VOLUME GROWTH AND CLOSED GEODESICS ON RIEMANNIAN MANIFOLDS OF HYPERBOLIC TYPE

JEAN-PIERRE EZIN AND CARLOS OGOUYANDJOU

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We study the volume growth function of geodesic spheres in the universal Riemannian covering of a compact manifold of hyperbolic type. Furthermore, we investigate the growth rate of closed geodesics in compact manifolds of hyperbolic type.

1. Introduction

In this paper, we investigate asymptotic properties of universal Riemannian covering of a compact manifold of hyperbolic type.

Definition 1.1. A compact Riemannian manifold (M,g) is called of hyperbolic type if there exists another Riemannian metric g_0 such that (M,g_0) has a strictly negative curvature.

Note that, in dimension 2, an orientable manifold M is of hyperbolic type if and only if its genus is greater than or equal to 2.

We say that a function $f:\mathbb{R}_+\to\mathbb{R}_+$ is of purely exponential type if there exist constants $a>1$ and $r_0>0$ such that

$$\frac{1}{a}\leq\frac{f(r)}{e^{hr}}\leq a\quad\forall r\geq r_0,\tag{1.1}$$

for some constant $h>0$. The real number h is called the exponential factor of f.

In 1969, Margulis proved, for suitable constant $h>0$, that

$$a(p):=\lim_{r\to\infty}\frac{\operatorname{vol}S(p,r)}{e^{hr}}\tag{1.2}$$

exists at each point p in manifolds of negative curvature and that the function a is continuous (see [18]). Clearly, this result implies purely exponential growth of volume of geodesic spheres.

If (M,g) is a compact Riemannian manifold, Manning has introduced an interesting asymptotic invariant h_g (volume entropy) which is defined as follows: if $\operatorname{vol}B_g(p,r)$

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denotes the volume of the geodesic ball $B_g(p,r)$ with centre p and radius r in the universal Riemannian covering X of (M,g), then

$$h_g := \lim_{r\to\infty} \frac{\log \operatorname{vol} B_g(p,r)}{r}, \tag{1.3}$$

where the limit on the right-hand side exists for all $p \in X$ and, in fact, is independent of p. Manning showed that, in the case of nonpositive curvature, h_g coincides with the topological entropy (see [17]).

In 1997, using the notions of Busemann density and Patterson Sullivan measure, G. Knieper proved the following result (see [16]): if (M,g_0) is a rank-1 compact Riemannian manifold of nonpositive curvature and X_0 its universal Riemannian covering, there exist constants $a_0 \geq 1$ and $r_0 \geq 0$ such that

$$\frac{1}{a_0} \leq \frac{\operatorname{vol} S_{g_0}(p,r)}{e^{h_{g_0} r}} \leq a_0 \quad \forall r \geq r_0, \tag{1.4}$$

where h_{g_0} is the volume entropy of (M,g_0) and $S_{g_0}(p,r)$ is the geodesic sphere in X_0 with centre p and radius r.

The main result of this paper is as follows.

Theorem 1.2. *Let (M,g) be a compact Riemannian manifold of hyperbolic type without conjugate points and let X be its universal Riemannian covering. Then the growth function of the volume of geodesic spheres of X is of purely exponential type with the volume entropy h_g as exponential factor.*

Remark 1.3. Note that the manifolds considered in Theorem 1.2 may have curvature of both signs (see [7] or [13, page 199]). This result yields a sufficient condition for the nonexistence of Riemannian metric with negative curvature on a compact manifold. In Theorem 1.2 by integration an analogous growth result holds if one replaces geodesic spheres by geodesic balls. Precisely the following holds.

Corollary 1.4. *Let (M,g) be a compact Riemannian manifold of hyperbolic type without conjugate points and let X be its universal Riemannian covering. Then the growth function of the volume of geodesic balls of X is of purely exponential type.*

Remark 1.5. Corollary 1.4 implies that the critical exponent of the deck transformations group of X is equal to the volume entropy of M. However, using a Coornaert's result ([4, Theorem 4.3]), we get an analogous result without the assumption of no conjugate points.

We also study the counting function $\mathcal{P}(t)$ of the number of closed geodesics of period less than or equal to t (up to free homotopy) in the compact quotient M.

In the case of negative curvature, Margulis showed that $\mathcal{P}(t) \sim e^{ht}/t$, where h is the volume entropy of X.

In this paper, we prove the following.

Theorem 1.6. *Let (M,g) be a compact Riemannian manifold of hyperbolic type without conjugate points. Then there are constants $a > 1$ and $t_0 > 0$ such that*

$$\frac{1}{a}\frac{e^{h_g t}}{t} \leq \mathcal{P}(t) \leq a e^{h_g t} \quad \forall t > t_0, \tag{1.5}$$

where h_g is the volume entropy of (M,g), $\mathcal{P}(t)$ the number of closed geodesics of period less than or equal to t in M.

The corresponding result for compact rank-1 manifolds was proven by Knieper [16].

The paper is organized as follows. In Section 2, we recall some basic facts about Gromov hyperbolic spaces. In particular, we study the ideal boundary and the Gromov boundary of a manifold of hyperbolic type. In Section 3, we introduce a notion of Busemann quasidensity, which is used to prove the so-called shadow lemma (see Lemma 3.6). In Section 4, we prove Theorem 1.2. Section 5 starts with some properties of closed geodesics of compact manifold. Then, we give a proof of Theorem 1.6.

2. Gromov and ideal boundaries of manifolds of hyperbolic type

We recall first some basic notions about a compactification of Hadamard manifolds.

Definition 2.1. A connected, simply connected, and complete Riemannian manifold is called Hadamard manifold.

Let (X_0,g_0) be a Hadamard manifold. Two geodesics $c_1, c_2 : \mathbb{R} \to X_0$ are said to be asymptotic, if there exists a constant $D \geq 0$ such that

$$d_{g_0}\big(c_1(t), c_2(t)\big) < D \quad \forall t \geq 0. \tag{2.1}$$

This defines an equivalence relation on the set of geodesics of X_0.

An equivalence class of this relation is called point at infinity of X_0. If $c : \mathbb{R} \to X_0$ is a geodesic, its class is denoted by $c(+\infty)$. Let $c^{-1} : \mathbb{R} \to X_0$ defined by $c^{-1}(t) := c(-t)$ for all $t \in \mathbb{R}$. The class of c^{-1} is denoted by $c(-\infty)$.

The ideal boundary $X_0(\infty)$ of X_0 is the coset of the geodesics of X_0.

One defines a natural topology on the set $\overline{X}_0 := X_0 \cup X_0(\infty)$ as follows: consider $B(x,1) = \{v \in T_x X_0 \mid \|v\| \leq 1\}$ and the bijection

$$\begin{aligned} \Phi_x : B(x,1) &\longrightarrow \overline{X}_0 = X_0 \cup X_0(\infty), \\ v &\longmapsto \begin{cases} \exp_x\left(\dfrac{\|v\|}{1-\|v\|}\right)v & \text{if } \|v\| < 1, \\ c_v(+\infty) & \text{if } \|v\| = 1, \end{cases} \end{aligned} \tag{2.2}$$

where c_v is the geodesic satisfying $c_v(0) = x$ and $\dot{c}_v(0) = v$. The following classic lemma will also be used.

