WITT GROUP OF HERMITIAN FORMS OVER A NONCOMMUTATIVE DISCRETE VALUATION RING

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We investigate Hermitian forms on finitely generated torsion modules over a *noncommutative* discrete valuation ring. We also give some results for lattices, which still are satisfied even if the base ring is not commutative. Moreover, for a *noncommutative discrete-valued division algebra D* with valuation ring R and residual division algebra $\bar{D}$, we prove that $W(\bar{D}) \cong WT(R)$, where $WT(R)$ denotes the Witt group of regular *Hermitian* forms on finitely generated torsion R-modules.

1. Introduction

There has been considerable interest in Witt group of bilinear forms over commutative rings. In particular, several authors are interested in commutative discrete valuation rings and it was clear that a similar theory could be developed in noncommutative settings. Therefore, most of the topics of this paper are generalizations of known results on Hermitian forms and Witt group over commutative discrete valuation rings. In Theorem 2.7, we present useful results on lattices and Hermitian forms which are a straightforward generalization of the commutative situation. For a commutative discrete valuation ring R with residual field k, it is well known that $W(k) \cong WT(R)$, where $WT(R)$ denotes the Witt group of symmetric regular bilinear forms on finitely generated torsion R-modules (see, e.g., [4, Theorem 2.1, page 207]). In Theorem 3.3, we extend this result to noncommutative case. More precisely, for a discrete-valued division algebra D with valuation ring R and residual division algebra $\bar{D}$, we prove that $W(\bar{D}) \cong WT(R)$. To conclude this paper, Theorem 3.6 explores an exact sequence which is known only for Dedekind domain.

Let D be a noncommutative finite-dimensional central division algebra over a field F and let $\star$ be an involution of the *first kind* on D, that is, $\star$ leaves F elementwise invariant. For clarity, it is interesting to elucidate some of the terminology to be used. If v is a discrete valuation on D with valuation ring R and if $\mathcal{M}$ denotes the maximal ideal of R, then $\bar{D} := R/\mathcal{M}$ is called the residual division algebra. Since every element $x \in D$ satisfies $x \in R$ or $x^{-1} \in R$, then every ideal of R is a two-sided ideal. Indeed, if π is a prime element of R, that is, a generator of $\mathcal{M}$, and if I is a one-sided ideal of R, then $I = \pi^n R = R\pi^n$,

where $n = \inf\{v(x)/x \in I\}$. In particular, R is then a *principal ideal domain*. As $aR = Ra$ for all $a \in R$, then elementary divisor theorem holds even if R is not commutative. In all that follows, $\mathcal{U}(R)$ will stand for the set of all invertible elements of R. Recall that if V is a D-vector space with dimension n, then a submodule L of V is called an R-*lattice* (or simply lattice) on V if $L = \oplus_{i=1}^{n} Re_i$ for some basis $(e_1,\ldots,e_n)$ of V. Moreover, if h is a Hermitian form on V, then the *dual* lattice of L is defined by $L^{\#} := \{x \in V/h(x,L) \subset R\}$. Finally, an R-module $J \subset D$ is said to be a fractional ideal if there exists $a \in R$ such that $aJ \subset R$. Furthermore, if $IJ = JI = R$ for some fractional ideal I, then J is invertible with inverse I. In particular, for $a \in R$, aR is an invertible fractional ideal with inverse $a^{-1}R = Ra^{-1}$. Moreover, if we set $[I : R] := \{x \in D/xI \subset R\}$, then $I^{-1} = [I : R]$.

2. Hermitian forms and lattices

In this section, we continue with the same notations as in the preceding section. For a finitely generated torsion left R-module M, its dual $M^* := \operatorname{Hom}_R(M, T = D/R)$ is endowed with a natural right R-module structure defined by $f \cdot r(m) = f(m)r$ for $f \in M^*$, $r \in R$, and $m \in M$.

