FIXED POINT THEOREMS FOR THE CLASS $Q(X,Y)$

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We study a new family of functions $Q(X,Y)$, research its properties, and get some fixed point theorems about this family.

1. Introduction and preliminaries

Kuratowski [6] showed that a continuous compact map $f : X \to X$ defined on a closed convex subset X of a Banach space has a fixed point. This theorem has enormous influence on fixed point theory, variational inequalities, and equilibrium problems. In 1968, Goebel [5] established the well-known coincidence theorem, and then there had been a lot of generalization and application (see, [1, 2, 5]).

Let X be a subset of a Hausdorff topological vector space E and Y a Hausdorff topological vector space, we define a new class $Q(X,Y)$ of set-valued maps from X into Y as follows. $T \in Q(X,Y)$ implies that for any compact convex subset K of X and any continuous function $f : T(K) \to K$, the composition $f(T|_K) : K \to 2^K$ has a fixed point.

Subclasses of $Q(X,Y)$ are the class of continuous functions $C(X,Y)$, the class of the Kakutani maps $K(X,Y)$ (with convex values and codomains being convex spaces), the class of the acyclic maps $V(X,Y)$ (with acyclic values), and the class of the approachable maps $\mathscr{A}_0(X,Y)$ (whose domains and codomains are subsets of topological vector spaces), and so forth.

A nonempty subset X of a Hausdorff topological vector space E is said to be nearly convex (see Wu [7]) if for every compact subset A of X and every neighborhood V of the origin 0 of E, there is a continuous mapping $h : A \to X$ such that $x - h(x) \in V$ for all $x \in A$ and $h(A)$ is contained in some convex subset of X.

Remark 1.1. It is clear that every convex set is nearly convex, but the converse is not true in general.

For a counterexample, let (M,d) be a metric space, where $M = \mathbb{R}^2$ and the metric $d : M \times M \to \mathbb{R}^+$ is denoted by $d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ for all $x = (x_1,x_2)$, $y = (y_1,y_2) \in M$. Then the set $B(0) = \{x \in M : d(x,0) < 1\} \cup \{x = (x_1,x_2) \in M : |x_1| = |x_2| = 1\}$ is a nearly convex subset of M, but it is not convex.

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Let E and F be topological vector spaces, let X be a nonempty subset of E, and let Y be a subset of F. We denote by 2^Y the class of all nonempty subsets of Y, and $\langle X\rangle$ denotes the class of all nonempty finite subsets of X. For a set-valued function $T: X \to 2^Y$, the following notations are used.

(i) $Tx = \{y \in Y \mid y \in Tx\}$.

(ii) $TA = \cup_{x\in A} Tx$.

(iii) $T^{-1}y = \{x \in X \mid y \in Tx\}$.

(iv) $T^{-1}B = \{x \in X \mid Tx \cap B \neq \phi\}$.

(v) T is said to be compact if the image TX of X under T is contained in a compact subset of Y.

(vi) T is said to be closed if its graph $\mathcal{G}_T = \{(x,y) \in X \times Y \mid y \in Tx$, for all $x \in X\}$ is a closed subset of $X \times Y$.

(vii) T is upper semicontinuous (usc) if $T^{-1}B$ is closed in X for each closed subset B of Y, it is well known that if Y is compact and T is closed, then T is usc.

(viii) $C(X,Y)$ denotes the class of all continuous single-valued functions from X to Y.

The authors Chang and Yen (see [4]) introduced the following concept of KKM property. Let X be a nonempty convex subset of a linear space and Y a topological space. If $T: X \to 2^Y$, and $F: X \to 2^Y$ are two multifunctions satisfying $T(\text{co}(A)) \subset F(A)$ for any $A \in \langle X\rangle$, where co(A) denotes the convex hull of A, then F is called a generalized KKM mapping with respect to T. If the multifunction $T: X \to 2^Y$ satisfies that for any generalized KKM mapping F with respect to T, the family $\{\overline{F(x)} : x \in X\}$ has the finite intersection property, then T is said to have the KKM property. The class KKM(X,Y) is defined to be the set $\{T: X \to 2^Y \mid T$ has the KKM property$\}$.

