BETA BESSEL DISTRIBUTIONS

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Three new distributions on the unit interval $[0,1]$ are introduced which generalize the standard beta distribution. These distributions involve the Bessel function. Expression is derived for their shapes, particular cases, and the nth moments. Estimation by the method of maximum likelihood and Bayes estimation are discussed. Finally, an application to consumer price indices is illustrated to show that the proposed distributions are better models to economic data than one based on the standard beta distribution.

1. Introduction

Beta distributions are very versatile and a variety of uncertainties can be usefully modeled by them. Many of the finite range distributions encountered in practice can be easily transformed into the standard distribution. In reliability and life testing experiments, many times the data are modeled by finite range distributions, see, for example, [2].

A random variable X is said to have the standard beta distribution with parameters ν and μ if its probability density function (pdf) is

$$f(x) = \frac{x^{\nu-1}(1-x)^{\mu-1}}{B(\nu,\mu)} \tag{1.1}$$

for $0 < x < 1$, $\nu > 0$, and $\mu > 0$, where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt \tag{1.2}$$

denotes the beta function. Many generalizations of (1.1) involving algebraic, exponential, and hypergeometric functions have been proposed in the literature. Some of these are (see [6, Chapter 25] and [5] for comprehensive accounts)

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(i) the four-parameter generalization given by

$$f(x) = \frac{1}{(d-c)B(a,b)} \left(\frac{x-c}{d-c}\right)^{a-1} \left(1 - \frac{x-c}{d-c}\right)^{b-1} \tag{1.3}$$

for $c \leq x \leq d$, $a > 0$, and $b > 0$ (see [7, Section 3.2] for a reparameterization of this);

(ii) the McDonald and Richards [9, 13] beta distribution given by

$$f(x) = \frac{p x^{ap-1} \{1 - (x/q)^p\}^{b-1}}{q^{ap} B(a,b)} \tag{1.4}$$

for $0 \leq x \leq q$, $a > 0$, $b > 0$, $p > 0$, and $q > 0$;

(iii) the Libby and Novick [8] beta distribution given by

$$f(x) = \frac{\lambda^a x^{a-1} (1-x)^{b-1}}{B(a,b) \{1 - (1-\lambda)x\}^{a+b}} \tag{1.5}$$

for $0 \leq x \leq 1$, $a > 0$, $b > 0$, and $\lambda > 0$;

(iv) the McDonald and Xu [10] beta distribution given by

$$f(x) = \frac{p x^{ap-1} \{1 - (1-c)(x/q)^p\}^{b-1}}{q^{ap} B(a,b) \{1 + c(x/q)^p\}^{a+b}} \tag{1.6}$$

for $0 \leq x^p \leq q^p/(1-c)$, where $a > 0$, $b > 0$, $0 \leq c \leq 1$, $p > 0$, and $q > 0$;

(v) the Gauss hypergeometric distribution given by

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{(1+zx)^{\gamma}} B(a,b) {}_2F_1(\gamma, a; a+b; -z) \tag{1.7}$$

for $0 < x < 1$, $a > 0$, $b > 0$, and $-\infty < \gamma < \infty$ (Armero and Bayarri [1]), where

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!} \tag{1.8}$$

denotes the Gauss hypergeometric function, where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial;

(vi) confluent hypergeometric distribution given by

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}\exp(-\gamma x)}{B(a,b)\,{}_1F_1(a;a+b;-\gamma)} \tag{1.9}$$

for $0 < x < 1$, $a > 0$, $b > 0$, and $-\infty < \gamma < \infty$ (Gordy [3]), where

$${}_1F_1(a;b;x) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k}\frac{x^k}{k!} \tag{1.10}$$

is the confluent hypergeometric function.

In this paper, we introduce the first generalizations of (1.1) involving the Bessel function. We refer to them as the beta Bessel (BB) distributions. We propose three BB distributions in all.