Lemma 2.1. *The restriction of $\star$ to R is an involution. Moreover, $\star$ induces an involution:*
  (1) *$\bar{\star}$ on $\bar{D}$ defined by $\bar{x}^{\bar{\star}} := \bar{x^{\star}}$;*
  (2) *$\star^{\cdot}$ on D/R defined by $(d + R)^{\star^{\cdot}} := d^{\star} + R$.*

Proof. From theory of valued division algebras, it is well known that the restriction of v to the field F extends in a unique way to D, moreover, this extension is v. To prove the restriction of $\star$ to R is an involution, it suffices to show that $v(x) = v(x^{\star})$ for all $x \in R$. For $x \in D$, setting $\omega(x) := v(x^{\star})$, then it is straightforward to check that ω is a valuation on D. Since $\star$ is an involution of the first kind, we deduce that $\omega/_F = v/_F$, hence $\omega = v$ so that $v(x) = v(x^*)$ for all $x \in R$. Consequently, (1) and (2) hold. $\square$

Since $\star$ is an involution on R, then M^* can again be regarded as a left module if one defines $r \cdot f(m) = f(m)r^*$.

Lemma 2.2. *$M \cong M^*$, and M is canonically isomorphic to M^{**}.*

Proof. For the nontrivial part, as R is in particular a left principal ring satisfying $aR = Ra$ for all $a \in R$, there exists an ascending sequence $(I_1,\ldots,I_n)$ of ideals of R such that $M \cong \oplus_{i=1}^{n} R/I_i$. Since $(M \oplus N)^* = M^* \oplus N^*$, it then suffices to prove the lemma for $M = R/I$. On the other hand, for each homomorphism $f \in M^*$, we have $f(\bar{r}) = rf(\bar{1})$ for all $r \in R$. Setting $\bar{\beta} = f(\bar{1})$, we then get $0 = f(\bar{x}) = x\beta + R$ for all $x \in I$, which yields $x\beta \in R$ so that $\beta \in \{d \in D/I d \subset R\} = \{d \in D/dI \subset R\} = I^{-1}$. Thus $M^* \cong I^{-1}/R$. From R principal, we deduce $I = aR = Ra$ for some $a \in R$, therefore $M^* \cong I^{-1}/R = a^{-1}R/R \cong R/aR = M$. $\square$

A *Hermitian form* on M is a biadditive map $h : M \times M \to D/R$ such that, for $x, y \in M$ and $r, \beta \in R : h(rx, \beta y) = rh(x,y)\beta^*$, $h(x,y) = h(y,x)^*$. Furthermore, h is called *regular* if $\hat{h} : M \to M^*$ defined by $\hat{h}(x) = h(x,\cdot)$ is an isomorphism of R-modules. For a submodule N of M, we define $N^{\perp} = \{x \in M/h(x,n) = 0 \ \forall n \in N\}$. In view of regularity of h, we obtain $(N^{\perp})^{\perp} = N$. Recall that (M,h) is called *weakly metabolic* if there exists a submodule N of M such that $N = N^{\perp}$.

In the sequel, $WT(R)$ denotes the *Witt group of regular Hermitian forms on finitely generated torsion R-modules*, where weakly metabolic forms are identified with 0.

PROPOSITION 2.3. *Let N be a submodule of (M,h) with $N \subset N^\perp$. Then h induces a regular Hermitian form $\bar{h}$ on $N^\perp/N$ such that $(M,h) \oplus (N^\perp/N, -\bar{h})$ is weakly metabolic.*

