A WEAK PERIODICITY CONDITION FOR RINGS

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A ring is called semi-weakly periodic if each element which is not in the center or the Jacobson radical can be written as the sum of a potent element and a nilpotent element. After discussing some basic properties of such rings, we investigate their commutativity behavior.

1. Introduction

An element x of the ring R is called *periodic* if there exist distinct positive integers m, n such that $x^m = x^n$; and x is *potent* if there exists $n > 1$ for which $x^n = x$. We denote the set of potent elements by P or $P(R)$, the set of nilpotent elements by N or $N(R)$, the center by Z or $Z(R)$, and the Jacobson radical by J or $J(R)$.

The ring R is called *periodic* if each of its elements is periodic, and R is called *weakly periodic* if $R = P + N$. It is easy to show that every periodic ring is weakly periodic, but whether the converse holds is apparently not known. It has long been known that periodic rings have nice commutativity behavior; in particular, Herstein [10] showed that if R is periodic and $N \subseteq Z$, then R is commutative—a result which extends easily to weakly periodic rings. Various generalized periodic and weakly periodic rings have been introduced in recent years, and their commutativity behavior has been explored [6, 7, 13, 14].

Define R to be *semi-weakly periodic* if $R \setminus (J \cup Z) \subseteq P + N$. Clearly the class of semi-weakly periodic rings is quite large; it contains all weakly periodic rings, all commutative rings, and all Jacobson radical rings. Our purpose is to point out some general properties of semi-weakly periodic rings and to investigate commutativity of such rings.

2. Preliminaries

We fix some more notation. If $x, y \in R$, the symbol $[x, y]$ denotes the commutator $xy - yx$; and if $S, T \subseteq R$, $[S, T]$ denotes the set $\{[s,t]/s \in S,\ t \in T\}$. For $x \in R$, the symbol $\langle x \rangle$ denotes the subring generated by x; and the symbols $C(R)$ and E stand for the commutator ideal and the set of idempotents of R. If $\wp$ is a ring property, a ring R having the property is called a $\wp$-*ring*.

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We also state some known results we require, the first of which is trivial.

Lemma 2.1. *If R is any ring and S is any proper additive subgroup of R, the centralizer of $R\backslash S$ is equal to $Z(R)$.*

Lemma 2.2 [9]. *If R is a ring such that for each $x \in R$, there exists an integer $n > 1$ such that $x^n - x \in Z$, then R is commutative. In particular, if $R = P \cup Z$, then R is commutative.*

Lemma 2.3. *If R has an ideal I such that I and R/I are both commutative, then N is an ideal and $C(R) \subseteq N$.*

Proof. Since R/I is commutative, $[x,y] \in I$ for all $x,y \in R$; and since I is commutative, R satisfies the polynomial identity $[[x,y],[z,w]] = 0$, which is not satisfied by the ring of 2×2 matrices over any $GF(p)$. The result now follows by [1, Theorem 1]. □

Lemma 2.4 [8]. *Let R be a ring such that for each $x \in R$, there exist a positive integer m and a polynomial $p(X)$ with integer coefficients for which $x^m = x^{m+1}p(x)$. Then R is periodic.*

Lemma 2.5 [4, Theorem 2]. *Let R be an arbitrary ring, and let $N^* = \{x \in R/x^2 = 0\}$. If N^* is commutative and N is multiplicatively closed, then $PN \subseteq N$.*

We conclude this section with a theorem stating some basic results on semi-weakly periodic rings.

Theorem 2.6. *Let R be a semi-weakly periodic ring.*

(a) Every ideal of R is semi-weakly periodic.

(b) Every epimorphic image of R is semi-weakly periodic.

(c) If N is an ideal, then for each $x \in R\backslash(J \cup Z)$, there exists $n > 1$ for which $x - x^n \in N$.

(d) R is weakly periodic if and only if Z is periodic and J is nil.

(e) If $N \subseteq J \subseteq Z$, then R is commutative.

Proof. (a) Let I be an ideal of R and $x \in I\backslash(J(I) \cup Z(I))$. Clearly $x \notin Z(R)$; and since $J(I) = I \cap J(R)$, $x \notin J(R)$. Therefore, $x = a + u$, where $u \in N$ and $a \in P$; and we may choose $n > 1$ such that $u^n = 0$ and $a^n = a$. It follows that $a = a^n = (x-u)^n \in I$ and hence $u \in I$. Consequently, I is semi-weakly periodic.

(b) Let S be a ring and let $\varphi : R \to S$ be an epimorphism. Let $y \in S\backslash(J(S) \cup Z(S))$, and let $x \in \varphi^{-1}(y)$. Since $\varphi(J(R)) \subseteq J(S)$, we have $x \notin J(R) \cup Z(R)$ and therefore $x \in P(R) + N(R)$. It follows that $y \in P(S) + N(S)$, hence S is semi-weakly periodic.