In general, $Q(X,Y)$ and KKM(X,Y) may not be comparable, we conclude the differences as follows.

Proposition 1.2. *Let X be a convex subset of a Hausdorff topological vector space and Y a normal space. Then* $Q(X,Y) \subset$ KKM(X,Y).

Proof. By [4, Proposition 3(ii), (iii)] of Chang and Yen, we complete the proof. □

Proposition 1.3. *Let X be a convex subset of a locally convex space and $T \in$* KKM(X,Y) *is closed. Then $T \in Q(X,Y)$.*

Proof. By [4, Proposition 3(i), (ii) and Theorem 2] of Chang and Yen, we complete the proof. □

2. Main results

The following is our new fixed point theorems for the class Q.

Theorem 2.1. *Let X be a nonempty nearly convex subset of a Hausdorff topological vector space E, let $T \in Q(X,X)$ be closed, and let $\overline{TX} \subset X$ be compact. Then T has a fixed point in X.*

Proof. Let $\mathbb{N} = \{U_\beta \mid \beta \in \Lambda\}$ be a local base of E such that U_β is symmetric and open for each $\beta \in \Lambda$, and let $V \in \mathbb{N}$. Since $\overline{TX}$ is a compact subset of the nearly convex set X, there exists a continuous mapping $h: \overline{TX} \to X$ such that $x - h(x) \in V$ for all $x \in \overline{TX}$ and $h(\overline{TX})$ is contained in some convex subset of X.

Let $Z = \text{co}(h(\overline{TX}))$, then $h(\overline{TX}) \subset Z \subset X$ and Z is compact and convex. Note that $h : \overline{TX} \to Z$ and $T|_Z : Z \to 2^{\overline{TX}}$. Since $T \in Q(X,X)$, Z is compact and convex, and $h|_{T(Z)} : T(Z) \to Z$ is continuous, the composition $h|_{T(Z)} \circ T|_Z : Z \to 2^Z$ has a fixed point, say x_V, then $x_V \in h(T(x_V))$. Let $x_V = h(y_V)$ for some $y_V \in T(x_V) \subset T(Z) \subset \overline{TX}$. Then we have $y_V - x_V = y_V - h(y_V) \in V$. Since $\overline{TX}$ is compact, we may assume that $\{y_V\}$ converges to $\overline{x}$ and then $\{x_V\}$ also converges to $\overline{x}$. The closedness of T implies that $\overline{x} \in T(\overline{x})$. □

Corollary 2.2. *Let X be a nonempty convex subset of a Hausdorff topological vector space E, let $T \in Q(X,X)$ be closed, and let $\overline{TX} \subset X$ be compact. Then T has a fixed point in X.*

Let $\mathbb{R}^+$ be the set of all nonnegative real numbers. A mapping $\Phi : B(E) \to \mathbb{R}^+$ is called a measure of noncompactness (see [6]) provided that the following conditions hold.

(i) $\Phi(\overline{\text{co}}(\Omega)) = \Phi(\Omega)$ for each $\Omega \in B(E)$, where $\overline{\text{co}}(\Omega)$ denotes the closure of the convex hull of Ω.

(ii) $\Phi(\Omega) = 0$ if and only if Ω is precompact.

(iii) $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$, for each $A, B \in B(E)$.

(iv) $\Phi(\lambda\Omega) = \lambda\Phi(\Omega)$, for each $\lambda \geq 0$, $\Omega \in B(E)$.

If X is a nonempty subset of E and $T : X \to 2^E$, then T is called Φ-condensing mapping provided that $\Phi(D) = 0$ for any $D \subset X$ with $\Phi(D) \leq \Phi(T(D))$.

The following Lemma is well known by many authors.

Lemma 2.3. *Let X be a nonempty closed convex subset of a topological vector space E and $T : X \to 2^X$ a Φ-condensing mapping. Then there exists a nonempty compact convex subset K of X such that $T(K) \subset K$.*

From Corollary 2.2 and Lemma 2.3, we have the following theorem.