Proof. Consider the map $\bar{h} : N^\perp/N \times N^\perp/N \to D/R$ defined by $\bar{h}(\bar{x},\bar{y}) = h(x,y)$. If x, x', y, y' are elements in $N^\perp$ such that $(\bar{x},\bar{y}) = (\bar{x}',\bar{y}')$, then $x = x' + n$ and $y = y' + n'$, where $n, n' \in N$. Using $N \subset N^\perp$, we then get $\bar{h}(\bar{x},\bar{y}) = h(x' + n, y' + n') = h(x',y')$ so that $\bar{h}$ is a well-defined map. Moreover, for $x, y \in N^\perp$, we have $\bar{h}(\alpha\bar{x}, \beta\bar{y}) = \bar{h}(\overline{\alpha x}, \overline{\beta y}) = \alpha h(x,y)\beta^* = \alpha\bar{h}(\bar{x},\bar{y})\beta^*$ for all $\alpha,\beta \in R$. For regularity of $\bar{h}$, if $x \in N^\perp$ is such that $\bar{h}(\bar{x},\bar{y}) = 0$ for all $y \in N^\perp$, then $h(x,y) = 0$ for all $y \in N^\perp$. Consequently, $x \in (N^\perp)^\perp = N$ and then $\bar{x} = 0$. Let $f \in (N^\perp/N)^*$, then $f \circ \pi \in (N^\perp)^*$, where $\pi : N^\perp \to N^\perp/N$ is the canonical projection. The surjectivity of $M \xrightarrow{\hat{h}} M^* \to (N^\perp)^*$ assures existence of $x \in M$ such that $f(\bar{y}) = f(y + N) = h(x,y)$ for all $y \in N^\perp$. In particular, for $y \in N$, $f(\bar{y}) = 0 = h(x,y)$, thus $x \in N^\perp$ and $f = \hat{\bar{h}}(x)$, which proves the regularity of $\bar{h}$. Finally, the submodule $\Delta = \{(x,\bar{x})/x \in N^\perp\}$ of $M \oplus N^\perp/N$ satisfies $\Delta^\perp = \Delta$. Thus $(M,h) \oplus (N^\perp/N, -\bar{h})$ is weakly metabolic. $\square$

In the classical theory of regular symmetric bilinear forms over a commutative field F, orthogonal basis plays a key role in studying the Witt group $W(F)$. Furthermore, existence of orthogonal basis is proved in noncommutative case (see [1, Theorem 3, page 153]). In our situation, we can give a short proof for this result as follows.

PROPOSITION 2.4. *Every regular Hermitian space (V,h) has an orthogonal basis.*

Proof. Suppose $h \neq 0$, otherwise each basis is clearly orthogonal. There exist $x, y \in V$ such that $h(x,y) = 1$. If h satisfies $h(z,z) = 0$ for all $z \in V$, then $h(x + \alpha y, x + \alpha y) = 0 = \alpha + \alpha^*$ so that $\alpha^* = -\alpha$ for all $\alpha \in D$, contradicting $1^* = 1$. Thus there exists $z \in V$ for which $h(z,z) \neq 0$ and $h(z,z)$ is then invertible in D. The regularity of Dz yields $V = Dz \perp (Dz)^\perp$. Reasoning by induction, this completes the proof. $\square$

Remark 2.5. As an application of the previous result, every regular Hermitian space contains a lattice L such that $L \subset L^\#$. Indeed, let $L = \oplus_{i=1}^n Re_i$ be a lattice on (V,h) and let $(e_i)_{1 \le i \le n}$ be an orthogonal basis of V. Hence, $h(e_i,e_j) = 0$ for $i \neq j$ and $h(e_i,e_i) = \lambda_i \in D$ and either $\lambda_i \in R$ or $\lambda_i^{-1} \in R$. Assume $\lambda_i^{-1} \in R$ for some $1 \le i \le n$, then $(e_1,\dots,\lambda_i^{-1}e_i,\dots,e_n)$ is an orthogonal basis of V satisfying $h(e_j,e_j) = \lambda_j \in R$ for $j \neq i$ and $h(\lambda_i^{-1}e_i,\lambda_i^{-1}e_i) = (\lambda_i^*)^{-1} \in R$, accordingly, $L \subset L^\#$. This remark will be used in the sequel.

The following lemma describes relationship between linear R-independence and linear D-independence for elements in V.

LEMMA 2.6. *Let $e_1,\dots,e_n$ be elements in V, then $e_1,\dots,e_n$ are linearly R-independent if and only if $e_1,\dots,e_n$ are linearly D-independent.*

Proof. For the nontrivial sense, suppose $e_1,\dots,e_n$ are not linearly D-independent, then there exist $d_1,\dots,d_m \in D - \{0\}$ $(m \le n)$ such that $\sum_{i=1}^m d_i e_i = 0$.