Lemma 2.2 (see [2, page 22] or [7]). *Let (X_0,g_0) be a Hadamard manifold, $x \in X_0$, and $\xi \in X_0(\infty)$. Then there exists a unique geodesic $c : \mathbb{R} \to X_0$ satisfying $c(0) = x$ and $c(+\infty) = \xi$.*

For $p \in X_0$, q_1 and q_2 in $\overline{X}_0 = X_0 \cup X_0(\infty)$ with $p \neq q_1$ and $p \neq q_2$, we define

$$\angle_p(q_1,q_2) := \angle(\dot{c}_{pq_1}(0),\dot{c}_{pq_2}(0)), \tag{2.3}$$

where $c_{pq_i} : \mathbb{R} \to X_0$ is the geodesic joining the points p and q_i and $\angle(\dot{c}_{pq_1}(0),\dot{c}_{pq_2}(0))$ is the angle subtended by the vectors $\dot{c}_{pq_1}(0)$ and $\dot{c}_{pq_2}(0)$.

For $p \in X_0$, $\xi \in X_0(\infty)$, $\epsilon > 0$, and $R > 0$, let

$$\Gamma_p(\xi,\epsilon,R) := \{q \in \overline{X}_0 = X_0 \cup X_0(\infty) \mid q \neq p,\ \angle_p(q,\xi) < \epsilon,\ d_{g_0}(p,q) > R\}. \tag{2.4}$$

For a fixed point $p \in X_0$, the set of all $\Gamma_p(\xi,\epsilon,R)$ and the open subsets of X_0 generate a topology on $\overline{X}_0 = X_0 \cup X_0(\infty)$. This topology is called the cône topology. With respect to this topology, the set $\overline{X}_0 := X_0 \cup X_0(\infty)$ is homeomorphic to a closed n-ball in $\mathbb{R}^n$ (see [2, page 22] or [7]). The induced topology on $X_0(\infty)$ is called the sphere topology.

Definition 2.3. Let (X_1,d_1) and (X_2,d_2) be two metric spaces. A map $\phi : X_1 \to X_2$ is called an (A,α)-quasi-isometric map for some constants $A > 1$ and $\alpha > 0$ if

$$\frac{1}{A}d_1(x,y) - \alpha \leq d_2(\phi(x),\phi(y)) \leq Ad_1(x,y) + \alpha \quad \forall x,y \in X_1. \tag{2.5}$$

In a metric space X, a (A,α)-quasigeodesic (resp., (A,α)-quasigeodesic ray) is a (A,α)-quasi-isometric map $\phi : \mathbb{R} \to X$ (resp., $\phi : \mathbb{R}^+ \to X$).

Definition 2.4. Let (X,d) be a metric space, E and F subsets of X. The Hausdorff distance d_H is defined by

$$d_H(E,F) := \inf\{r > 0,\ E \subset T_r(F),\ F \subset T_r(E)\}, \tag{2.6}$$

where

$$T_r(G) := \{x \in X,\ d(x,G) \leq r\} \quad \forall G \subset X. \tag{2.7}$$

Theorem 2.5 (Morse lemma, see [14]). *Let (X_0,g_0) be a Hadamard manifold with sectional curvature $K_{X_0} \leq -k_0^2 < 0$ for some constant $k_0 > 0$. Then for each (A,α)-quasigeodesic (resp., (A,α)-quasigeodesic ray) $\phi : \mathbb{R} \to X_0$ (resp., $\phi : \mathbb{R}^+ \to X_0$), there exist a real number $r_0 > 0$ and a geodesic (resp., geodesic ray) $c : \mathbb{R} \to X_0$ (resp., $c : \mathbb{R}^+ \to X_0$) such that $d_H(c(\mathbb{R}),\phi(\mathbb{R})) \leq r_0$ (resp., $d_H(c(\mathbb{R}^+),\phi(\mathbb{R}^+)) \leq r_0$); r_0 depends only on A, α, and k_0.*

Definition 2.6. Let (X,d) be a metric space with a reference point x_0. The Gromov product of the points x and y of X with respect to x_0 is the nonnegative real number $(x \cdot y)_{x_0}$ defined by

$$(x \cdot y)_{x_0} = \frac{1}{2}\{d(x,x_0) + d(y,x_0) - d(x,y)\}. \tag{2.8}$$

A metric space (X,d) is said to be a δ-hyperbolic space for some constant $\delta \geq 0$, if

$$(x \cdot y)_{x_0} \geq \min\{(x \cdot z)_{x_0};\ (y \cdot z)_{x_0}\} - \delta \tag{2.9}$$

for all x, y, z and every choice of reference point x_0. X is a Gromov hyperbolic space if it is a δ-hyperbolic space for some $\delta \geq 0$. The usual hyperbolic space $\mathbb{H}^n$ is a δ-hyperbolic space, where $\delta = \log 3$. More generally, every Hadamard manifold with sectional curvature less than or equal to $-k^2$ for some constant $k > 0$ is a δ-hyperbolic space, where $\delta = k^{-1}\log 3$ (see [1, 5, 10] or [11]).

Lemma 2.7 (see [5, page 20] or [4]). *Let (X,d) be a complete geodesic δ-hyperbolic space, x_0 a reference point in X, x and y two points of X. Then*

$$d(x_0,\gamma_{xy}) - 4\delta \leq (x \cdot y)_{x_0} \leq d(x_0,\gamma_{xy}) \tag{2.10}$$

for every geodesic segment γ_{xy} joining x and y.

Now let X be a Gromov hyperbolic manifold, x_0 a reference point in X. We say that the sequence $(x_i)_{i\in\mathbb{N}}$ of points in X converges at infinity if

$$\lim_{i,j\to\infty} (x_i \cdot x_j)_{x_0} = \infty. \tag{2.11}$$

If x_1 is another reference point in X,

$$(x \cdot y)_{x_0} - d(x_0,x_1) \leq (x \cdot y)_{x_1} \leq (x \cdot y)_{x_0} + d(x_0,x_1). \tag{2.12}$$

Then the definition of the sequence that converges at infinity does not depend on the choice of the reference point. We recall the following equivalence relation $\mathcal{R}$ on the set of sequences of points in X that converge at infinity:

$$(x_i)\mathcal{R}(y_j) \Longleftrightarrow \lim_{i,j\to\infty} (x_i \cdot y_j)_{x_0} = \infty. \tag{2.13}$$

The Gromov boundary $X^G(\infty)$ of X is the coset of sequences that converge at infinity.

Let X be a simply connected Riemannian manifold which is a Gromov hyperbolic space. One defines on the set $X \cup X^G(\infty)$ a topology as follows (see [5, page 22] or [10, page 122]):

(1) if $x \in X$, a sequence $(x_i)_{i\in\mathbb{N}}$ converges to x with respect to the topology of X,
(2) if $(x_i)_{i\in\mathbb{N}}$ defines a point $\xi \in X^G(\infty)$, $(x_i)_{i\in\mathbb{N}}$ converges to ξ,
(3) for $\eta \in X^G(\infty)$ and $k > 0$, let

$$V_k(\eta) := \{y \in X \cup X^G(\infty),\ (y \cdot \eta)_{x_0} > k\}, \tag{2.14}$$

where

$$(x \cdot y)_{x_0} = \inf\left\{\liminf_{i\to\infty} (x_i \cdot y_i)_{x_0},\ x_i \to x,\ y_i \to y\right\} \tag{2.15}$$

for x and y elements of $X \cup X^G(\infty)$.

The set of all $V_k(\eta)$ and the open metric balls of X generate a topology on $X \cup X^G(\infty)$. With respect to this topology, X is dense in $X \cup X^G(\infty)$ and $X \cup X^G(\infty)$ is compact.