For each of the three BB distributions, we derive various particular cases, an expression for the nth moment as well as estimation procedures by the method of maximum likelihood and Bayes method (Sections 2 to 4). We also present an application of the proposed models to consumer price indices (Section 5). The calculations involve several special functions, including the modified Bessel function of the first kind defined by

$$I_m(x) = \frac{x^m}{\sqrt{\pi}2^m\Gamma(m+1/2)}\int_{-1}^{1}\left(1-t^2\right)^{m-1/2}\exp(xt)dt, \tag{1.11}$$

the ${}_2F_2$ hypergeometric function defined by

$${}_2F_2(a,b;c,d;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k(d)_k}\frac{x^k}{k!}, \tag{1.12}$$

and the ${}_2F_3$ hypergeometric function defined by

$${}_2F_3(a,b;c,d,e;x) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k(d)_k(e)_k}\frac{x^k}{k!}, \tag{1.13}$$

where $(f)_k = f(f+1)\cdots(f+k-1)$ denotes the ascending factorial. The properties of the above special functions can be found in [4, 11, 12].

2. BB distribution I

The first generalization of (1.1) is given by the pdf

$$f(x) = Cx^{\alpha-1}(1-x)^{\beta-1}I_\nu(cx) \tag{2.1}$$

for $0 < x < 1$, $\nu > 0$, $\alpha > 0$, $\beta > 0$, and $c \geq 0$, where C denotes the normalizing constant. Application of [12, equation (2.15.2.1)] shows that one can determine C as

$$\frac{1}{C} = \frac{c^\nu\Gamma(\alpha+\nu)\Gamma(\beta)}{2^\nu\Gamma(\alpha+\beta+\nu)\Gamma(\nu+1)}\,{}_2F_3\left(\frac{\alpha+\nu}{2}, \frac{\alpha+\nu+1}{2}; \nu+1, \frac{\alpha+\nu+\beta}{2}, \frac{\alpha+\nu+\beta+1}{2}; \frac{c^2}{4}\right). \tag{2.2}$$

The standard beta pdf (1.1) arises as the particular case of (2.1) for $c = 0$ and $\nu = 0$. Several other particular cases of (2.1) can be obtained using special properties of $I_\nu(\cdot)$. Note that

$$\begin{aligned}
I_{3/2}(x) &= \sqrt{\frac{2}{\pi}}\frac{x\cosh(x)-\sinh(x)}{x^{3/2}},\\
I_{5/2}(x) &= \sqrt{\frac{2}{\pi}}\frac{(x^2+3)\sinh(x)-3x\cosh(x)}{x^{5/2}},\\
I_{7/2}(x) &= \sqrt{\frac{2}{\pi}}\frac{x(x^2+15)\cosh(x)-3(2x^2+5)\sinh(x)}{x^{7/2}},\\
I_{9/2}(x) &= \sqrt{\frac{2}{\pi}}\frac{(x^4+45x^2+105)\sinh(x)-5x(2x^2+21)\cosh(x)}{x^{9/2}}.
\end{aligned} \tag{2.3}$$

More generally, if $\nu - 1/2 \geq 1$ is an integer, then

$$\begin{aligned}
I_\nu(x) = {} & \sqrt{2}\sqrt{x\pi}\exp\left\{\frac{\pi i}{2}\left(\frac{1}{2}-\nu\right)\right\}\\
& \times\left[\sinh\left(\frac{\pi x}{2}\left(\frac{1}{2}-\nu\right)-x\right)\times\sum_{k=0}^{[(2|\nu|-1)/4]}\frac{(|\nu|+2k-1/2)!}{(2k)!(|\nu|-2k-1/2)!(2x)^{2k}}\right.\\
& \left.+\cosh\left(\frac{\pi x}{2}\left(\frac{1}{2}-\nu\right)-x\right)\sum_{k=0}^{[(2|\nu|-3)/4]}\frac{(|\nu|+2k+1/2)!(2x)^{-2k-1}}{(2k+1)!(|\nu|-2k-3/2)!}\right].
\end{aligned} \tag{2.4}$$

