DIMENSIONS OF SUBSETS OF CANTOR-TYPE SETS

FANG-XIONG ZHEN

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We define the Cantor-type set E first, and then the Besicovitch subset $B_{\mathbf{p}}$ of E. We mainly show the dimensions of subsets of Cantor-type set E in compatible case and incompatible case.

1. Introduction

Let $I = [0,1]$ be the unit interval on the real line and $m > 1$ be an integer. Let $J = \{0,1,\dots,m-1\}$. For every point $x \in I$, there is a unique base-m representation $x = \Sigma_{k=1}^{\infty} j_k m^{-k}$ with $j_k \in J$ except for countable many points. Since countable sets do not interfere with our work, we neglect them here.

For each $j \in J$, $x \in [0,1]$, and $n \in \mathbb{N}$, let $\tau_j(x,n) = \sharp\{k : i_k = j,\ 1 \le k \le n\}$, then the limit $\tau_j(x) = \lim_{n\to\infty}(1/n)\tau_j(x,n)$ is called the frequency of number j in the base-m representation of x. Here and in the following context, the notation "$\sharp A$" denotes the number of elements in set A.

A classical result of Borel [3] says that for Lebesgue almost every $x \in [0,1]$, we have $\tau_j(x) = 1/m$. As for another problem, for a given probability vector $\mathbf{p} = (p_0, p_1, \dots, p_{m-1})$ such that $\Sigma_{j\in J} p_j = 1$, consider the set

$$\Lambda_m(p_0,p_1,\dots,p_{m-1}) = \{x \in [0,1] : \tau_j(x) = p_j, \text{ for } j \in J\}. \tag{1.1}$$

That is, the set $\Lambda_m(p_0,p_1,\dots,p_{m-1})$ is composed of the number in $[0,1]$ having a ratio p_j of digits equal to j in its base-m representation for each j. A precursor theory is due to Besicovitch, when he showed in [1] that if $p \in (0,1/2)$, then

$$\dim \Lambda_2(p,1-p) = \frac{p\log p + (1-p)\log(1-p)}{-\log 2}. \tag{1.2}$$

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Eggleston generalized this formula for $m > 2$. He showed in [4] that

$$\dim \Lambda_m(p_0, p_1, \dots, p_{m-1}) = -\frac{\Sigma_{j\in J} p_j \log p_j}{\log m}. \tag{1.3}$$

From then on, $\Lambda_m(p_0, p_1, \dots, p_{m-1})$ is called Besicovitch set. And later, Billingsley furthered their work and the related information can be found in his book [2].

The research developed in this paper is motivated by Morán and Rey's work, and some of the proofs are edified by the methods mentioned in their article [7]. Let $M = \{1,2,\dots,m\}$ and for any $j \in M$, $\omega = i_1 i_2 \cdots i_n \cdots \in M^\infty$, $n \in \mathbb{N}$, we define

$$\delta_j(\omega, n) = \sharp\{k : i_k = j,\ 1 \le k \le n\}. \tag{1.4}$$

Whenever there exists the limit

$$\delta_j(\omega) = \lim_{n\to\infty} \frac{1}{n}\delta_j(\omega, n), \tag{1.5}$$

it is called the frequency of number j in infinite length word ω. Then for a given probability vector $\mathbf{p} = (p_1, p_2, \dots, p_m)$, we define the Besicovitch set $B_\mathbf{p}$ to be the subset of Cantor-type set E (see Section 2) given by

$$B_\mathbf{p} = \{\varphi(\omega) : \delta_j(\omega) = p_j,\ \omega \in M^\infty, \text{ for any } j \in M\}. \tag{1.6}$$

Here φ is a bijective between M^∞ and the Cantor-type set E and M^∞ is the set of all infinite length sequences consisted by M. A remark from [7] says that $B_\mathbf{p}$ is a Borel set. In this paper, we give the Hausdorff dimension of $B_\mathbf{p}$ for any given probability vector $\mathbf{p} = (p_1, p_2, \dots, p_m)$ with $p_j > 0$ for $j \in M$ as the following theorem.

Theorem 1.1. *Denote*

$$s = \frac{\sum_{j\in M} p_j \log p_j}{\sum_{j\in M} p_j \log r_j}, \tag{1.7}$$

$\dim B_\mathbf{p} = s$*, and* $s \le \alpha$*, where* α *is uniquely decided by* $\Sigma_{j\in M} r_j^\alpha = 1$*. The equality is attained when* $\mathbf{p}$ *coincides with* $(r_1^\alpha, r_2^\alpha, \dots, r_m^\alpha)$*.*

Furthermore, we will show that there exists a subset B of Besicovitch set $B_\mathbf{p}$ such that B has full $\mu_\mathbf{p}$-measure (see Section 2) but zero Hausdorff measure when $\mathbf{p} \ne (r_1^\alpha, r_2^\alpha, \dots, r_m^\alpha)$, which is described as follows.