(c) Let $x \in R\backslash(J \cup Z)$; and write $x = a + u$, where $u \in N$ and $a^n = a$, $n > 1$. Then $x - x^n = a - a^n + u - w$ where $w \in N$, hence $x - x^n \in N$.

(d) Clearly, if Z is periodic and J is nil, then R is weakly periodic. Conversely, suppose R is weakly periodic. By the argument in the proof of (a), J is weakly periodic; and since J contains no nonzero idempotents and hence no nonzero potent elements, J is nil. Let $z \in Z$ and write $z = a + u$ with $u \in N$ and $a^n = a$, $n > 1$. Then $[a,u] = 0$ and hence $z - z^n$ is a sum of commuting nilpotent elements, so that $z - z^n \in N$. It follows from Lemma 2.4 that Z is periodic.

(e) Since $N \subseteq Z$, N is an ideal; and by (c), for each $x \in R\backslash(J \cup Z)$, there exists $n > 1$ for which $x - x^n \in N \subseteq Z$. Since $J \subseteq Z$, Lemma 2.2 now implies that R is commutative. □

3. Commutativity results

It is proved in [2] that if R is periodic and N is commutative, then N is an ideal. Surprisingly, this result extends to semi-weakly periodic rings.

Theorem 3.1. *Let R be a semi-weakly periodic ring with $R \neq J$. If N is commutative, then N is an ideal.*

Proof. Since N is commutative, N is an additive subgroup and is closed under multiplication. By Lemma 2.5, $PN \subseteq N$; and it follows that $(R\backslash(J \cup Z))N \subseteq N$. Since $ZN \subseteq N$, we have $(R\backslash J)N \subseteq N$. Now let $u \in N$ and $y \in J$, and let $x \in R\backslash J$. Then $x + y \in R\backslash J$, and hence $yu = (x + y)u - xu \in N$. Therefore, $JN \subseteq N$ and hence $RN \subseteq N$. □

Corollary 3.2. *If R is a semi-weakly periodic ring in which J is commutative and N is commutative, then N is an ideal and $C(R) \subseteq N$.*

Proof. If $R = J$, then R is commutative and the conclusion is immediate. If $R \neq J$, N is an ideal by Theorem 3.1 and hence $N \subseteq J$. Therefore, in R/J every element is either potent or central, so that R/J is commutative by Lemma 2.2. Our result now follows from Lemma 2.3. □

Herstein's theorem on commutativity of periodic rings with $N \subseteq Z$ also has an extension to semi-weakly periodic rings.

Theorem 3.3. *Let R be a semi-weakly periodic ring with $R \neq J$. If $N \subseteq Z$, then R is commutative.*

Proof. Since $N \subseteq Z$, N is an ideal and hence $N \subseteq J$. By Theorem 2.6(c), for each $x \in R\backslash(J \cup Z)$, there exists $n > 1$ such that $x^n - x \in N \subseteq Z$; moreover, since $ex - exe$ and $xe - exe$ are in N for all $x \in R$ and all $e \in E$, we can use a standard argument to show that $E \subseteq Z$.

By Theorem 2.6(e), we need only show that $J \subseteq Z$. Assume first that R has 1, and suppose that $w \in J\backslash Z$. Since $1 \notin J$, we see at once that $1 - w \notin J \cup Z$. It follows that there exist at most one prime q such that $q(1 - w) \in J$ and at most one prime q such that $q(1 - w) \in Z$, hence there exists a prime p such that $p(1 - w) \notin J \cup Z$. Thus, there exists $n > 1$ such that $(1 - w)^n - (1 - w) \in N$ and $(p(1 - w))^n - p(1 - w) \in N$; consequently, $(p^n - p)(1 - w) \in N$. Since $1 - w$ is invertible, this yields $k > 1$ such that $(p^n - p)^k R = \{0\}$; and this fact, together with the fact that $(1 - w)^n - (1 - w) \in N$, shows that $\langle w \rangle$ is finite and hence w is periodic. But the only periodic elements in J are nilpotent, so we have contradicted our hypothesis that $w \in J\backslash Z$. Thus, $J \subseteq Z$ as required.

Now suppose R does not have 1. If $R = J \cup Z$, then we must have $R = Z$, since a group cannot be the union of two proper subgroups; therefore we may suppose that $R \neq J \cup Z$, in which case $P \neq \{0\}$, since $N \subseteq J$. Let $a \in P\backslash\{0\}$ with $a^n = a$, $n > 1$. Then $e = a^{n-1}$ is an idempotent, necessarily central; and by Theorem 2.6(a), eR is semi-weakly periodic with multiplicative identity e and nilpotent elements central. Hence, by the argument above, $J(eR) \subseteq Z(eR)$. If $u \in J(R)$, then $eu \in eR \cap J(R) = J(eR)$, hence $[ea, eu] = 0 = e[a,u] = [a,u]$. Thus, $[J(R), P + N] = 0 = [J(R), R\backslash(J(R) \cup Z(R))] = [J(R), R\backslash J(R)]$, so $J(R) \subseteq Z(R)$ by Lemma 2.1. □

Our next theorem may be regarded as an extension of [3, Theorem 2].