Lemma 2.8 [4]. *Let X be a δ-hyperbolic space. Then*

(1) *each geodesic $\gamma:\mathbb{R}\to X$ defines two distinct points at infinity $\gamma(+\infty)$ and $\gamma(-\infty)$,*

(2) *for each $(\eta,x)\in X^G(\infty)\times X$, there exists a geodesic ray γ such that $\gamma(0)=x$ and $\gamma(+\infty)=\eta$. For any other geodesic ray γ' with $\gamma'(0)=x$ and $\gamma'(+\infty)=\eta$, $d(\gamma'(t),\gamma(t))\le 4\delta$ for all $t\ge 0$.*

Definition 2.9. Let $\xi\in X^G(\infty)$ and $c:\mathbb{R}_+\to X$ be a minimal geodesic ray satisfying $c(+\infty)=\xi$. The function

$$b_c(x):=\lim_{t\to\infty}\big(d(x,c(t))-t\big) \tag{2.16}$$

is well-defined on X and is called the Busemann function for the geodesic c.

Lemma 2.10 [4]. *Let X be a δ-hyperbolic space, $\xi\in X^G(\infty)$, $x,y\in X$, and c a geodesic ray with $c(0)=x$ and $c(+\infty)=\xi$. Then there exists a neighbourhood $\mathcal{V}$ of ξ in $X\cup X^G(\infty)$ such that*

$$\big|\,b_c(y)-\big(d(z,y)-d(z,x)\big)\,\big|\le K\quad \forall z\in\mathcal{V}\cap X, \tag{2.17}$$

where b_c is the Busemann function for the geodesic c and K is a constant depending only on δ.

Lemma 2.11 [5]. *Let X_1 be a metric space and let (X_2,d_2) be a geodesic Gromov hyperbolic space. If there exists a quasi-isometric map $\phi:X_1\to X_2$, then X_1 is also a Gromov hyperbolic space. Moreover, if the map*

$$x\longmapsto d_2\big(x,\phi(X_1)\big) \tag{2.18}$$

is bounded above, $X_1^G(\infty)\simeq X_2^G(\infty)$, that is, $X_1^G(\infty)$ is homeomorphic to $X_2^G(\infty)$.

Now let (M,g) be a compact Riemannian manifold of hyperbolic type and let X be its universal Riemannian covering. Let g_0 denote an associated metric of strictly negative curvature on M. The universal Riemannian covering X_0 of (M,g_0) is a Hadamard manifold satisfying $K_{X_0}\le -k_0^2<0$ for some constant $k_0>0$. Then X_0 and X are Gromov hyperbolic spaces. Moreover, $X^G(\infty)\simeq X_0^G(\infty)$.

Two geodesic rays c and c' are said to be asymptotic if there exists a constant $D\ge 0$ such that $d_H(c(\mathbb{R}_+),c'(\mathbb{R}_+))\le D$. This defines an equivalence relation on the set of minimizing g-geodesic rays of X. Let $X(\infty)$ be the coset of asymptotic minimizing g-geodesic rays. For each minimizing g-geodesic ray c of X, it follows from Morse lemma that there exists a g_0-geodesic ray c_0 such that $d_H(c(\mathbb{R}_+),c_0(\mathbb{R}_+))\le r_0$, where r_0 is the constant in Morse lemma. Let $[c]$ be the equivalence class of minimizing g-geodesic ray c and let $[c_0]$ be the equivalence class of the g_0-geodesic c_0. The map f defined by

$$\begin{gathered} f:X(\infty)\longrightarrow X_0(\infty),\\ [c]\longmapsto [c_0] \end{gathered} \tag{2.19}$$

is bijective. Then f defines on $X(\infty)$ a natural topology with respect to which $X(\infty)$ and $X_0(\infty)$ are homeomorphic, that is, $X(\infty)\simeq X_0(\infty)$ (see [8]).

Lemma 2.12 [3]. *Let X_0 be a Hadamard manifold with sectional curvature $K_{X_0} \leq -k_0^2 < 0$ for some constant $k_0 > 0$. There exists a natural homeomorphism*

$$\phi : X_0 \cup X_0^G(\infty) \longrightarrow X_0 \cup X_0(\infty). \tag{2.20}$$

In particular, $X_0^G(\infty) \simeq X_0(\infty)$.

Using Morse lemma, Lemma 2.12 and the properties of the ideal boundaries, we obtain the following lemma.

Lemma 2.13. *Let (M,g) be a compact Riemannian manifold of hyperbolic type, and let X be its universal Riemannian covering. Let g_0 be an associated metric of strictly negative curvature on M and let X_0 be the universal Riemannian covering of (M,g_0). It holds that*

$$X(\infty) \simeq X_0(\infty) \simeq X_0^G(\infty) \simeq X^G(\infty). \tag{2.21}$$

3. Busemann quasidensities

Let (X,d) be a metric space and let Γ be a discrete and infinite subgroup of the isometry group $\mathrm{Iso}(X)$ of X. For $x_0, x \in X$ and $s \in \mathbb{R}$,

$$P_s(x,x_0) := \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma x_0)} \tag{3.1}$$

denotes the Poincaré series associated to Γ. The number

$$\alpha := \inf\{s \in \mathbb{R}/P_s(x,x_0) < \infty\} \tag{3.2}$$

is called the critical exponent of Γ and is independent of x and x_0. The group Γ is called of divergence type if $P_\alpha(x,x_0)$ diverges. The following lemma introduces a useful modification (due to Patterson) of the Poincaré series if Γ is not of divergence type.

Lemma 3.1 [19]. *Let Γ be a discrete group with critical exponent α. There exists a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous, nondecreasing, and such that*

$$\forall a > 0, \quad \lim_{r \to +\infty} \frac{f(r+a)}{f(r)} = 1, \tag{3.3}$$

and the modified series

$$\widetilde{P}_s(x,x_0) := \sum_{\gamma \in \Gamma} f(d(x,\gamma x_0))e^{-sd(x,\gamma x_0)} \tag{3.4}$$

converges for $s > \alpha$ and diverges for $s \leq \alpha$.

Now let (M,g) be a compact Riemannian manifold of hyperbolic type and let X be its universal Riemannian covering. Let g_0 denote a metric of negative curvature on M. The universal Riemannian covering X_0 of (M,g_0) is a Hadamard manifold satisfying $K_{X_0} \leq -k_0^2 < 0$ for some constant $k_0 > 0$. Let Γ be the group of deck transformations of X and let α^{g_0} be its critical exponent with respect to the metric g_0. It follows from [16, Theorem 5.1] that

$$\alpha^{g_0} = h_{g_0} := \lim_{r\to\infty} \frac{\log \operatorname{vol} B_{g_0}(p,r)}{r}. \tag{3.5}$$

The fact that M is compact implies the existence of a constant $\lambda \geq 1$ such that the critical exponent α^g of Γ with respect to the metric g belongs to $[\lambda^{-1}h_{g_0}, \lambda h_{g_0}] \subset \mathbb{R}_+^*$ (see [15]).

Lemma 3.2. *Let (M,g) be a compact Riemannian manifold of hyperbolic type and let X be its universal Riemannian covering. Let Γ be the group of deck transformations of X and for a given $x \in X$ the set $\Lambda^g(\Gamma,x)$ of the accumulation points of the orbit Γx in $X^G(\infty)$. Then*
(1) $\Lambda^g(\Gamma,x) = \overline{\Gamma x} \cap X^G(\infty)$,
(2) $\gamma(\Lambda^g(\Gamma,x)) = \Lambda^g(\Gamma,x)$ *for all* $\gamma \in \Gamma$ *and* $x \in X$,
(3) $\Lambda^g(\Gamma,x)$ *is independent of* x,
(4) $\Lambda^g(\Gamma,x) = X^G(\infty)$.

