A NOTE INVOLVING p-VALENTLY BAZILEVIĆ FUNCTIONS

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A theorem involving p-valently Bazilević functions is considered and then its certain consequences are given.

1. Introduction and definitions

Let $\mathcal{A}_n(p)$ be the class of normalized functions of the form

$$f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (n, p \in \mathbb{N} = \{1,2,3,\dots\}), \tag{1.1}$$

which are analytic and p-valent in the unit disc $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in \mathcal{A}_n(p)$ is said to be in the class $\mathcal{S}_n(p,\alpha)$ if it satisfies the inequality

$$\Re e\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leq \alpha < p,\ p \in \mathbb{N},\ z \in \mathcal{U}). \tag{1.2}$$

Also a function $f \in \mathcal{A}_n(p)$ is said to be a p-valently Bazilević function of type β ($\beta \geq 0$) and order γ ($0 \leq \gamma < p;\ p \in \mathbb{N}$) if there exists a function g belonging to the class $\mathcal{S}_n(p) := \mathcal{S}_n(p,0)$ such that

$$\Re e\left\{\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}\right\} > \gamma \quad (z \in \mathcal{U}). \tag{1.3}$$

We denote the class of all such functions by $\mathcal{B}_n(p,\beta,\gamma)$. In particular, when $\beta = 1$, a function $f \in \mathcal{K}_n(p,\gamma) := \mathcal{B}_n(p,1,\gamma)$ is said to be p-valently close-to-convex of order γ in $\mathcal{U}$. Moreover, $\mathcal{B}_n(p,0,\gamma) =: \mathcal{S}_n(p,\gamma)$ when $\beta = 0$.

2. Main results and their consequences

We begin with the following lemma due to Jack [2].

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Lemma 2.1. *Let $\omega(z)$ be nonconstant and regular in $\mathcal{U}$ with $\omega(0)=0$. If $|\omega(z)|$ attains its maximum value on the circle $|z|=r$ $(0<r<1)$ at the point z_0, then $z_0\omega'(z_0)=c\omega(z_0)$, where $c\geq 1$.*

With the aid of the above lemma, we prove the following result.

Theorem 2.2. *Let $f\in\mathcal{A}_n(p)$, $w\in\mathbb{C}\setminus\{0\}$, $\beta\geq 0$, $0\leq\alpha<p$, $p\in\mathbb{N}$, $z\in\mathcal{U}$, and also let the function $\mathcal{H}$ be defined by*

$$\mathcal{H}(z)=\left(\frac{zf'(z)}{zf'(z)-p[f(z)]^{1-\beta}[g(z)]^{\beta}}\right)\cdot\left(1+\frac{zf''(z)}{f'(z)}-(1-\beta)\frac{zf'(z)}{f(z)}-\beta\frac{zg'(z)}{g(z)}\right), \tag{2.1}$$

where $g\in\mathcal{S}_n(p)$. If $\mathcal{H}(z)$ satisfies one of the following conditions:

$$\Re e\{\mathcal{H}(z)\}\begin{cases}<|w|^{-2}\Re e\{w\} & \text{when } \Re e\{w\}>0,\\ \neq 0 & \text{when } \Re e\{w\}=0,\\ >|w|^{-2}\Re e\{w\} & \text{when } \Re e\{w\}<0,\end{cases} \tag{2.2}$$

or

$$\Im m\{\mathcal{H}(z)\}\begin{cases}<|w|^{-2}\Im m\{\overline{w}\} & \text{when } \Im m\{\overline{w}\}>0,\\ \neq 0 & \text{when } \Im m\{\overline{w}\}=0,\\ >|w|^{-2}\Im m\{\overline{w}\} & \text{when } \Im m\{\overline{w}\}<0,\end{cases} \tag{2.3}$$

then

$$\left|\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}-p\right)^{w}\right|<p-\alpha, \tag{2.4}$$

where the value of complex power in (2.4) is taken to be as its principal value.

Proof. We define the function Ω by

$$\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}-p\right)^{w}=(p-\alpha)\Omega(z), \tag{2.5}$$

where $\beta\geq 0$, $w\in\mathbb{C}\setminus\{0\}$, $0\leq\alpha<p$, $p\in\mathbb{N}$, $z\in\mathcal{U}$, $f\in\mathcal{A}_n(p)$, and $g\in\mathcal{S}_n(p)$.

We see clearly that the function Ω is regular in $\mathcal{U}$ and $\Omega(0)=0$. Making use of the logarithmic differentiation of both sides of (2.5) with respect to the known complex variable z, and if we make use of equality (2.5) once again, then we find that

$$wz\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}-p\right)^{-1}\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}-p\right)'=\frac{z\Omega'(z)}{\Omega(z)}, \tag{2.6}$$

which yields

$$\mathcal{H}(z) := \frac{\overline{w}}{|w|^2}\frac{z\Omega'(z)}{\Omega(z)} \quad (w \in \mathbb{C} \setminus \{0\};\ z \in \mathcal{U}). \tag{2.7}$$

Assume that there exists a point $z_0 \in \mathcal{U}$ such that

$$\max_{|z| \le |z_0|} |\Omega(z)| = |\Omega(z_0)| = 1 \quad (z \in \mathcal{U}). \tag{2.8}$$