Let $v(d_s) = \inf\{v(d_i)/1 \le i \le n\}$, then for all $1 \le i \le m$, we have $d_i = r_i d_s$ for some $r_i \in R - \{0\}$. Accordingly, $\sum_{i=1}^m r_i d_s e_i = 0$ so that $\sum_{i=1}^m d_s^{-1} r_i d_s e_i = 0$ proving our proposition, since $d_s^{-1} r_i d_s \in R - \{0\}$. $\square$

In the following theorem, we collect some useful results on lattices. Our approach is based on commutative case with different proof due to the use of valuation.

THEOREM 2.7. *Let L_1 and L_2 be two lattices on V. Then*
  (1) *there exist $a, b \in R$ such that $aL_1 \subset L_2$ and $bL_2 \subset L_1$,*
  (2) *there exist a basis $(e_1, \ldots, e_n)$ of V and fractional ideals $I_1, \ldots, I_n$ such that $L_1 = \oplus_{i=1}^n Re_i$ and $L_2 = \oplus_{i=1}^n I_i e_i$,*
  (3) *$(L_1 \cap L_2)^{\#} = L_1^{\#} + L_2^{\#}$ and $(L_1 + L_2)^{\#} = L_1^{\#} \cap L_2^{\#}$,*
  (4) *if $L_1 \subset L_2$, then $L_2^{\#} \subset L_1^{\#}$.*

Proof. (1) Writing $L_1 = \oplus_{i=1}^n Re_i$ and $L_2 = \oplus_{i=1}^n Rf_i$ where both $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ are D-bases of V. We fix $1 \le s \le n$, there exist $d_{s_1}, \ldots, d_{s_n} \in D$ such that $f_s = \oplus_{j=1}^n d_{s_j} e_j$. Consider $v(d_{s_t}) = \inf\{v(d_{s_j})/1 \le j \le n\}$, then for all $1 \le j \le n$, we have $d_{s_j} = r_j d_{s_t}$, where $r_j \in R$. If $d_{s_t} \in R$, then $d_{s_j} \in R$ for all j, otherwise $d_{s_t}^{-1} \in R$ so that $d_{s_t}^{-1} d_{s_j} = d_{s_t}^{-1} r_j d_{s_t} \in R$ for all $j \ne t$ and $d_{s_t}^{-1} d_{s_t} = 1 \in R$ which proves existence of $r_s \in R$ such that $r_s d_{s_j} \in R$ for all $1 \le j \le n$. Hence, $r_s f_s \in L_1$ for all $1 \le s \le n$. Setting $b = \prod_{i=1}^n r_i \in R$, we then get $bRf_i = r_1 \cdots r_i \cdots r_n R r_i^{-1} r_i f_i \subset Rr_i f_i \subset L_1$ for all i, hence $bL_2 \subset L_1$. A similar reasoning yields $aL_1 \subset L_2$ for some element $a \in A$.

(2) Let a be a nonzero element in R such that $aL_2 \subset L_1$. As aL_2 is a lattice on V, then aL_2 is a finitely generated free R-module with rank $n = [V : D]$. Applying elementary divisor theorem to the submodule aL_2 of L_1, there exist an R-basis $(e_1, \ldots, e_n)$ of L_1 and elements $b_1, \ldots, b_n \in R - \{0\}$ such that $(b_1 e_1, \ldots, b_n e_n)$ is an R-basis of aL_2. Therefore, $L_1 = \oplus_{i=1}^n Re_i$ and $aL_2 = \oplus_{i=1}^n Rb_i e_i$. Consequently, $L_2 = \oplus_{i=1}^n I_i e_i$, where $I_i = a^{-1} R b_i = R a^{-1} b_i$ is a fractional ideal. Finally, Lemma 2.6 implies that $(e_1, \ldots, e_n)$ is a D-basis of V.

(3) and (4) are clear. $\qquad\square$

Remark 2.8. If $L = \oplus_{i=1}^n Re_i$ is a lattice on a regular Hermitian space, then $(L^{\#})^{\#} = L$. Furthermore, it is interesting to note that $L^{\#} = \oplus_{i=1}^n Re_i'$, where $(e_1', \ldots, e_n')$ is the dual basis of $(e_1, \ldots, e_n)$.