Proof. Using the definition of $\Lambda^g(\Gamma,x)$, we can easily check (1) and (2).

(3) For all $\xi \in \Lambda^g(\Gamma,x)$, by definition there is a sequence $(\gamma_n)_n$ of points of Γ such that $\lim_{n\to\infty} \gamma_n x = \xi$. Then

$$\lim_{m,n\to\infty} (\gamma_n x \cdot \gamma_m x)_{x_0} = +\infty. \tag{3.6}$$

For all $y \in X$, we have

$$\begin{aligned} 2(\gamma_n x \cdot \gamma_n y)_{x_0} &= d(\gamma_n x, x_0) + d(\gamma_n y, x_0) - d(\gamma_n x, \gamma_n y) \\ &\geq d(\gamma_n x, x_0) + d(\gamma_n y, x_0) - d(x,y) \\ &\geq d(\gamma_n x, x_0) + d(x,y). \end{aligned} \tag{3.7}$$

Hence,

$$\lim_{n\to\infty} (\gamma_n x \cdot \gamma_n y)_{x_0} = +\infty, \qquad \lim_{n\to\infty} \gamma_n y = \xi. \tag{3.8}$$

(4) Let g_0 denote a metric of strictly negative curvature on M. The universal Riemannian covering X_0 of (M,g_0) is a Hadamard manifold satisfying $K_{X_0} \leq -k_0^2 < 0$ for some constant $k_0 > 0$. Then $\Lambda^{g_0}(\Gamma,x) = X_0(\infty)$ (see [15]). Finally, using Lemma 2.11 we obtain that $\Lambda^g(\Gamma,x) = X^G(\infty)$. □

Definition 3.3. Let X be a Gromov hyperbolic manifold, $\alpha \in \mathbb{R}_+$, and let Γ be a discrete and infinite subgroup of $\operatorname{Iso}(X)$. A family $\{\mu_x\}_{x\in X}$ of finite nontrivial Borel measures on $X \cup X^G(\infty)$ is an α-dimensional Busemann quasidensity with reference point $x_0 \in X$ if
(1) $\sup \mu_x \subset \Lambda(\Gamma,x)$, where $\Lambda(\Gamma,x)$ is the limit set of the orbit Γx in $X^G(\infty)$,
(2) $\mu_{\gamma x}(\gamma A) = \mu_x(A)$ for all $\gamma \in \Gamma$, $A \subset X^G(\infty)$, A measurable, $x \in X$,
(3) there exists a constant $\lambda \geq 1$ such that for all $x \in X$,

$$\lambda^{-1} e^{-\alpha b_c(x_0)} \leq \frac{d\mu_{x_0}}{d\mu_x}(\xi) \leq \lambda e^{-\alpha b_c(x_0)} \tag{3.9}$$

for almost all $\xi \in X^G(\infty)$, where c is a geodesic satisfying $c(0) = x$, $c(\infty) = \xi$ and b_c is the Busemann function for the geodesic c.

The next lemma states the existence of a Busemann quasidensity.

Lemma 3.4. *Let (M,g) be a compact Riemannian manifold of hyperbolic type and let X be its universal Riemannian covering. Let Γ be the group of deck transformations of X and let α^g be its critical exponent. Then there exists an α^g-dimensional Busemann quasidensity $\{\mu_x\}_{x\in X}$ on $X \cup X^G(\infty)$.*

Proof. We have to construct a family of measure $\{\mu_x\}_{x\in X}$ which satisfies the axiomatic Definition 3.3.

Construction of $\{\mu_x\}_{x\in X}$. A natural way to obtain Busemann quasidensity was given by Patterson (see [19]) in the case of Fuchsian groups.

Let x_0 be a reference point of the Gromov hyperbolic manifold X. For $s > \alpha^g$ and $x \in X$, we consider the measure

$$\mu_{s,x_0,x} := \frac{\sum_{\gamma\in\Gamma} f\big(d(x,\gamma x_0)\big)e^{-sd(x,\gamma x_0)}\delta_{\gamma x_0}}{\widetilde{P}_s(x_0,x_0)}, \tag{3.10}$$

where f is a useful modification function (due to Patterson) of the Poincaré series if Γ is not of divergence type and

$$\widetilde{P}_s(x_0,x_0) = \sum_{\gamma\in\Gamma} f\big(d(x_0,\gamma x_0)\big)e^{-sd(x_0,\gamma x_0)}. \tag{3.11}$$

Let $(s_n)_n$ be a sequence with $s_n > \alpha^g$ and $s_n \to \alpha^g$ such that $\mu_{s_n,x_0,x}$ converges weakly, as well to the measure μ_x. For every $x \notin \Gamma x_0$, we choose a subsequence of $(s_n)_n$, denoted by (s_n^x), such that the measure $\mu_{s_n^x,x_0,x}$ is also weakly convergent. For all points of the same orbit Γx we can choose the same subsequence, that is, $s_n^{x'} = s_n^x$ if $x' \in \Gamma x$. These choices yield a family $\{\mu_x\}_{x\in X}$ of measures.

$\{\mu_x\}_{x\in X}$ is an α^g-dimensional Busemann quasidensity. (i) Using the triangle inequality and the fact that $1/2 \le f(d(x,\gamma x_0))/f(d(x_0,\gamma x_0)) \le 3/2$ for almost all $\gamma \in \Gamma$, we deduce that

$$ae^{-sd(x,x_0)} \le \mu_{s,x_0,x} \le be^{-sd(x,x_0)}, \tag{3.12}$$

where a and b depend only on $d(x_0,x)$. This implies that $\{\mu_x\}_{x\in X}$ is a family of finite nontrivial Borel measures on $X \cup X^G(\infty)$.

(ii) For all $z \in X \cup X^G(\infty)\backslash\Lambda^g(\Gamma,x)$, there is an open neighbourhood $\mathcal{U}$ of z with $\Gamma x \cap \mathcal{U}\backslash\{z\} = \varnothing$. Then

$$\mu_{s_n,x_0,x}(\mathcal{U}) \le \frac{f\big(d(x,z)\big)e^{-s_n d(x,z)}}{\widetilde{P}_{s_n}(x_0,x_0)}. \tag{3.13}$$

Since $\widetilde{P}_s(x_0,x_0)$ diverges for $s = \alpha^g$, we obtain $\mu_x(\mathcal{U}) = 0$.

(iii) Let $\eta \in \Gamma$, and let A be a measurable subset of $X \cup X^G(\infty)$. Then

$$\begin{aligned}\mu_{s,x_0,\eta x}(\eta A) &= \frac{\sum_{\gamma \in \Gamma,\ \gamma x_0 \in \eta A} f\big(d(\eta x, \gamma x_0)\big) e^{-sd(\eta x, \gamma x_0)}}{\widetilde{P}_s(x_0, x_0)} \\ &= \frac{\sum_{\gamma' \in \Gamma,\ \gamma' x_0 \in A} f\big(d(x, \gamma' x_0)\big) e^{-sd(x, \gamma' x_0)}}{\widetilde{P}_s(x_0, x_0)} \\ &= \mu_{s,x_0,x}(A).\end{aligned} \tag{3.14}$$

Thus $\mu_{\eta x}(\eta A) = \mu_x(A)$ for all $\eta \in \Gamma$.