Theorem 2.4. *Let X be a nonempty convex subset of a Hausdorff topological vector space E, let $T \in Q(X,X)$ be a closed Φ-condensing mapping. Then T has a fixed point in X.*

Proof. By Lemma 2.3, there exists a nonempty compact convex subset K of X such that $T(K) \subset K$. It is easy to show that $T|_K \in Q(K,K)$. Hence, by Corollary 2.2, we have that $T|_K$ has a fixed point in K. This completes the proof. □

Let X, Y be subsets of topological vector spaces E and F, respectively, and let $T : X \to 2^Y$. $N_E(x)$ will denote a filter of neighborhoods of a given point $x \in E$. Given $U \in N_E(0)$ and $V \in N_F(0)$, a function $s : X \to Y$ is said to be a (U,V)-selection of T if for any $x \in X$, $s(x) \in (T[(x+U) \cap X] + V) \cap Y$. T is said to be approachable if it has a continuous (U,V)-selection for any $U \in N_E(0)$ and any $V \in N_F(0)$. The classes of approachable mappings are defined as

(i) $\mathcal{A}_0(X,Y) = \{T : X \to 2^Y \mid \text{T is approachable}\}$,

(ii) $\mathcal{A}(X,Y) = \{T \in \mathcal{A}_0 \mid \text{T is usc and compact-valued}\}$,

(iii) $\mathcal{A}_c(X,Y) = \{T = T_m \circ T_{m-1} \circ \cdots \circ T_1 \mid T_i \in \mathcal{A}(X,Y) \text{ for } i = 1,2,\ldots,m\}$.

Lemma 2.5. *Let X be a subset of a Hausdorff topological vector space, and let Y, Z be two topological spaces. If $T \in Q(X,Y)$ and $f \in C(Y,Z)$, then $fT \in Q(X,Z)$.*

Proof. Let K be any compact convex subset of X and $h : f(T(K)) \to K$ any continuous function. Let $h' = hf : T(K) \to K$, then h' is continuous, and since $T \in Q(X,Y)$, we have that the composition $h'(T|_K): K \to 2^K$ has a fixed point $x \in h'(T(x)) = hf(T(x))$. This implies that the composition $h(fT|_K): K \to 2^K$ has a fixed point. So $fT \in Q(X,Z)$. □

By using a result of Ben-El-Mechaiekh and Deguire (see [3]), we get some fixed point theorems and a generalized Fan's matching theorem.

Theorem 2.6. *Let X be a nonempty nearly convex subset of a Hausdorff topological vector space E, and let Y be a compact subset of a topological vector space F. Suppose that $T \in Q(X,Y)$ is closed. Then for any $G \in \mathcal{A}_c(Y,X)$, TG has a fixed point in Y.*

Proof. Since $T \in Q(X,Y)$, for any $f \in C(Y,X)$, by Lemma 2.5, $fT \in Q(X,X)$. By using the fact that fT is compact and closed, we conclude via Theorem 2.1 that fT has a fixed point in X. Hence $\mathcal{G}_f \cap \mathcal{G}_{T^{-1}} \neq \phi$, and thus, by Ben-El-Mechaiekh and Deguire [3, Corollary 7.5], we have $\mathcal{G}_G \cap \mathcal{G}_{T^{-1}} \neq \phi$ for each $G \in \mathcal{A}_c(Y,X)$. Therefore, TG has a fixed point in Y. □

A family $\mathfrak{I}$ of subsets of a topological space is locally finite if and only if each point of the space has a neighborhood which intersects only finitely many members of $\mathfrak{I}$. It follows immediately from the definition that a point is an accumulation point of the union $\bigcup\{A : A \in \mathfrak{I}\}$ if and only if it is an accumulation point of some member of $\mathfrak{I}$, and hence the closure of the union is the union of the closures. It is also evident that the family of all closures of members of $\mathfrak{I}$ is locally finite.

We now deduce a matching theorem for a covering by using the results in the previous theorem.