3. Main results

From [3, Corollary 1.4.4, page 396], it follows that every finitely generated torsion-free module over a principal ideal domain is a free module. The following proposition gives another proof for this result due to the use of valuation's arguments.

PROPOSITION 3.1. *Every finitely generated torsion-free module is a free module.*

Proof. Let $(m_1, \ldots, m_n)$ be a minimal set of generators of M with $\sum_{i=1}^n a_i m_i = 0$ and let $v(a_i) = \inf\{v(a_j)/1 \le j \le n\}$. For all j, we have $v(a_j) = v(a_i) + s_j = v(a_i \pi^{s_j})$, and thus $a_j = a_i \pi^{s_j} u_j$, where u_j is an invertible element of R. If $a_i = 0$, then $a_j = 0$ for all j which implies that $(m_1, \ldots, m_n)$ is free. Assume $a_i \ne 0$, as M is torsion-free, the fact that $a_i(\pi^{s_1} u_1 m_1 + \cdots + m_i + \pi^{s_n} u_n m_n) = 0$ implies $\pi^{s_1} u_1 m_1 + \cdots + m_i + \pi^{s_n} u_n m_n = 0$ so that $(m_1, \ldots, m_{i-1}, m_{i+1}, \ldots, m_n)$ generated M, contradicting minimality of $(m_1, \ldots, m_n)$. Accordingly, $(m_1, \ldots, m_n)$ is free. $\qquad\square$

PROPOSITION 3.2. *Assume $2 \in \mathcal{U}(R)$. Then every regular Hermitian module (M, h) has an orthogonal basis.*

Proof. Suppose existence of an element $x \in M$ such that $h(x,x) \in \mathcal{U}(R)$. It is easily seen that Rx is a regular submodule of M, hence $M = N \perp N^{\perp}$. Reasoning by induction, (M,h) admits an orthogonal basis $(e_1 = x, e_2, \ldots, e_n)$.

Now to prove existence of $x \in M$ with $h(x,x) \in \mathcal{U}(R)$, let $x, y \in M$ such that $h(x,y) = r \neq 0$. If $h(z,z) = 0$ for all $z \in M$, then $h(x + uy, x + uy) = 0$ so that $ru^* + ur^* = 0$ for all $u \in R$. In particular, for $u = 1$, we get $ru^* = ur$ for all $u \in R$. Consequently, for $u, v \in R$, we then have $uvr = r(uv)^* = rv^*u^* = vru^* = vur$ which yields $uv = vu$ since R is a domain and therefore contradicts the noncommutativity of R. Hence, M contains an element z satisfying $h(z,z) \neq 0$. Furthermore, we claim existence of $x \in M$ such that $h(x,x) \in \mathcal{U}(R)$. Indeed, otherwise $h(x,x) \in \mathcal{M}$ for all $x \in M$. Let $(e_1, \ldots, e_n)$ denote a basis of M over R and consider the map $\phi : M \rightarrow R$ defined by $\phi(e_i) = \delta_{i1}$. It is clear that $\phi \in M^*$ and the regularity of h implies $\phi = \hat{h}(x)$ with $x \in M$. From $1 = \phi(e_1) = h(x, e_1)$, we deduce $2 = h(x, e_1) + h(e_1, x) \in \mathcal{M}$ which completes the proof. $\qquad\square$

THEOREM 3.3. $WT(R) \cong W(\bar{D})$ *is an isomorphism which depends on the choice of the prime element π.*