Theorem 3.4. *Let R be a semi-weakly periodic ring with $R \neq J$. If N is commutative and each element of $R\backslash(J \cup Z)$ is uniquely expressible as a sum of a potent element and a nilpotent element, then R is commutative.*

Proof. We begin by showing that $E \subseteq Z$—a fact which will enable us to pass from the case of R with 1 to the general case by an argument similar to that used in the proof of Theorem 3.3.

Suppose $e \in E\backslash Z$. Then if $[e,x] \neq 0$, either $ex - exe \neq 0$ or $xe - exe \neq 0$; and we assume $ex - exe \neq 0$, in which case the nonzero idempotent $f = e + ex - exe$ is not in $J \cup Z$. Then we have $(e + ex - exe) + 0 = e + (ex - exe)$—two representations of f as a sum of a potent element and a nilpotent element. Therefore, $ex - exe = 0$—a contradiction.

It does not seem necessary to write out the details of the case R without 1, so we assume henceforth that R has 1. In view of Theorem 3.3, we need only show that $N \subseteq Z$. Suppose that $w \in N\backslash Z$. The same argument used in the previous proof shows that $(R,+)$ is a torsion group and there exists $n > 1$ such that $(1+w)^n - (1+w) \in N$; and it follows that $\langle 1+w \rangle$ is finite. Thus, $1+w$ is a periodic invertible element, that is, a potent element. Now $1+w \notin J \cup Z$ and $(1+w)+0 = 1+w$, where 1 and $1+w$ are in P, and 0 and w are in N; therefore $w = 0$, contradicting our assumption that $w \notin Z$. Thus, $N \subseteq Z$ as required. □

It appears from our proofs that the serious work of establishing commutativity of semi-weakly periodic rings is proving the result for R with 1. This observation suggests the following general theorem.

Theorem 3.5. *Let $\wp$ be a ring property which is inherited by ideals and which implies that N is an ideal, and suppose that every semi-weakly periodic $\wp$-ring with 1 is commutative. If R is any semi-weakly periodic $\wp$-ring in which $E \subseteq Z$ and J is commutative, then R is commutative.*

Proof. If $R = J \cup Z$, then $R = J$ or $R = Z$ and hence R is commutative. Otherwise, if $a \in P\backslash\{0\}$, let $a^n = a$ and $a^{n-1} = e$. Then e is a central idempotent; and eR is commutative, since it is a semi-weakly periodic $\wp$-ring with 1. Thus $[ea, ex] = 0$ for all $x \in R$, that is, $a^{n-1}[a,x] = 0 = [a,x]$ for all $x \in R$. Therefore, $P \subseteq Z$ and in particular $[P,J] = 0$. Since $N \subseteq J$, $[N,J] = 0$ and we conclude that $[R\backslash J, J] = 0$, so that $J \subseteq Z$ by Lemma 2.1. Commutativity of R now follows from Theorem 2.6(e). □

Of course there are many applications of this theorem, since there are many conditions known to imply commutativity of rings with 1 but not of arbitrary rings. We conclude with one example.

Theorem 3.6. *Let $n > 1$ be a fixed positive integer and let R be an $n(n-1)$-torsion-free semi-weakly periodic ring with $E \subseteq Z$ and J commutative. If*

$(*)$

$$(xy)^n = x^n y^n \quad \forall x, y \in R\backslash N, \tag{3.1}$$

then R is commutative.

Proof. Let $\wp$ be the following property: R is $n(n-1)$-torsion-free with J commutative and R satisfies $(*)$. It is proved in [5] that any $n(n-1)$-torsion-free ring with 1 which satisfies

$(*)$ is commutative. Hence, our theorem follows from Theorem 3.5 once we show that property $\wp$ forces N to be an ideal. In fact, we show that $(*)$ together with the hypothesis that J is commutative forces N to be an ideal. Consider R/J, which is a subdirect product of primitive rings R_α satisfying $(*)$. Since $(*)$ is inherited by subrings and epimorphic images, it follows from Jacobson's density theorem that either all R_α are division rings, or there exist a division ring D and an integer $m > 1$ for which the ring of $m \times m$ matrices over D satisfies $(*)$. A simple substitution shows that the second alternative cannot occur, so each R_α is a division ring such that $(xy)^n = x^n y^n$ for all $x, y \in R_\alpha$. But such division rings are commutative by a well-known theorem of Herstein [11, 12], so R/J is commutative and hence N is an ideal by Lemma 2.3. □

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