Theorem 2.7. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space E, and let $\{A_i : i \in I\}$ be a locally finite family of closed subsets of X such that $X = \bigcup_{i\in I} A_i$. If $T \in Q(X,X)$ is closed, then for any subset $\{x_i : i \in I\}$ of X indexed by the same set I, there exists a nonempty subset J of I such that*

$$T(\operatorname{co}\{x_i : i \in J\}) \cap \left(\bigcap_{i\in J} A_i\right) \neq \phi. \tag{2.1}$$

Proof. For any $x \in X$, since $\{A_i : i \in I\}$ is a locally finite family of closed subsets of X, by Zorn's lemma, we may choose a maximal neighborhood $N(x)$ of x which intersects only finitely many members of $\{A_i : i \in I\}$. Now we let $I(x) = \{i \in I : x \in A_i\}$. Since $\{A_i : i \in I\}$ covers X, $I(x) \neq \phi$ for each $x \in X$, and since $\{A_i : i \in I\}$ is a locally finite family of closed subsets of X, so $I(x)$ is a finite subset of I.

Next, we define a multifunction $G : X \to 2^X$ by $G(x) = \operatorname{co}\{x_i : i \in I(x)\}$ for $x \in X$, then each Gx is a nonempty compact convex subset of X. Also, if $z \in N(x)$, then $I(z) \subset I(x)$ which implies that $G(z) \subset G(x)$. This shows that G is upper semicontinuous. Therefore, by [3, Proposition 4.1], $G \in \mathcal{A}(X,X) \subset \mathcal{A}_c(X,X)$, and so, in view of Theorem 2.6, TG has a fixed point x_0 in X. Hence, $x_0 \in T(\operatorname{co}\{x_i : i \in I(x_0)\}) \cap (\cap_{i\in I(x_0)} A_i)$. This completes the proof. □

The above matching theorem can reduce to the following results of Ky Fan's matching theorem.

Corollary 2.8. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space E. Assume that $T \in Q(X,X)$ is closed and $G : X \to 2^X$ satisfies that*

(i) *for each $x \in X$, Gx is open,*

(ii) *for any $\{x_1,x_2,\dots,x_n\} \in \langle X\rangle$, $T(\mathrm{co}\{x_1,x_2,\dots,x_n\}) \subset \bigcup_{i=1}^n Gx_i$.*

Then the family $\{Gx : x \in X\}$ has the finite intersection property.

Proof. On the contrary, we assume that there exists a finite subset $\{x_1,x_2,\dots,x_n\}$ of X such that $\bigcap_{i=1}^n Gx_i = \phi$. Define $F : X \to 2^X$ by $Fx = G^c x$ for $x \in X$, then each Fx is closed, and hence $\{Fx_i\}_{i=1}^n$ is a family of closed subsets of X with $\bigcup_{i=1}^n Fx_i = X$. Therefore, by Theorem 2.7, there is a subset $\{x_{i_1},x_{i_2},\dots,x_{i_m}\}$ of $\{x_1,x_2,\dots,x_n\}$ such that $T(\mathrm{co}\{x_{i_1},x_{i_2},\dots,x_{i_m}\})\bigcap(\bigcap_{j=1}^m Fx_{i_j}) \neq \phi$. It follows that $T(\mathrm{co}\{x_{i_1},x_{i_2},\dots,x_{i_m}\}) \nsubseteq \bigcup_{j=1}^m Gx_{i_j}$, and we have a contradiction. This completes the proof. □

Let X be a compact Hausdoff space and let $\{A_1,A_2,\dots,A_n\}$ be a finite family of open subsets of X such that $X = \bigcup_{i=1}^n A_i$. Then there exist continuous functions $\lambda_1,\lambda_2,\dots,\lambda_n$ on X satisfying the following:

(i) $0 \leqslant \lambda_i(x) \leqslant 1$ for all i, $1 \leqslant i \leqslant n$, and for each $x \in X$,

(ii) $\Sigma_{i=1}^n \lambda_i(x) = 1$, for all $x \in X$,

(iii) $\lambda_i(x) = 0$ if $x \notin A_i$.

We call the family $\{\lambda_1,\lambda_2,\dots,\lambda_n\}$ a partition of unity corresponding to $\{A_1,A_2,\dots,A_n\}$.

We now have the following coincidence theorem.

Theorem 2.9. *Let X be a nonempty convex subset of a Hausdorff topological vector space E, and let $T,G : X \to 2^X$ be two set-valued mappings satisfying*

(i) *$T \in Q(X,X)$ and $\overline{TX}$ is a compact subset of X,*

(ii) *for each $y \in T(X)$, $G^{-1}y$ is convex,*

(iii) *$\{\mathrm{int}\, Gx : x \in X\}$ covers $\overline{TX}$.*

Then there exists an $x_0 \in X$ such that $Tx_0 \bigcap Gx_0 \neq \phi$.