Proof. (1) Since the action $r \cdot (d + R) = rd + R$ endows D/R with a left R-module structure, the fact that $(R, \star)$ is a ring with involution implies that D/R can be transformed into a right module via $(d + R)r = r^*d + R$. On the other hand, if M is a finite-dimensional $\bar{D}$-vector space, then M is a finite generated torsion R-module ($\pi \cdot m = 0$ for all $m \in M$, π the nonzero generator of $\mathcal{M}$). Now, consider a regular Hermitian form $h : M \times M \rightarrow \bar{D}$. For $x, y \in M$, $\pi^{-1}h(x,y) \in D/R$ and $\pi^{-1}h(x,y) = \pi^{-1}h(x,y)\pi^*(\pi^*)^{-1} = h(x,y)(\pi^*)^{-1}$. From $\pi^{-1}h(\alpha x, \beta y) = h(\alpha x, \beta y)(\pi^*)^{-1} = \alpha h(x, \beta y)(\pi^*)^{-1} = \alpha \pi^{-1}h(x, \beta y) = \alpha \pi^{-1}h(x,y)\beta^*$ and $\pi^{-1}h(y,x) = \pi^{-1}h(x,y)^* = h(x,y)^*(\pi^*)^{-1} = (\pi^{-1}h(x,y))^*$ for all $\alpha, \beta \in R$, it follows that $\pi^{-1}h$ is a Hermitian form. To prove regularity of $\pi^{-1}h$, suppose $\pi^{-1}h(x,y) = 0$ for all $y \in M$, then $h(x,y) \in \pi R$ in such a way that $x = 0$. Moreover, if $f : M \rightarrow D/R$, then $f(\pi x) = \pi f(x) = 0$ for all $x \in M$, and thus $\pi f(x) \in R$. From $M \xrightarrow{\pi f} R \xrightarrow{s} R/\pi R$, the regularity of h then assures existence of $x \in M$ such that $s \circ \pi f = \hat{h}(x)$, that is, $\pi f(y) + \pi R = h(x,y)$ for all $y \in M$. Therefore, $f(y) = \pi^{-1}h(x, \cdot)(y)$ proving that

$$\Psi : W(\bar{D}) \longrightarrow WT(R), \qquad (V, h) \longmapsto (V, \pi^{-1}h) \qquad (3.1)$$

is a well-defined map.

(2) Let M be a finitely generated torsion R-module and let $h : M \times M \rightarrow D/R$ be a regular Hermitian form. Hence, for $\alpha, \beta \in R$, $h(\alpha x, \beta y) = \alpha h(x,y)\beta^* = \beta \alpha h(x,y) = h(\alpha \beta x, y)$ for all $x, y \in M$. Then we have to distinguish two cases.

    (i) $\pi M = \{0\}$. It is readily verified that M is a left $\bar{D}$-vector space by the action $\bar{r} \cdot m = rm$. In view of $\pi h(\alpha x, \beta y) = h(\alpha x, \beta y)\pi^* = \alpha h(x, \beta y)\pi^* = \alpha \pi h(x, \beta y) = \alpha \pi h(x,y)\beta^*$, we deduce that $\pi h : M \times M \rightarrow \bar{D}$ is a Hermitian form. To prove regularity of πh, let $x \in M$ be such that $\pi h(x,y) = 0$ for all $y \in M$, then $h(x,y) \in R$ and the regularity of h yields $x = 0$. Furthermore, if $f : M \rightarrow R/\pi R$, then from $\pi^{-1}f(\alpha x) = \pi^{-1}f(\alpha x)\pi^*(\pi^*)^{-1} = (\pi^{-1}f(\alpha x)\pi^*)(\pi^*)^{-1} = f(\alpha x)(\pi^*)^{-1} = \alpha f(x)(\pi^*)^{-1} = \alpha \pi \pi^{-1}f(x)(\pi^*)^{-1} = \alpha \pi (\pi^{-1}f(x)(\pi^*)^{-1}) = \alpha \pi (\pi^2 f(x)) = \alpha \pi^{-1}f(x)$, we get $\pi^{-1}f \in \mathrm{Hom}_R(M, D/R)$ so that $\pi^{-1}f = \hat{h}(x)$ for some $x \in M$. Thus $f = \widehat{\pi h}(x)$ proving regularity of πh in this case.