(iv) We now consider $\xi \in X^G(\infty)$ and a sequence $(U_n)_n$ of open sets in $X \cup X^G(\infty)$ with $\lim_{n\to\infty} U_n = \xi$. By Lemma 2.10, there exists $n_0 \in \mathbb{N}$ such that

$$\big| b_c(x_0) - \big(d(\gamma x_0, x_0) - d(x, x_0)\big) \big| \le K \tag{3.15}$$

for all $n \ge n_0$ and $\gamma x_0 \in U_n$, where c is a geodesic joining x and ξ, b_c a Busemann function for the geodesic c, and K a constant depending only on the metric g_0. Then, using Lemma 3.1, we deduce the existence of a constant $\lambda \ge 1$ such that

$$\lambda^{-1} e^{-\alpha b_c(x_0)} \le \frac{d\mu_{x_0}}{d\mu_x}(\xi) \le \lambda e^{-\alpha b_c(x_0)}. \tag{3.16}$$

For a given $y \in X \cup X^G(\infty)$, $x \in X$, and $\rho \ge 0$, we introduce the shadow $\mathbb{O}_y^g(x,\rho)$ (of the ball $B_g(x,\rho)$ viewed from the point y) as follows: $\mathbb{O}_y^g(x,\rho)$ consists of all points $\xi \in X^G(\infty)$ such that all geodesic rays $c_{y\xi}$ connecting y and ξ satisfy $c_{y\xi} \cap B_g(x,\rho) \neq \emptyset$. □

Lemma 3.5. *Let (M,g), X, Γ, and $\{\mu_x\}_{x\in X}$ be as in Lemma 3.4. Then there exist constants $R_1 > 0$ and $l > 0$ such that for all $\rho \ge R_1$,*

$$\mu_x\big(\mathbb{O}_y^g(x,\rho)\big) \ge l \quad \forall x, y \in X. \tag{3.17}$$

Proof. Let g_0 be a metric of negative curvature on M and X_0 the universal Riemannian covering of (M, g_0). For $v \in S_x X_0$ we define

$$C_\varepsilon^{g_0}(v) = \{c_w(\infty),\ w \in S_x X_0,\ (v,w) < \varepsilon\}, \tag{3.18}$$

where c_w is the g_0-geodesic satisfying $\dot{c}_w(0) = w$.

Let $\mathcal{F}$ be a fundamental domain in X. It follows from [16, Proposition 3.6] the existence of constants $R_0 > 0$ and $\varepsilon > 0$ such that for all $x \in \mathcal{F}$ and $y \in X$, $C_\varepsilon^{g_0}(v) \subset \mathbb{O}_y^{g_0}(x, R_0)$ for some $v \in S_x X_0$. Hence, using Morse lemma we obtain a constant $R_1 > 0$ with

$$C_\varepsilon^{g_0}(v) \subset \mathbb{O}_y^{g_0}(x, R_1). \tag{3.19}$$

Finally, because of

$$\sup \mu_x = X^G(\infty) \simeq X_0^G(\infty), \qquad \gamma\big(\mathbb{O}_y^g(x,\rho)\big) = \mathbb{O}_{\gamma y}^g(\gamma x, \rho) \tag{3.20}$$

for all $\gamma \in \Gamma$, there exists a constant $l > 0$ such that for all $\rho \geq R_1$,

$$\mu_x(\mathbb{O}^g_y(x,\rho)) \geq l \quad \forall x, y \in X. \tag{3.21}$$

□

The shadow lemma was proven by Sullivan in the case of the usual hyperbolic space (see [20]). Our version generalizes this result to all compact manifolds of hyperbolic type.

Lemma 3.6 (shadow lemma). *Let (M,g) be a compact Riemannian manifold of hyperbolic type and let X be its universal Riemannian covering. Let Γ be the group of deck transformations of X, let α^g be its critical exponent, and let $\{\mu_x\}_{x\in X}$ be a Patterson-Sullivan density associated to Γ on $X \cup X^G(\infty)$. Then there exist a constant $R_1 > 0$ and a function $b \geq 1$ such that for all $\rho \geq R_1$ and $x \in X$,*

$$\frac{1}{b(\rho)} e^{-\alpha^g d(x,x_0)} \leq \mu_{x_0}(\mathbb{O}^g_{x_0}(x_0,\rho)) \leq b(\rho) e^{-\alpha^g d(x,x_0)}. \tag{3.22}$$

Proof. It follows from Lemma 3.4 that there exists a constant $\lambda \geq 1$ such that for all $\xi \in X^G(\infty)$ and $x \in X$,

$$\lambda^{-1} \int_{\mathbb{O}^g_{x_0}(x_0,\rho)} e^{-\alpha^g b_c(x_0)} d\mu_x(\xi) \leq \mu_{x_0}(\mathbb{O}^g_{x_0}(x_0,\rho)) \leq \lambda \int_{\mathbb{O}^g_{x_0}(x_0,\rho)} e^{-\alpha^g b_c(x_0)} d\mu_x(\xi), \tag{3.23}$$

where c is a geodesic joining x and ξ, b_c the Busemann function for the geodesic c.

Morse lemma and the definition of $\mathbb{O}^g_{x_0}(x_0,\rho)$ imply the existence of constant $D > 0$ such that

$$d(x,x_0) - D \leq b_c(x_0) \leq d(x,x_0) + D \quad \forall x \in X. \tag{3.24}$$

Therefore

$$\mu_{x_0}(\mathbb{O}^g_{x_0}(x_0,\rho)) \leq \lambda e^{-\alpha^g(d(x,x_0)-2D)} \mu_x(\mathbb{O}^g_{x_0}(x,\rho)) \leq b' e^{2\alpha_g D} e^{\alpha^g d(x,x_0)}, \tag{3.25}$$

where $b' = \sup_{x\in X} \mu_x(X^G(\infty))$. Moreover,

$$\mu_{x_0}(\mathbb{O}^g_{x_0}(x_0,\rho)) \geq \lambda^{-1} e^{-2\alpha^g D} e^{-\alpha^g d(x,x_0)} \mu_x(\mathbb{O}^g_{x_0}(x,\rho)). \tag{3.26}$$

Then using Lemma 3.4, we obtain

$$\mu_{x_0}(\mathbb{O}^g_{x_0}(x_0,\rho)) \geq l\lambda^{-1} e^{-2\alpha^g D} e^{-\alpha^g d(x,x_0)}. \tag{3.27}$$

□

4. The growth rate of volume of spheres in manifolds of hyperbolic type

A Riemannian manifold M is said to be without conjugate points if every nonzero Jacobi field vanishes at most one point. It is well known that if M has no conjugate points, for each point $p \in M$ the exponential map $\exp_p : T_pM \to M$ is a covering map. Moreover, if M is simply connected, $\exp_p$ is a diffeomorphism and any two points of M can be joined by a unique geodesic segment.

Theorem 4.1. *Let (M,g) be a compact Riemannian manifold of hyperbolic type without conjugate points and let X be its universal Riemannian covering. Let $S(x_0,r)$ be the geodesic sphere about $x_0 \in X$ of radius r and let h_g be the volume entropy of (M,g). Then there exist constants $a \geq 1$ and $r_0 > 0$ such that*

$$\frac{1}{a} \leq \frac{\operatorname{vol} S(x_0,r)}{e^{h_g r}} \leq a \quad \forall r \geq r_0, \tag{4.1}$$

that is, the growth function of the volume of the geodesic spheres $S(x_0,r)$ is of purely exponential type.