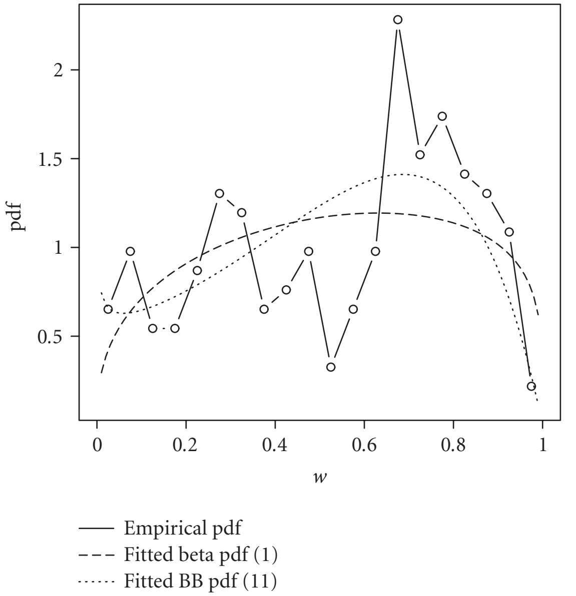

Figure 2.1. The empirical and fitted densities for the consumer price indices of the United States and the United Kingdom (X = consumer price index of the United States and Y = consumer price index of the United Kingdom).

Thus, several particular forms of (2.1) can be obtained for half-integer values of ν. For example, if $\nu = 3/2$, then (2.1) reduces to

$$f(x) = C\sqrt{\frac{2}{\pi c^3}}x^{\alpha-5/2}(1-x)^{\beta-1}\{cx\cosh(cx) - \sinh(cx)\}. \tag{2.5}$$

If $\nu = 5/2$, then (2.1) reduces to

$$f(x) = C\sqrt{\frac{2}{\pi c^5}}x^{\alpha-7/2}(1-x)^{\beta-1}I_\nu(cx)\{(c^2x^2+3)\sinh(cx) - 3cx\cosh(cx)\}. \tag{2.6}$$

The modes of (2.1) are the solutions of

$$\frac{\alpha-1}{x} - \frac{\beta-1}{x} + \frac{cI_{\nu-1}(cx)}{I_\nu(cx)} = \frac{\nu}{c}. \tag{2.7}$$

There could be more than one mode (see Figures 2.1 and 2.2). The nth moment of (2.1) can be written as

$$E(X^n) = C\int_0^1 x^{n+\alpha-1}(1-x)^{\beta-1}I_\nu(cx)dx \tag{2.8}$$

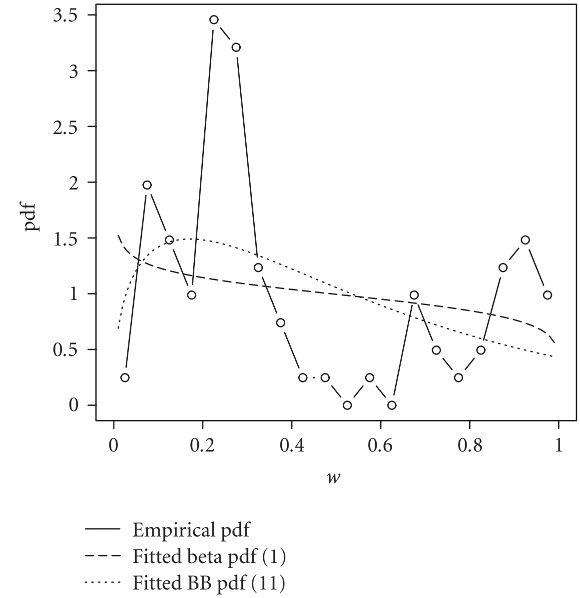

Figure 2.2. The empirical and fitted densities for the consumer price indices of the United States and Germany (X = consumer price index of the United States and Y = consumer price index of Germany).

and an application of [12, equation (2.15.2.1)] shows that (2.8) reduces to

$$E(X^n) = \frac{Cc^{\nu}\Gamma(n+\alpha+\nu)\Gamma(\beta)}{2^{\nu}\Gamma(n+\alpha+\beta+\nu)\Gamma(\nu+1)} \times {}_2F_3\left(\frac{n+\alpha+\nu}{2}, \frac{n+\alpha+\nu+1}{2}; \nu+1, \frac{n+\alpha+\nu+\beta}{2}, \frac{n+\alpha+\nu+\beta+1}{2}; \frac{c^2}{4}\right). \tag{2.9}$$