Applying Lemma 2.1, we can then write

$$z_0\Omega'(z_0) = c\Omega(z_0) \quad (c \ge 1). \tag{2.9}$$

Then (2.7) yields

$$\Re e\{\mathcal{H}(z_0)\} = \Re e\left\{\frac{\overline{w}}{|w|^2}\frac{z_0\Omega'(z_0)}{\Omega(z_0)}\right\} = \Re e\{c\,\overline{w}\,|w|^{-2}\}, \tag{2.10}$$

so that

$$\Re e\{\mathcal{H}(z_0)\} = \frac{c}{|w|^2}\Re e\{\overline{w}\}\begin{cases} \ge |w|^{-2}\Re e\{w\} & \text{if } \Re e\{w\} > 0, \\ = 0 & \text{if } \Re e\{w\} = 0, \\ \le |w|^{-2}\Re e\{w\} & \text{if } \Re e\{w\} < 0, \end{cases} \tag{2.11}$$

$$\Im m\{\mathcal{H}(z_0)\} = \frac{c}{|w|^2}\Im m\{\overline{w}\}\begin{cases} \ge |w|^{-2}\Im m\{\overline{w}\} & \text{if } \Im m\{\overline{w}\} > 0, \\ = 0 & \text{if } \Im m\{\overline{w}\} = 0, \\ \le |w|^{-2}\Im m\{\overline{w}\} & \text{if } \Im m\{\overline{w}\} < 0. \end{cases} \tag{2.12}$$

But the inequalities in (2.11) and (2.12) contradict, respectively, the inequalities in (2.2) and (2.3). Hence, we conclude that $|\Omega(z)| < 1$ for all $z \in \mathcal{U}$. Consequently, it follows from (2.5) that

$$\left|\left(\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}} - p\right)^{w}\right| = (p-\alpha)|\Omega(z)| < p - \alpha. \tag{2.13}$$

Therefore, the desired proof is completed. □

This theorem has many interesting and important consequences in analytic function theory and geometric function theory. We give some of these with their corresponding geometric properties.

First, if we choose the value of the parameter w as a real number with $w := \delta \in \mathbb{R} \setminus \{0\}$ in the theorem, then we obtain the following corollary.

Corollary 2.3. *Let $f \in \mathcal{A}_n(p)$, $\delta \in \mathbb{R} \setminus \{0\}$, $\beta \ge 0$, $0 \le \alpha < p$, $p \in \mathbb{N}$, $z \in \mathcal{U}$, and let the function $\mathcal{H}$ be defined by (2.1). Also, if $\mathcal{H}$ satisfies the following conditions:*

$$\Re e\{\mathcal{H}(z)\}\begin{cases} < |\delta|^{-2} & \text{when } \delta > 0, \\ > |\delta|^{-2} & \text{when } \delta < 0, \end{cases} \tag{2.14}$$

then

$$\Re e\left\{\frac{zf'(z)}{[f(z)]^{1-\beta}[g(z)]^{\beta}}\right\} > p-(p-\alpha)^{1/\delta}. \tag{2.15}$$

Putting $w = 1$ in the theorem, we get the following corollary.

Corollary 2.4. *Let $f \in \mathcal{A}_n(p)$, $g \in \mathcal{S}_n(p)$, $\beta \geq 0$, $0 \leq \alpha < p$, $p \in \mathbb{N}$, $z \in \mathcal{U}$, and let the function $\mathcal{H}$ be defined by (2.1). If $\mathcal{H}(z)$ satisfies one of the following conditions:*

$$\Re e\{\mathcal{H}(z)\} < 1 \quad or \quad \Im m\{\mathcal{H}(z)\} \neq 0, \tag{2.16}$$

then $f \in \mathcal{B}_n(p,\beta,\alpha)$, that is, f is a p-valently Bazilević function of type β and order α in $\mathcal{U}$.

Setting $w = 1$ and $\beta = 0$ in the theorem, we have the following corollary.

Corollary 2.5. *Let $f \in \mathcal{A}_n(p)$, $0 \leq \alpha < p$, $p \in \mathbb{N}$, $z \in \mathcal{U}$, and let the function $\mathcal{G}$ be defined by*

$$\mathcal{G}(z) = \left(\frac{zf'(z)}{zf'(z)-pf(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right). \tag{2.17}$$

If $\mathcal{G}(z)$ satisfies one of the following conditions:

$$\Re e\{\mathcal{G}(z)\} < 1 \quad or \quad \Im m\{\mathcal{G}(z)\} \neq 0, \tag{2.18}$$

then $f \in \mathcal{S}_n(p,\alpha)$, that is, f is p-valently starlike of order α in $\mathcal{U}$.

By taking $w = 1$ and $\beta = 1$ in the theorem, we obtain the following corollary.

Corollary 2.6. *Let $f \in \mathcal{A}_n(p)$, $g \in \mathcal{S}_n(p)$, $0 \leq \alpha < p$, $p \in \mathbb{N}$, $z \in \mathcal{U}$, and let the function $\mathcal{F}$ be defined by*

$$\mathcal{F}(z) = \left(\frac{zf'(z)}{zf'(z)-pg(z)}\right)\left(1+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\right). \tag{2.19}$$

If $\mathcal{F}(z)$ satisfies one of the following conditions:

$$\Re e\{\mathcal{F}(z)\} < 1 \quad or \quad \Im m\{\mathcal{F}(z)\} \neq 0, \tag{2.20}$$

then $f \in \mathcal{K}_n(p,\alpha)$, that is, f is p-valently close-to-convex of order α in $\mathcal{U}$.

Lastly, if we take $p = 1$ in Corollaries 2.4, 2.5, and 2.6, then we easily obtain the three important results involving Bazilević functions of type β ($\beta \geq 0$) and order α ($0 \leq \alpha < 1$) in $\mathcal{U}$, starlike functions of order α ($0 \leq \alpha < 1$) in $\mathcal{U}$, and close-to-convex functions of order α ($0 \leq \alpha < 1$) in $\mathcal{U}$, respectively, (see, e.g., [1, 3, 4, 5]).

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