The following lemmas will be useful for the proof of Theorem 4.1. Their proofs use similarly arguments like those given in [3].

Lemma 4.2. *Let (M,g) be a compact Riemannian manifold of hyperbolic type without conjugate points, let X be its universal Riemannian covering, and let $n = \dim X$. Let $S(x_0,r)$ be the geodesic sphere about $x_0 \in X$ of radius r. Then for all $\rho \leq (1/2)r$, there exists a constant $l_1(\rho) > 0$ such that all $(n-1)$-dimensional subdomains B in $S(x_0,r)$ with* $\operatorname{diam} B = \rho$ *satisfy*

$$\operatorname{vol}_{n-1}(B) \leq l_1(\rho). \tag{4.2}$$

Proof. We will use in $T_{x_0}X$ the geodesic polar coordinate system (t,θ), where $\theta \in S_{x_0}X$. Since the Riemannian manifold X is simply connected without conjugate points, the exponential map $\exp_{x_0}$ realizes a diffeomorphism from $T_{x_0}X$ to X. Let $(D\exp_{x_0})(t\theta)$ denote the differential of $\exp_{x_0}$ evaluated at a point $(t,\theta) \in T_{x_0}X$. The fact that M is compact implies the existence of a constant $k > 0$ with $\operatorname{Ric}(X) \geq -(n-1)k^2$. Let $X^n_{-k^2}$ denote the simply connected space form with constant sectional curvature $-k^2$. Using Bishop-Gromov theorem (see [12]), we obtain

$$\det\left(D\exp_{x_0}\right)(s_1\theta) \leq \left[\frac{\sinh(ks_1)}{\sinh(ks_2)}\right]\det\left(D\exp_{x_0}\right)(s_2\theta) \tag{4.3}$$

for all $s_1 \geq s_2 > 0$. We consider a $(n-1)$-dimensional subdomain B in the geodesic sphere $S(x_0,r)$ with $\operatorname{diam} B = \rho$ and the following set:

$$\mathbb{F} := \bigcup_{r-\rho \leq t \leq r} \mathbb{P}_t(B) \quad \text{where } \mathbb{P}_t(y) = \exp_{x_0}\left[\frac{t}{r}\exp_{x_0}^{-1}(y)\right] \tag{4.4}$$

for all $y \in S(x_0,r)$. For each point $x \in B$, the set $\mathbb{F}$ is contained in the geodesic ball $B(x,2\rho)$. Therefore using Bishop-Gunther theorem (see [9, page 140]), we obtain a constant $t_0 \in [r-\rho,r]$ such that

$$\operatorname{vol}_n \mathbb{P}_{t_0}(B) \leq \frac{1}{\rho}V_{-k^2}(2\rho) \quad \text{where } V_{-k^2}(2\rho) \tag{4.5}$$

is the volume of a ball with radius 2ρ in the space form $X^n_{-k^2}$. Then using (4.3), we obtain

$$\operatorname{vol}_{n-1}(B) \leq \left[\frac{\sinh(2k\rho)}{\sinh(k\rho)}\right]^{n-1}\frac{V_{-k^2}(2\rho)}{\rho}. \tag{4.6}$$

□

Let $B(x_0,r)$ be the open geodesic ball of radius r about a point x_0 in X. For $x,y \in X\backslash B(x_0,r)$, we define

$$d_r(x,y) := \inf\{l(\sigma),\ \sigma \text{ is a piecewise smooth curve connecting } x,y,\sigma \subset X\backslash B(x_0,r)\}. \tag{4.7}$$

For $x \in S(x_0,r)$, let

$$B^r_\rho(x) := \{y \in S(x_0,r),\ d_r(x,y) < \rho\}. \tag{4.8}$$

Lemma 4.3. *Let (M,g) be a compact Riemannian manifold of hyperbolic type without conjugate points, let X be its universal Riemannian covering, and let $n = \dim X$. Suppose that X is a δ-hyperbolic manifold. A constant $K > 0$ can be found such that for all $\rho \geq K$ and $r \geq 2\rho$, there exists a constant $l_2(\rho) > 0$ with*

$$\operatorname{vol}_{n-1}\left(B^r_\rho(x)\right) \geq l_2(\rho) \tag{4.9}$$

for all $x \in S(x_0,r)$.

Proof. We consider the set

$$\mathbb{H} := \bigcup_{r\leq t\leq r+4\rho} \mathbb{P}_t\left(B^r_\rho(x)\right). \tag{4.10}$$

Using (4.3) in Lemma 4.2, we obtain

$$\operatorname{vol}_n\left(\mathbb{P}_t\left(B^r_\rho(x)\right)\right) \leq \left[\frac{\sinh(kt)}{\sinh(kr)}\right]^{n-1} \operatorname{vol}_{n-1}\left(B^r_\rho(x)\right). \tag{4.11}$$

Hence,

$$\operatorname{vol}_{n-1}\left(B^r_\rho(x)\right) \geq \frac{\operatorname{vol}_n(\mathbb{H})}{4\rho}\left[\frac{\sinh(kt)}{\sinh(kr+4k\rho)}\right]^{n-1}. \tag{4.12}$$

But there exist some point $z \in \mathbb{H}$ and a constant $K > 0$ such that $B(z,\rho/4) \subset \mathbb{H}$ for all $\rho \geq K$. Therefore

$$\operatorname{vol}_{n-1}\left(B^r_\rho(x)\right) \geq \frac{\operatorname{vol}_n\left(B(z,\rho/4)\right)}{4\rho}\left[\frac{\sinh(kt)}{\sinh(kr+4k\rho)}\right]^{n-1}. \tag{4.13}$$

Since M is compact, there exists a constant $k_1 > 0$ with $K_X \leq k_1$. Then using Bishop-Gunther theorem (see [9, page 140]), we obtain

$$\operatorname{vol}_n\left(B\left(z,\frac{\rho}{4}\right)\right) \geq V_{k_1}\left(\frac{\rho}{4}\right), \tag{4.14}$$

where $V_{k_1}(\rho/4)$ is the volume of a ball of radius $\rho/4$ in the space form $X^{k_1}_n$. Hence,

$$\operatorname{vol}_{n-1}\left(B^r_\rho(x)\right) \geq \frac{V_{k_1}(\rho/4)}{4\rho}\left[\frac{\sinh(2k\rho)}{\sinh(6k\rho)}\right]^{n-1} \quad \forall r \geq 2\rho. \tag{4.15}$$

□

Proof of Theorem 4.1. Choose $\rho = \max\{6R_1, 3K, 13\delta\}$, where R_1 is as in Lemma 3.6, K is as in Lemma 4.3, and $\delta > 0$ such that X is a δ-hyperbolic space. Let $x_1, x_2, \dots, x_m$ be a maximal ρ-separating set in $S(x_0, r)$. Then

$$X^G(\infty) = \bigcup_{i=1}^{m} \mathbb{O}^g_{x_0}(x_i, \rho + 4\delta). \tag{4.16}$$