For a random sample $w_1,\dots,w_n$, the maximum-likelihood estimators (MLEs) of the four parameters in (2.1) are the solutions of

$$\begin{aligned} \sum_{i=1}^{n} \operatorname{In} w_i &= -\frac{n}{C}\frac{\partial C}{\partial \alpha}, \\ \sum_{i=1}^{n} \operatorname{In}(1-w_i) &= -\frac{n}{C}\frac{\partial C}{\partial \beta}, \\ \sum_{i=1}^{n} \frac{w_i I_{\nu-1}(cw_i)}{I_\nu(cw_i)} &= \frac{n\nu}{c} - \frac{n}{C}\frac{\partial C}{\partial c}, \\ \sum_{i=1}^{n} \frac{\partial I_\nu(cw_i)/\partial \nu}{I_\nu(cw_i)} &= -\frac{n}{C}\frac{\partial C}{\partial \nu}. \end{aligned} \tag{2.10}$$

Assuming (2.1) as the prior, the Bayes estimate of the binomial parameter, say p, is

$$E(p \mid x) = \frac{Cc^{\nu}\Gamma(x+1+\alpha+\nu)\Gamma(n-x+\beta)}{2^{\nu}\Gamma(n+1+\alpha+\beta+\nu)\Gamma(\nu+1)} \times {}_2F_3\left(\frac{1+x+\alpha+\nu}{2}, \frac{x+\alpha+\nu}{2}+1; \nu+1, \frac{n+1+\alpha+\beta+\nu}{2}, \frac{n+\alpha+\beta+\nu}{2}+1; \frac{c^2}{4}\right), \tag{2.11}$$

where n is the number of trials and x is the number of successes.

3. BB distribution II

The second generalization of (1.1) is given by the pdf

$$f(x) = Cx^{\alpha-1}(1-x)^{\beta-1}\exp(cx)I_{\nu}(cx) \tag{3.1}$$

for $0 < x < 1$, $\nu > 0$, $\alpha > 0$, $\beta > 0$, and $c \geq 0$, where C denotes the normalizing constant. Application of [12, equation (2.15.4.1)] shows that one can determine C as

$$\frac{1}{C} = \frac{c^{\nu}\Gamma(\alpha+\nu)\Gamma(\beta)}{2^{\nu}\Gamma(\alpha+\beta+\nu)\Gamma(\nu+1)} {}_2F_2\left(\nu+\frac{1}{2}, \alpha+\nu; 2\nu+1, \alpha+\beta+\nu; 2c\right). \tag{3.2}$$

The standard beta pdf (1.1) arises as the particular case of (3.1) for $c = 0$ and $\nu = 0$. Further particular cases of (3.1) can be obtained using (2.4). The modes of (3.1) are the solutions of

$$\frac{\alpha-1}{x} - \frac{\beta-1}{x} + \frac{cI_{\nu-1}(cx)}{I_{\nu}(cx)} = \frac{\nu}{c} - c. \tag{3.3}$$

There could be more than one mode (see Figures 2.1 and 2.2). The nth moment of (3.1) can be written as

$$E(X^n) = C\int_0^1 x^{n+\alpha-1}(1-x)^{\beta-1}\exp(cx)I_{\nu}(cx)dx \tag{3.4}$$

and an application of [12, equation (2.15.4.1)] shows that the above reduces to

$$E(X^n) = \frac{Cc^{\nu}\Gamma(n+\alpha+\nu)\Gamma(\beta)}{2^{\nu}\Gamma(n+\alpha+\beta+\nu)\Gamma(\nu+1)} {}_2F_2\left(\nu+\frac{1}{2}, n+\alpha+\nu; 2\nu+1, n+\alpha+\beta+\nu; 2c\right). \tag{3.5}$$

