

Corollary 4.3. (a) *If $f, g \in \mathrm{PS}^*(\rho)$ for $\rho \geq 1/2$, then $f * g \in \mathrm{PS}^*(\rho)$.*
(b) *If $f \in C$ and $g \in \mathrm{PS}^*(\rho)$, then $f * g \in \mathrm{PS}^*(\rho)$.*

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References

[1] R. Aghalary and S. R. Kulkarni, *Certain properties of parabolic starlike and convex functions of order ρ*, Bull. Malays. Math. Sci. Soc. (2) **26** (2003), no. 2, 153–162.
[2] R. M. Ali, *Coefficients of the inverse of strongly starlike functions*, Bull. Malays. Math. Sci. Soc. (2) **26** (2003), no. 1, 63–71.
[3] R. M. Ali and V. Singh, *Coefficients of parabolic starlike functions of order ρ*, Computational Methods and Function Theory 1994 (Penang), Ser. Approx. Decompos., vol. 5, World Scientific Publishing, New Jersey, 1995, pp. 23–36.

[4] A. W. Goodman, *On uniformly convex functions*, Ann. Polon. Math. **56** (1991), no. 1, 87–92.
[5] S. K. Lee, *Characterizations of parabolic starlike functions and the generalized uniformly convex functions*, Master's thesis, Universiti Sains Malaysia, Penang, Malaysia, 2000.
[6] W. C. Ma and D. Minda, *Uniformly convex functions*, Ann. Polon. Math. **57** (1992), no. 2, 165–175.
[7] ———, *Uniformly convex functions. II*, Ann. Polon. Math. **58** (1993), no. 3, 275–285.
[8] ———, *A unified treatment of some special classes of univalent functions*, Proceedings of the Conference on Complex Analysis (Tianjin, 1992), Conf. Proc. Lecture Notes Anal., I, International Press, Massachusetts, 1994, pp. 157–169.
[9] F. Rønning, *On starlike functions associated with parabolic regions*, Ann. Univ. Mariae Curie-Skłodowska Sect. A **45** (1991), 117–122.
[10] ———, *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc. **118** (1993), no. 1, 189–196.
[11] S. Ruscheweyh, *Convolutions in Geometric Function Theory*, Séminaire de Mathématiques Supérieures, vol. 83, Presses de l'Université de Montréal, Quebec, 1982.
[12] T. N. Shanmugam and V. Ravichandran, *Certain properties of uniformly convex functions*, Computational Methods and Function Theory 1994 (Penang), Ser. Approx. Decompos., vol. 5, World Scientific Publishing, New Jersey, 1995, pp. 319–324.

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