(ii) $\pi^k M = \{0\}$ and $\pi^{k-1}M = M_1 \neq \{0\}$ for some $k \geq 2$. The fact that $h(\alpha x, \beta y) = h(\beta \alpha x, y)$ then yields $h(\pi^{k-1}x, \pi^{k-1}y) = h(\pi^{2(k-1)}x, y) = 0$, and therefore $M_1 \subset M_1^\perp$. Setting $\overline{M_1} = M_1^\perp/M_1$ and applying Proposition 2.3, we get $(M,h) = (\overline{M_1}, \bar{h})$ in $WT(R)$. Let $h_1 = \pi h$ and $M_2 = \pi^{k-1}M_1$, then clearly $M_2 \subset M_2^\perp$. Furthermore, $\pi^{k-1}\overline{M_1} = \{0\}$ because $\pi^{k-1}M_1^\perp \subset \pi^{k-1}M = M_1$. We then construct $(\overline{M_r}, \bar{h})$ such that $(M,h) = (\overline{M_r}, \bar{h})$ in $WT(R)$ and $\pi\overline{M_r} = \{0\}$. Therefore, $(\overline{M_r}, \pi\bar{h}) \in W(\bar{D})$. Consequently,

$$\Phi : WT(R) \longrightarrow W(\bar{D}),$$
$$(M,h) \longmapsto (\overline{M_r}, \pi\bar{h}) \tag{3.2}$$

is a well-defined map. Moreover, $\Phi \circ \Psi = \mathrm{Id}_{W(\bar{D})}$ and $\Psi \circ \Phi = \mathrm{Id}_{WT(R)}$.   □

Let L be a finitely generated projective R-module with rank n. It is clear that $L \otimes_R D$ is a right D-vector space with dimension n. Moreover, every Hermitian form $h : L \times L \to R$ induces a Hermitian form h_L on $L \otimes_R D$ defined by $h_L(x \otimes d, y \otimes d') = d^*h(x,y)d'$.

LEMMA 3.4. *With the notations above, if h is regular, then h_L is regular too.*

Proof. Let $\{e_i\}_{1 \leq i \leq n}$ be an R-basis of L and let $\{e_i'\}_{1 \leq i \leq n}$ denote its dual basis defined by $h(e_i, e_j') = \delta_{ij}$. It is easy to verify that $\{e_i \otimes 1\}_{1 \leq i \leq n}$ is a D-basis of $L \otimes D$ with dual basis $\{e_i' \otimes 1\}_{1 \leq i \leq n}$. Let $x = \sum_{i=1}^n e_i \otimes d_i \in L \otimes D$ such that $h_L(x,y) = 0$ for all $y \in L \otimes D$, hence $h_L(x, e_j' \otimes 1) = 0 = \sum_{i=1}^n d_i^* \delta_{ij} = d_j^*$ for all j so that $x = 0$, thus h_L is regular.   □

LEMMA 3.5. *Let (V,h) be a Hermitian form and let L be a lattice on V with $L \subset L^\#$. Then $(L^\#/L, \bar{h}) \in WT(R)$ does not depend on the choice of the lattice L.*

Proof. Let L_0 be a lattice on V such that $L_0 \subset L$. Using Theorem 2.7, we obtain $L_0 \subset L \subset L^\# \subset L_0^\#$. Moreover, it is clear that L/L_0 is a torsion submodule of $L_0^\#/L_0$ which satisfies $(L/L_0)^\perp = \{x + L_0, \ x \in L_0^\#/h(x,y) \in R$ for all $y \in L\} = L_0^\# \cap L^\#/L_0 = L^\#/L_0$ so that $L/L_0 \subset (L/L_0)^\perp$. From Proposition 2.3, we then get $(L_0^\#/L, h_{L_0}) = ((L^\#/L_0)/(L/L_0), \bar{h}_{L_0}) = (L^\#/L, h_L)$ in $WT(R)$. Now, consider two lattices L_1 and L_2 on V and set $L_0 = L_1 \cap L_2$, then $L_0 \subset L_1$ and $L_0 \subset L_2$. Moreover, $(L_0^\#/L_0, h_{L_0}) = (L_1^\#/L_1, h_{L_1}) = (L_2^\#/L_2, h_{L_2})$.   □

A regular Hermitian R-module (M,h) is called *metabolic* if there exists a direct summand N of M satisfying $N^\perp = N$. However, it suffices that M admits a submodule N such that $N^\perp = N$ (see [2, Lemma 1.2, page 122]). In what follows, $W(R)$ denotes the *Witt group of regular Hermitian forms on finitely generated projective R-modules*, where metabolic modules are identified with 0.