For a random sample $w_1,\dots,w_n$, the MLEs of the four parameters in (3.1) are the solutions of

$$\begin{aligned}\sum_{i=1}^{n} \operatorname{In} w_i &= -\frac{n}{C}\frac{\partial C}{\partial \alpha},\\ \sum_{i=1}^{n} \operatorname{In}(1-w_i) &= -\frac{n}{C}\frac{\partial C}{\partial \beta},\\ \sum_{i=1}^{n} \frac{w_i I_{\nu-1}(cw_i)}{I_\nu(cw_i)} + \sum_{i=1}^{n} w_i &= \frac{n\nu}{c} - \frac{n}{C}\frac{\partial C}{\partial c},\\ \sum_{i=1}^{n} \frac{\partial I_\nu(cw_i)/\partial \nu}{I_\nu(cw_i)} &= -\frac{n}{C}\frac{\partial C}{\partial \nu}.\end{aligned} \tag{3.6}$$

Assuming (3.1) as the prior, the Bayes estimate of the binomial parameter, say p, is

$$\begin{aligned}E(p \mid x) = {} & \frac{Cc^\nu \Gamma(1+x+\alpha+\nu)\Gamma(n-x+\beta)}{2^\nu \Gamma(n+1+\alpha+\beta+\nu)\Gamma(\nu+1)}\\ & \times {}_2F_2\left(\nu+\frac{1}{2}, 1+x+\alpha+\nu; 2\nu+1, n+1+\alpha+\beta+\nu; 2c\right),\end{aligned} \tag{3.7}$$

where n is the number of trials and x is the number of successes.

4. BB distribution III

The third and final generalization of (1.1) is given by the pdf

$$f(x) = Cx^{\alpha-1}(1-x)^{\beta-1}\exp(-cx)I_\nu(cx) \tag{4.1}$$

for $0 < x < 1$, $\nu > 0$, $\alpha > 0$, $\beta > 0$, and $c \geq 0$, where C denotes the normalizing constant. Application of [12, equation (2.15.4.1)] shows that one can determine C as

$$\frac{1}{C} = \frac{c^\nu \Gamma(\alpha+\nu)\Gamma(\beta)}{2^\nu \Gamma(\alpha+\beta+\nu)\Gamma(\nu+1)} {}_2F_2\left(\nu+\frac{1}{2}, \alpha+\nu; 2\nu+1, \alpha+\beta+\nu; -2c\right). \tag{4.2}$$

The standard beta pdf (1.1) arises as the particular case of (4.1) for $c = 0$ and $\nu = 0$. Further particular cases of (4.1) can be obtained using (2.4). The modes of (4.1) are the solutions of

$$\frac{\alpha-1}{x} - \frac{\beta-1}{x} + \frac{cI_{\nu-1}(cx)}{I_\nu(cx)} = \frac{\nu}{c} + c. \tag{4.3}$$

There could be more than one mode (see Figures 2.1 and 2.2). The nth moment of (4.1) can be written as

$$E(X^n) = C\int_0^1 x^{n+\alpha-1}(1-x)^{\beta-1}\exp(-cx)I_\nu(cx)dx \tag{4.4}$$

and an application of [12, equation (2.15.4.1)] shows that the above reduces to

$$E(X^n) = \frac{Cc^\nu\Gamma(n+\alpha+\nu)\Gamma(\beta)}{2^\nu\Gamma(n+\alpha+\beta+\nu)\Gamma(\nu+1)}{}_2F_2\left(\nu+\frac{1}{2}, n+\alpha+\nu; 2\nu+1, n+\alpha+\beta+\nu; -2c\right). \tag{4.5}$$

For a random sample $w_1,\dots,w_n$, the MLEs of the four parameters in (4.1) are the solutions of

$$\begin{gathered}\sum_{i=1}^{n}\operatorname{In} w_i = -\frac{n}{C}\frac{\partial C}{\partial\alpha},\\ \sum_{i=1}^{n}\operatorname{In}(1-w_i) = -\frac{n}{C}\frac{\partial C}{\partial\beta},\\ \sum_{i=1}^{n}\frac{w_iI_{\nu-1}(cw_i)}{I_\nu(cw_i)} - \sum_{i=1}^{n}w_i = \frac{n\nu}{c} - \frac{n}{C}\frac{\partial C}{\partial c}, \sum_{i=1}^{n}\frac{\partial I_\nu(cw_i)/\partial\nu}{I_\nu(cw_i)} = -\frac{n}{C}\frac{\partial C}{\partial\nu}.\end{gathered} \tag{4.6}$$