Since $\rho \geq 6R_1$, Lemma 3.6 implies the existence of a constant $b(\rho + 4\delta)$ with

$$m \geq \frac{b_0 e^{\alpha^g r}}{b(\rho + 4\delta)} \quad \text{where } b_0 = \mu_{x_0}(X^G(\infty)), \tag{4.17}$$

and α^g is the critical exponent of the group of deck transformations. Note that the balls $B^r_{\rho/3}(x_i)$ are pairwise disjoint subsets of $S(x_0, r)$. Then since $\rho \geq 3K$, by Lemma 4.3 we obtain a constant $l_2(\rho/3) > 0$ such that

$$\operatorname{vol} S(x_0, r) \geq \frac{b_0 l_2(\rho/3) e^{\alpha^g r}}{b(\rho + 4\delta)} \quad \forall r \geq \frac{2\rho}{3}. \tag{4.18}$$

Furthermore, Lemma 4.2 implies the existence of a constant $l_1(\rho) > 0$ with

$$\operatorname{vol} S(x_0, r) \leq m l_1(\rho) \tag{4.19}$$

for all $r \geq 2\rho$. Since $\rho \geq 13\delta$, the shadows $\mathbb{O}^g_{x_0}(x_i, \rho/6)$ are pairwise disjoint subsets of $X^G(\infty)$. Because of $\rho \geq 6R_1$, Lemma 3.6 implies that there exists a constant $b(\rho/6)$ with

$$b_0 \geq \frac{m}{b(\rho/6) e^{\alpha^g r}}. \tag{4.20}$$

Finally, since

$$\operatorname{vol} B(x_0, r) = \int_0^r \operatorname{vol} S(x_0, t)\, dt, \tag{4.21}$$

there exist constants $a_1 \geq 1$ and $r_1 > 0$, such that

$$\frac{1}{a_1} \leq \frac{\operatorname{vol} B(x_0, r)}{e^{\alpha_g r}} \leq a_1 \quad \forall r \geq r_1. \tag{4.22}$$

Hence $\alpha^g = h_g$. □

COROLLARY 4.4. *Let (M, g) be a compact orientable surface of genus greater than or equal to 2, without conjugate points and let X be its universal Riemannian covering. Then the growth function of the volume of geodesic spheres of X is of pure exponential type.*

COROLLARY 4.5. *Let (M, g) be a compact manifold of hyperbolic type without conjugate points and let X be its universal Riemannian covering. Then the growth function of geodesic balls of X is of purely exponential type with the volume entropy as exponential factor.*

5. Closed geodesics in compact manifolds of hyperbolic type

Let M be a complete, simply connected manifold and let d be the induced metric of the Riemannian structure. A geodesic $c : \mathbb{R} \to M$ is closed, if there exists a constant $u > 0$ such that $c(t+u) = c(u)$ for all $t \in \mathbb{R}$. The period $\operatorname{Per}(c)$ of c is the smallest constant $u > 0$ satisfying this property.

Definition 5.1. Consider two closed geodesics c_1 of period t_1 and c_2 of period t_2 as equivalent, if there exist $n_1, n_2 \in \mathbb{N}$ such that $c_{1|_{[0,n_1t_1]}}$ and $c_{2|_{[0,n_2t_2]}}$ or $c_{1|_{[0,n_1t_1]}}$ and $c^{-1}_{2|_{[0,n_2t_2]}}$ are freely homotopic, where $c_2^{-1}(t) = c_2(-t)$ for all $t \in \mathbb{R}$.

Let $[c]$ denote the equivalence class of the closed geodesic c,

$$\begin{aligned} l([c]) &= \inf\{\operatorname{Per}(c_0),\ c_0 \in [c]\}, \\ \mathcal{P}(t) &= \#\{[c],\ l([c]) \le t\}. \end{aligned} \tag{5.1}$$

Let (M,g) be a compact manifold, let X be its universal Riemannian covering, let $\pi : X \to M$ be the covering map, and let Γ be the group of deck transformations; $\Gamma \simeq \pi_1(M)$. For all $\gamma \in \Gamma$, since the manifold M is compact, there exists $p_0 \in X$ such that $d(p_0, \gamma(p_0)) =: l(\gamma)$. The geodesic c connecting p_0 and $\gamma(p_0)$ is called an axis of γ and the projection $\pi \circ c$ is a closed geodesic of M of period $l(\gamma)$.

Definition 5.2. Two elements γ_1 and γ_2 of Γ are equivalent ($\gamma_1 \sim \gamma_2$), if there exist $m, n \in \mathbb{Z}$ and an isometry $\beta \in \Gamma$ such that $\gamma_1^n = \beta\gamma_2^m\beta^{-1}$.

The projections of the axes of two equivalent elements γ_1 and γ_2 of Γ define two equivalent closed geodesics on M. Conversely, the lifts of two equivalent closed geodesics are axes of two equivalent isometries. Hence, we obtain the following well-known result.

Proposition 5.3 [16]. *The coset of closed geodesics is in one-to-one correspondence with the equivalence classes of the elements in the fundamental group.*

Lemma 5.4. *Let (M,g) be a compact Riemannian manifold of hyperbolic type without conjugate points and let X be its universal Riemannian covering. Let $\mathcal{P}(t)$ denote the number of equivalence classes of closed geodesics of M with length less than or equal to t. Then there exist constants $a > 1$ and $t_0 > 0$ such that $\mathcal{P}(t) \le a e^{h_g t}$ for all $t > t_0$, where h_g is the volume entropy of X.*

Proof. Let Γ be the group of deck transformations of X and $\mathcal{F} \subset X$ a fundamental domain of Γ with $\operatorname{diam}\mathcal{F} = D$. Using Proposition 5.3, we obtain for a fixed p in $\mathcal{F}$,

$$\mathcal{P}(t) \le \#\{\gamma \in \Gamma,\ \gamma\mathcal{F} \subset B_{2D+t}(p)\}. \tag{5.2}$$

Since the $\gamma_i\mathcal{F}$ are pairwise disjoint, we obtain by Corollary 4.5

$$\mathcal{P}(t) \le \frac{\operatorname{vol} B_{2D+t}(p)}{\operatorname{vol}\mathcal{F}} \le \frac{1}{\operatorname{vol}\mathcal{F}} a_0 e^{h_g t}. \tag{5.3}$$

□

Lemma 5.5. *Let (M,g) be a compact Riemannian manifold of hyperbolic type without conjugate points, X its universal Riemannian covering, and Γ the group of deck transformations* *of X. For $p \in X$ and $r \geq 0$, let*

$$\Gamma_t^r(p) := \{\gamma \in \Gamma,\ r < d(p,\gamma(p)) \leq t\}. \tag{5.4}$$

Then there exist constants $b > 0$ and $t_0 > 0$ such that $\#\Gamma_t^r(p) \geq be^{h_g t}$ for all $t \geq t_0$, where h_g is the volume entropy of X.