Recall that if R is a Dedekind domain with quotient field K, then the sequence

$$0 \longrightarrow W(R) \xrightarrow{i} W(D) \xrightarrow{\delta} WT(R) \tag{3.3}$$

defined by $i(L,h) = (L \otimes K, h_K)$ and $\delta(V,h) = (L^\#/L, \bar{h})$ is exact. The aim of the following theorem is to extend this result to noncommutative discrete valuation ring.

Theorem 3.6. *With the hypothesis above, we have the following exact sequence:*

$$0 \longrightarrow W(R) \xrightarrow{i} W(D) \xrightarrow{\delta} WT(R),$$

$$(L,h) \longmapsto (L \otimes D, h_L), \tag{3.4}$$

$$(V,h) \longmapsto \left(\frac{L^{\#}}{L}, \bar{h}\right).$$

Proof. Concerning the injectivity of i, suppose $(L \otimes D, h_L)$ is a hyperbolic space, then there is a D-subspace W of $V = L \otimes D$ such that $W = W^{\perp}$. If $\{e_1, \dots, e_{2n}\}$ denotes an R-basis of L, then $\{e_i \otimes 1\}_i$ is a D-basis of V and we can assume $(e_1 \otimes 1, \dots, e_n \otimes 1)$ is a D-basis of W. Setting $M = \oplus_{i=1}^n Re_i$, from $h_L(e_i \otimes 1, e_j \otimes 1) = 0$, we have $h(e_i, e_j) = 0$ for all i, j so that $M \subset M^{\perp}$. Reciprocally, let $x = \sum_{i=1}^{2n} r_i e_i \in M^{\perp}$, hence $0 = h(x, e_j) = \sum_{i=1}^{2n} r_i h(e_i, e_j) = h_L(\sum_{i=1}^{2n} e_i \otimes r_i, e_j \otimes 1)$ for all $1 \le j \le n$. As $\sum_{i=1}^n h_L(e_i \otimes r_i, e_j \otimes 1) = 0$, then $h_L(\sum_{i=n+1}^{2n} e_i \otimes r_i, e_j \otimes 1) = 0$ for all $1 \le j \le n$ in such a way that $\sum_{i=n+1}^{2n} e_i \otimes r_i \in W^{\perp} = W = \oplus_{i=1}^n (e_i \otimes 1)D$, therefore $\sum_{i=n+1}^{2n} e_i \otimes r_i = 0$ and thus $r_i = 0$ for all $n + 1 \le i \le 2n$. Hence, $x = \sum_{i=1}^n r_i e_i \in M$ and $M = M^{\perp}$. From [2, Lemma 1.2, page 122], it then follows that (L,h) is metabolic.

To prove $\operatorname{Im} i \subset \ker \delta$, observe that L is a lattice on $V = L \otimes D$ which satisfies $L = L^{\#}$. Indeed, let $(e_i)_{1 \le i \le n}$ be an R-basis of L with dual basis $(e_i')_{1 \le i \le n}$. If $x = \sum_i e_i \otimes d_i \in L^{\#}$, then $h(x, e_j' \otimes 1) = d_j \in R$ for all j. As $L \hookrightarrow L \otimes D$, then $L = L^{\#}$, and therefore $\delta \circ i(L,h) = 0$. Conversely, if $\delta(V,h) = 0$, then there exists a lattice L on V such that $(L^{\#}/L, \bar{h})$ is weakly metabolic. Hence, there is an R-module N with $L \subset N \subset L^{\#}$ and $N/L = (N/L)^{\perp}$. Then $(N/L)^{\perp} = N^{\#} \cap L^{\#}/L = N^{\#}/L$ so that $N = N^{\#}$. Thus $V = N \otimes_R D$ and $(V,h) = i(N, h/N)$. $\square$

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