Proof. Since $\overline{TX}$ is a compact subset of X and $\overline{TX} \subset \bigcup_{x\in X} \mathrm{int}\, Gx$, there exists a finite subset $\{x_1,x_2,\dots,x_n\}$ of X such that $\overline{TX} \subset \bigcup_{i=1}^n \mathrm{int}\, Gx_i$. Let $\{\lambda_i\}_{i=1}^n$ be the partition of the unity subordinated to $\{\mathrm{int}\, Gx_i : i = 1,2,\dots,n\}$ and let $P = \mathrm{co}\{x_1,x_2,\dots,x_n\}$. Define $f : \overline{TX} \to P$ by $f(y) = \Sigma_{i=1}^n \lambda_i(y)x_i = \Sigma_{i\in N_y}\lambda_i(y)x_i$, for each $y \in \overline{TX}$, where $i \in N_y$ if and only if $\lambda_i(y) \neq 0$ if and only if $y \in \mathrm{int}\, Gx_i \subset Gx_i$. Then $x_i \in G^{-1}y$ for each $i \in N_y$. Clearly, f is continuous, and by (ii), we have $f(y) \in \mathrm{co}\{x_i : i \in N_y\} \subset G^{-1}y$ for each $y \in \overline{TX}$. Since P is a compact convex subset of X and $T \in Q(X,X)$, $(f|_{T(P)})(T|_P) : P \to P$ has a fixed point $x_0 \in P \subset X$. So $x_0 \in fTx_0$ and $f^{-1}(x_0) \subset Gx_0$, and we have $Tx_0 \cap Gx_0 \neq \phi$. □

Using the above theorems, we have the following fixed point theorem of Leray-Schauder type.

Theorem 2.10. *Let X be a convex subset of a Hausdorff topological vector space E with $0 \in X$, U a neighborhood of 0, and let $T \in Q(X,X)$ such that $T|_{\overline{U}\cap X}$ is compact and closed. If T satisfies*

(LS)

$$Tx \cap \{\lambda x : \lambda > 1\} = \phi \quad \text{for each } x \in Bd_X U, \tag{2.2}$$

then T has a fixed point in $\overline{U} \cap X$.

Proof. Let p be a Minkowski function of U. Since $0 \in U$, P is continuous. We define $r : E \to \overline{U}$ by

$$r(x) = \begin{cases} x, & x \in \overline{U}, \\ \dfrac{x}{p(x)}, & x \notin \overline{U}, \end{cases} \tag{2.3}$$

that is,

$$r(x) = \frac{x}{\max\{1, p(x)\}}. \tag{2.4}$$

Then r is a continuous retraction of E on $\overline{U}$. Let f be the retraction on r to X. Since X is convex and $0 \in X$, $f(x) \in \overline{U} \cap X$ and so $f \in C(X, \overline{U} \cap X)$. Hence $fT \in Q(\overline{U} \cap X, \overline{U} \cap X)$, and fT is compact and closed. It follows from Corollary 2.2 that fT has a fixed point in $\overline{U} \cap X$, that is, there exists a $z \in \overline{U} \cap X$ such that $z \in fT(z)$. Choose $y \in T(z)$ such that $z = f(y)$. We have either $z \in U$ or $z \in Bd_X(\overline{U})$.

Case 1. If $z \in U$, then $1 > p(z) = p(f(y)) = p(y)/\max\{1, p(y)\}$, and so $p(y) < 1$, which implies that $y = f(y)$. Thus $z = f(y) = y \in T(z)$.

Case 2. If $z \in Bd_X(\overline{U})$, then $1 = p(z) = p(f(y)) = p(y)/\max\{1, p(y)\}$, from which we see that $p(y) \geq 1$. If $p(y) > 1$, we have $z = f(y) = y/p(y)$, and then $y = p(y)z$, which contradicts the condition (LS). So $p(y) = 1$, and thus $z = f(y) = y \in T(z)$. This completes the proof. □

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