Proof. Let $\mathcal{F}$ be a fundamental domain of Γ in X with $\operatorname{diam}\mathcal{F} = D$. For all $p \in \mathcal{F}$, using the definition of $\Gamma_t^r(p)$ and the triangle inequality, we have

$$B_t(p)\backslash B_r(p) \subset \cup_{\gamma \in \Gamma_{t+D}^{r-D}(p)} \gamma(B_D(p)). \tag{5.5}$$

Let r_0 be as in Theorem 4.1 and $r_1 = \max(r, r_0)$. We have

$$\operatorname{vol} B_t(p)\backslash B_r(p) \geq \operatorname{vol} B_t(p)\backslash B_{r_1}(p) \geq \frac{e^{h_g t}}{a}\left[1 - \frac{a^2 e^{h_g r_1}}{e^{h_g t}}\right]. \tag{5.6}$$

Then there exist constants $A > 0$ and $t_0 > 0$ such that

$$\operatorname{vol} B_t(p)\backslash B_{r_1}(p) \geq Ae^{h_g t} \tag{5.7}$$

for all $t \geq t_0$. □

Lemma 5.6. *Let (M,g) be a compact Riemannian manifold of hyperbolic type without conjugate points, X its universal Riemannian covering, and Γ the group of deck transformations of X. Let g_0 be a metric of negative curvature on M and X_0 the universal Riemannian covering of (M,g_0). Let $\eta \in \Gamma$, let $c : \mathbb{R} \to X_0$ be a g_0-axis of η, and let $p_0 = c(0)$. Then there exist constants $r,k > 0$ and neighbourhoods $\mathcal{U}$ of $c_0(-\infty)$ and $\mathcal{V}$ of $c_0(+\infty)$ in $X_0 \cup X_0(\infty)$ such that*

$$\#\{\gamma \in \Gamma_{t+r}^0(p_0),\ \gamma(\mathcal{V}) \cap \mathcal{U} = \varnothing\} \geq \frac{1}{4}\Gamma_t^k(p_0), \tag{5.8}$$

where

$$\Gamma_t^k(p) := \{\gamma \in \Gamma,\ k < d(p,\gamma(p)) \leq t\}. \tag{5.9}$$

Proof. Using Morse lemma and [16, Lemma 5.6], there exist $\beta \in \Gamma$ and neighbourhoods $\mathcal{U}$ of $c_0(-\infty)$ and $\mathcal{V}$ of $c_0(+\infty)$ such that

$$\{\beta c(-\infty), \beta c(+\infty)\} \cap \{c(-\infty), c(+\infty)\} = \varnothing. \tag{5.10}$$

Then using Morse lemma, we find neighbourhoods $\mathcal{U}$ of $c(-\infty)$ and $\mathcal{V}$ of $c(+\infty)$ such that

(1) $(\overline{\beta(\mathcal{U})} \cap \overline{\beta(\mathcal{V})}) \cap (\overline{\mathcal{U}} \cap \overline{\mathcal{V}}) = \varnothing$,
(2) there is a constant $L > 0$ such that for all $x \in \mathcal{U}$ and $y \in \mathcal{V}$, there is a g-geodesic h connecting x and y satisfying $d(h, p_0) \leq L$.

For $t \in \mathbb{R}$, let

$$\begin{aligned} A(\mathcal{U},\mathcal{V},t) &= \{\gamma \in \Gamma_t^0(p_0),\ \gamma(\mathcal{V}) \cap \mathcal{U} = \varnothing\}, \\ A(t) &= A(\mathcal{U},\mathcal{V},t) \cup A(\mathcal{V},\mathcal{U},t) \cup A(\mathcal{U}',\mathcal{V}',t) \cup A(\mathcal{V}',\mathcal{U}',t). \end{aligned} \tag{5.11}$$

Using Morse lemma and the triangle inequality, we prove that

$$\#A(t) = 4\#\{\gamma \in \Gamma^0_{t+r}(p_0),\ \gamma(\mathcal{V}) \cap \mathcal{U} = \varnothing\}. \tag{5.12}$$

Moreover, there is a constant $k > 0$ such that

$$A(t) \subset \Gamma^0_t(p_0) \backslash \Gamma^0_k(p_0) = \Gamma^k_t(p_0). \tag{5.13}$$

□

Lemma 5.7. *Let (M,g), (M,g_0), X, X_0, Γ, η, c, and p_0 be as in Lemma 5.6. Then there exist $n \in \mathbb{N}$, neighbourhoods $\mathcal{U}$ of $c_0(-\infty)$ and $\mathcal{V}$ of $c_0(+\infty)$ in $X_0 \cup X_0(\infty)$ and some constants ρ, $a > 0$ such that the endpoints of each element*

$$\beta \in \mathcal{D}(t) := \{\eta^n \gamma \eta^n,\ \gamma(\mathcal{V}) \cap \mathcal{U} = \varnothing,\ \gamma \in \Gamma^0_t(p_0)\} \tag{5.14}$$

belong to $\mathcal{U}$, respectively, $\mathcal{V}$ and $l(\beta) \le \rho + t$.

Proof. The fact that $c_0(-\infty) \neq c_0(+\infty)$ implies the existence of neighbourhoods $\mathcal{U}$ of $c_0(-\infty)$ and $\mathcal{V}$ of $c_0(+\infty)$ and $n \in \mathbb{N}$ such that

$$\begin{gathered} \eta^n(\overline{X} \setminus \mathcal{U}) \subset \mathcal{V}, \qquad \eta^{-n}(\overline{X} \setminus \mathcal{V}) \subset \mathcal{U}, \\ \overline{\mathcal{V}} \subset \overline{X} \setminus \mathcal{U}, \qquad \overline{\mathcal{U}} \subset \overline{X} \setminus \mathcal{V}. \end{gathered} \tag{5.15}$$

Let $\gamma \in \Gamma$ such that $\gamma(\mathcal{V}) \cap \mathcal{U} = \varnothing$ and $d(h,\gamma(p_0)) \le t$. We have

$$\begin{gathered} \eta^n \gamma \eta^n(\overline{\mathcal{V}}) \subset \mathcal{V}, \qquad \eta^{-n}\gamma^{-1}\eta^{-n}(\overline{\mathcal{U}}) \subset \mathcal{U}, \\ d(p_0, \eta^n \gamma \eta^n(p_0)) \le \rho + t. \end{gathered} \tag{5.16}$$

Finally, using [16, Lemma 5.6] we obtain the result. □

Theorem 5.8. *Let (M,g) be a compact Riemannian manifold of hyperbolic type without conjugate points and let X be its universal Riemannian covering. Let $\mathcal{P}(t)$ be the number of equivalence classes of closed geodesics of M of period less than or equal to t. Then there exist constant $b > 1$ and $t_0 > 0$ such that*

$$\frac{1}{b}\frac{e^{h_g t}}{t} \le \mathcal{P}(t) \le b e^{h_g t} \tag{5.17}$$

for all $t > t_0$, where h_g is the volume entropy of X.

Proof. Let $\mathcal{D}(t)$ be as in Lemma 5.7. If $\beta \in \mathcal{D}(t)$, we have $d(p_0,\beta(p_0)) \le \rho + t$ for some constant $\rho > 0$. Then, $l([\beta]) \le \rho + t$. Hence,

$$\mathcal{P}(t+\rho) \ge \#\{\gamma \in \Gamma,\ \gamma \in \mathcal{D}(t)\} \ge \frac{\#\mathcal{D}(t)}{\max_{\gamma \in \mathcal{D}(t)} \#[\gamma]}. \tag{5.18}$$

Finally, using Lemma 5.6, there exist constants $r,s>0$ such that

$$\mathcal{P}(t) \geq \frac{1}{4a(t-\rho)} \#\Gamma^{s}_{t-r-\rho}(p) \tag{5.19}$$

for some constant $a>1$. □

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Jean-Pierre Ezin: Institut de Mathématiques et de Sciences Physiques (IMSP), Université d'Abomey-Calavi, BP 613 Porto-Novo, Benin
E-mail address: jp.ezin@imsp-uac.org

Carlos Ogouyandjou: Institut de Mathématiques et de Sciences Physiques (IMSP), Université d'Abomey-Calavi, BP 613 Porto-Novo, Benin
Current address: Institut Universitaire de Technologie, Université de Parakou, République du Bénin
E-mail address: ogouyandjou@imsp-uac.org