Assuming (4.1) as the prior, the Bayes estimate of the binomial parameter, say p, is

$$\begin{aligned}E(p\mid x) = {} & \frac{Cc^\nu\Gamma(1+x+\alpha+\nu)\Gamma(n-x+\beta)}{2^\nu\Gamma(n+1+\alpha+\beta+\nu)\Gamma(\nu+1)}\\ & \times {}_2F_2\left(\nu+\frac{1}{2}, 1+x+\alpha+\nu; 2\nu+1, n+1+\alpha+\beta+\nu; -2c\right),\end{aligned} \tag{4.7}$$

where n is the number of trials and x is the number of successes.

5. Application

We now illustrate an application of the proposed beta distributions to consumer price index data. We collected the data on this index for the six countries: United States, United Kingdom, Japan, Canada, Germany, and Australia. The data were extracted from the website http://www.globalfindata.com/ (go to "Sample Data" under "Database" and then look under "Consumer Price Indices" for the closing value of the index) and the range of data for each country is shown in Table 5.1.

Taking the ratio $W = X/(X+Y)$, we attempted to model the relative economic performance of each country against another over the range of overlapping years. This yields 15 data sets for the variable W. As expected, some of the data for W appeared to concentrate to a subinterval of $[0,1]$ and so suitable location-scale transformations were applied to make the data span from 0 to 1. For each data set, we fitted the standard beta distribution given by (1.1) and the BB III distribution given by (4.1) with ν fixed as $\nu = 1$. The

Table 5.1. Countries and years of data.

Country	Range of data
Australia	1901 to 2003
Canada	1910 to 2003
Germany	1923 to 2003
Japan	1868 to 2003
United Kingdom	1800 to 2003
United States	1820 to 2003

two distributions were fitted by the method of maximum likelihood. The MLEs of the two parameters in (1.1) are obtained by solving the equations

$$\begin{aligned} \sum_{i=1}^{n} \ln w_i &= n\Psi(\alpha) - n\Psi(\alpha+\beta), \\ \sum_{i=1}^{n} \ln(1-w_i) &= n\Psi(\beta) - n\Psi(\alpha+\beta), \end{aligned} \tag{5.1}$$

where $\Psi(x) = d\ln\Gamma(x)/dx$ is the digamma function. The MLEs of (α,β,c) in (4.1) with ν fixed as $\nu = 1$ are obtained by solving (4.6).

The results of the fits were remarkable. In each fit, the maximized log-likelihood for (4.1) turned up significantly higher than that for the standard beta model. Here, we give details for just two of the 15 data sets.

(i) For the (United States, United Kingdom) data set shown in Table A.1 of the appendix the fitted estimates were $\hat{\alpha} = 1.392$, $\hat{\beta} = 1.230$ with $\log L = 5.145$ for the standard beta model (1.1); and $\hat{\alpha} = 0.820$, $\hat{\beta} = -3.180$, $\hat{c} = 1.571$ with $\log L = 7.647$ for the BB III model (4.1). The corresponding fitted densities superimposed with the empirical density are shown in Figure 2.1 (the empirical density computed using the hist command in the *R* software package).

(ii) For the (United States, Germany) data set shown in Table A.2 of the appendix the fitted estimates were $\hat{\alpha} = 0.914$, $\hat{\beta} = 1.130$ with $\log L = 1.494$ for the standard beta model (1.1); and $\hat{\alpha} = 1.405$, $\hat{\beta} = 2.370$, $\hat{c} = 7.828 \times 10^{-6}$ with $\log L = 5.198$ for the BB III model (4.1). The corresponding fitted densities superimposed with the empirical density are shown in Figure 2.2 (the empirical density computed using the hist command in the *R* software package).

So, we can conclude at least in this situation that the beta Bessel models are better than the one based on the standard beta distribution.

Appendix

Tables A.1 and A.2 provide the data on consumer price indices for the United States and the United Kingdom (years of overlap: 1820–2003) and for the United States and Germany (years of overlap: 1923–2003).

Table A.1. Consumer price index data for the United States and the United Kingdom for the years 1820–2003.

Year	US CPI	UK CPI	Year	US CPI	UK CPI	Year	US CPI	UK CPI
1820	6.2	4.9	1882	7.7	4.1	1944	17.8	7.4
1821	5.9	4.3	1883	7.5	4.3	1945	18.2	7.5
1822	6.1	3.7	1884	7.3	3.9	1946	21.5	7.5
1823	5.8	4.0	1885	7.2	3.7	1947	23.4	7.8
1824	5.5	4.3	1886	7.2	3.4	1948	24.1	8.1
1825	5.6	5.0	1887	7.3	3.4	1949	23.6	8.4
1826	5.3	4.8	1888	7.3	3.4	1950	25.0	8.6
1827	5.5	4.5	1889	7.3	3.4	1951	26.5	9.6
1828	5.5	4.3	1890	7.3	3.4	1952	26.7	10.2
1829	5.6	4.3	1891	7.3	3.6	1953	26.9	10.4
1830	5.2	4.1	1892	7.3	3.6	1954	26.7	10.8
1831	5.4	4.5	1893	7.3	3.3	1955	26.8	11.3
1832	5.5	4.2	1894	7.1	3.5	1956	27.6	11.7
1833	5.6	3.9	1895	7.0	3.5	1957	28.4	12.2
1834	4.9	3.6	1896	7.0	3.4	1958	28.9	12.4
1835	5.7	3.7	1897	7.0	3.5	1959	29.4	12.4
1836	6.5	4.1	1898	7.1	3.5	1960	29.8	12.6
1837	6.9	4.2	1899	7.3	3.4	1961	30.0	13.2
1838	6.8	4.2	1900	7.4	3.4	1962	30.4	13.5
1839	6.8	4.5	1901	7.6	3.3	1963	30.9	13.8
1840	5.7	4.6	1902	7.8	3.3	1964	31.2	14.5
1841	5.7	4.5	1903	7.8	3.4	1965	31.8	15.1
1842	5.3	4.2	1904	7.9	3.4	1966	32.9	15.7
1843	4.9	3.7	1905	8.1	3.4	1967	33.9	16.1
1844	5.0	3.7	1906	8.5	3.4	1968	35.5	17.0
1845	5.2	3.9	1907	8.8	3.5	1969	37.7	17.8
1846	5.6	4.0	1908	8.8	3.4	1970	39.8	19.2
1847	5.6	4.5	1909	9.3	3.5	1971	41.1	20.9
1848	4.9	4.0	1910	9.3	3.5	1972	42.5	22.5
1849	5.2	3.7	1911	9.5	3.6	1973	46.2	24.9

Table A.1. Continued.

Year	US CPI	UK CPI	Year	US CPI	UK CPI	Year	US CPI	UK CPI
1850	4.9	3.5	1912	9.8	3.7	1974	51.9	29.6
1851	5.7	3.5	1913	10.0	3.8	1975	55.5	37.0
1852	5.7	3.5	1914	10.1	4.1	1976	58.2	42.6
1853	6.1	4.1	1915	10.3	5.0	1977	62.1	47.8
1854	6.1	4.6	1916	11.6	6.1	1978	67.7	51.8
1855	6.4	4.6	1917	13.7	6.8	1979	76.7	60.7
1856	6.5	4.5	1918	16.5	8.1	1980	86.3	69.9
1857	6.7	4.6	1919	18.9	8.3	1981	94.0	78.3
1858	6.6	4.3	1920	19.4	9.8	1982	97.6	82.5
1859	6.0	4.4	1921	17.3	7.1	1983	101.3	86.9
1860	5.8	4.7	1922	16.9	6.6	1984	105.3	90.9
1861	6.0	4.7	1923	17.3	6.5	1985	109.3	96.0
1862	6.6	4.6	1924	17.3	6.6	1986	110.5	99.6
1863	7.5	4.1	1925	17.9	6.4	1987	115.4	103.3
1864	9.1	4.3	1926	17.7	6.4	1988	120.5	110.3
1865	9.8	4.5	1927	17.3	6.2	1989	126.1	118.8
1866	9.9	4.7	1928	17.1	6.1	1990	133.8	129.9
1867	9.8	4.8	1929	17.2	6.1	1991	137.9	135.7
1868	9.4	4.6	1930	16.1	5.6	1992	141.9	139.2
1869	9.1	4.5	1931	14.6	5.4	1993	145.8	141.9
1870	8.7	4.5	1932	13.1	5.2	1994	149.7	146.0
1871	8.5	4.8	1933	13.2	5.2	1995	153.5	150.7
1872	8.6	5.0	1934	13.4	5.3	1996	158.6	154.4
1873	8.4	5.2	1935	13.8	5.4	1997	161.3	160.0
1874	8.4	5.1	1936	14.0	5.6	1998	163.9	164.4
1875	8.0	4.7	1937	14.4	5.9	1999	168.3	167.3
1876	7.7	4.9	1938	14.0	5.7	2000	174.0	172.2
1877	7.3	4.8	1939	14.0	6.4	2001	176.7	173.4
1878	6.8	4.6	1940	14.1	7.2	2002	180.9	178.5
1879	7.3	4.4	1941	15.5	7.4	2003	184.3	183.5
1880	7.5	4.2	1942	16.9	7.3	—	—	—
1881	7.9	4.4	1943	17.4	7.3	—	—	—

Table A.2. Consumer price index data for the United States and Germany for the years 1923–2003.

Year	US CPI	DE CPI	Year	US CPI	DE CPI	Year	US CPI	DE CPI
1923	17.3	17.8326	1950	25	22.8529	1977	62.1	55.7936
1924	17.3	17.2442	1951	26.5	25.7652	1978	67.7	57.1635
1925	17.9	17.9829	1952	26.7	26.9921	1979	76.7	60.2769
1926	17.7	18.3777	1953	26.9	26.2559	1980	86.3	63.5772
1927	17.3	19.2692	1954	26.7	26.9921	1981	94	67.8739
1928	17.1	19.4475	1955	26.8	27.3546	1982	97.6	70.9873
1929	17.2	19.4348	1956	27.6	27.8025	1983	101.3	72.8554
1930	16.1	18.0339	1957	28.4	28.3949	1984	105.3	74.2876
1931	14.6	16.6584	1958	28.9	28.7063	1985	109.3	75.5953
1932	13.1	15.0537	1959	29.4	29.3289	1986	110.5	74.7858
1933	13.2	15.3593	1960	29.8	29.5781	1987	115.4	75.533
1934	13.4	15.5631	1961	30	30.3875	1988	120.5	76.7784
1935	13.8	15.7159	1962	30.4	31.2593	1989	126.1	79.2839
1936	14	15.8306	1963	30.9	32.3179	1990	133.8	81.4561
1937	14.4	15.8894	1964	31.2	33.0029	1991	137.9	84.1
1938	14	15.9679	1965	31.8	34.3105	1992	141.9	86.9
1939	14	16.1052	1966	32.9	35.3069	1993	145.8	90.6
1940	14.1	16.6545	1967	33.9	35.4936	1994	149.7	92.9
1941	15.5	16.9487	1968	35.5	36.3032	1995	153.5	94.3
1942	16.9	17.3018	1969	37.7	37.0504	1996	158.6	95.7
1943	17.4	17.6353	1970	39.8	38.5448	1997	161.3	97.6
1944	17.8	17.9688	1971	41.1	40.662	1998	163.9	98
1945	18.2	18.8093	1972	42.5	43.2774	1999	168.3	99.1
1946	21.5	20.745	1973	46.2	46.6399	2000	174	101.2
1947	23.4	21.318	1974	51.9	49.3175	2001	176.7	102.8
1948	24.1	25.4258	1975	55.5	51.995	2002	180.9	104
1949	23.6	23.6096	1976	58.2	53.9254	2003	184.3	105.1

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