Theorem 1.2. *There exists a Borel subset* B *of* $B_\mathbf{p}$ *such that* $\mu_\mathbf{p}(B) = 1$ *and* $\mathcal{H}^s(B) = 0$ *in the incompatible case, that is,* $\mathbf{p} \ne (r_1^\alpha, r_2^\alpha, \dots, r_m^\alpha)$*.*

As a corollary of Theorem 1.2, we can easily have that $\mu_\mathbf{p}$, $\mathcal{H}^\alpha$, and $\mathcal{P}^\alpha$ are equivalent on E in the compatible case, that is, $\mathbf{p} = (r_1^\alpha, r_2^\alpha, \dots, r_m^\alpha)$ (see Corollary 3.7). Furthermore, we can get another byproduct (see Corollary 3.8) that the set $E \setminus B_\mathbf{p}$ has zero $\mathcal{H}^\alpha$ and $\mathcal{P}^\alpha$-measure when $\mathbf{p}$ is substituted by $(r_1^\alpha, r_2^\alpha, \dots, r_m^\alpha)$.

If the gap condition (see Section 2) holds, we will see that not only $B_\mathbf{p}$ and the Cantor-type set E have the same Hausdorff measure and packing measure, but also both of them are α-sets in the case of $\mathbf{p} = (r_1^\alpha, r_2^\alpha, \dots, r_m^\alpha)$ (see Corollary 3.11). This can be regarded as a corollary from Corollary 3.7. When it comes to the incompatible case, the Hausdorff measure of $B_\mathbf{p}$ is infinity (Proposition 3.12), and so is the packing measure.

From Corollary 3.8, we see that the complementary of $B_\mathbf{p}$ with respect to E has zero $\mathcal{H}^\alpha$-measure in the compatible case, which implies that it is rather small from the viewpoint of measure theory.

The following section will give the basic definitions and notations of the paper, and the proofs of the above results are displayed in Section 3.

2. Definitions and notations

Let $I = [0,1]$ be the unit interval and $M = \{1,2,\dots,m\}$. Let Ω_n be the set of all sequences of length n with each sequence consisting of letters in M and let $\Omega = \bigcup_{n=1}^\infty \Omega_n$. Suppose that a sequence of closed subsets $\{I_\omega : \omega \in \Omega_n\}$ of I satisfies

(1) $I_{\omega * i} \subset I_\omega$ for any $\omega \in \Omega$ and $i \in M$,

(2) $I_{\omega * i} \cap I_{\omega * j} = \phi$ for any $\omega \in \Omega$ and $i, j \in M$ with $i \neq j$,

(3) $|I_{\omega * i}|/|I_\omega| = r_i$ with $r_i \in (0,1)$ for any $\omega \in \Omega$ and $i \in M$.

Then the set $E = \bigcap_{n=1}^\infty \bigcup_{\omega \in \Omega_n} I_\omega$ is called a Cantor-type set.

Notice that $\bigcap_{n=1}^\infty I_{i_1 i_2 \cdots i_n}$ consists of a single point which we will denote by $\varphi(\omega)$ since $\omega \in \Omega_n$, and we write $I_{i_1 i_2 \cdots i_n}$ for I_ω sometimes. We can easily see that $\varphi : M^\infty \to E$ is 1-1 and onto.

We denote by $\nu_\mathbf{p}$ the probability measure on M^∞ defined by the infinite product $\mathbf{p} \times \mathbf{p} \times \mathbf{p} \times \cdots$. Let $\mu_\mathbf{p}$ be the projected measure supported by E, that is, $\mu_\mathbf{p} = \nu_\mathbf{p} \circ \varphi^{-1}$, then for any n-level basic interval $I_\omega (\omega = i_1 i_1 \cdots i_n \in \Omega_n)$, we have $\mu_\mathbf{p}(I_\omega) = \Pi_{j=1}^n p_{i_j}$. By the *strong law of large numbers*, we see that $\mu_\mathbf{p}(B_\mathbf{p}) = 1$ and since $B_\mathbf{p} \subset E$, thus $\mu_\mathbf{p}(B_\mathbf{p}) = \mu_\mathbf{p}(E) = 1$. Such a measure $\mu_\mathbf{p}$ is called Bernoulli measure.

For the use in next section, we give some notations here.

Let $\mathcal{F} = \{I_\omega : \omega \in \Omega\}$, we define

$$\mathcal{H}^s_{\delta,\mathcal{F}}(E) = \inf\left\{\Sigma \left| I^j_\omega \right|^s : \{I^j_\omega\} \subset \mathcal{F},\ \left| I^j_\omega \right| \le \delta,\ E \subset \bigcup_j I^j_\omega\right\},$$
$$\mathcal{H}^s_{\mathcal{F}}(E) = \lim_{\delta \to 0} \mathcal{H}^s_{\delta,\mathcal{F}}(E). \tag{2.1}$$

Similar to the definition of Hausdorff dimension, we define

$$\dim_{\mathcal{F}} E = \sup\{s \ge 0 : \mathcal{H}^s_{\mathcal{F}}(E) = \infty\}$$
$$= \inf\{s \ge 0 : \mathcal{H}^s_{\mathcal{F}}(E) = 0\}. \tag{2.2}$$

Now, we would like to define $\mu^s_\mathbf{p}$-measure as follows:

$$\mu^s_\mathbf{p}(E,\delta) = \inf\left\{\Sigma \mu_\mathbf{p}(U_i)^s : \mu_\mathbf{p}(U_i) \le \delta,\ E \subset \bigcup_i U_i\right\},$$
$$\mu^s_\mathbf{p}(E) = \lim_{\delta \to 0} \mu^s_\mathbf{p}(E,\delta) \tag{2.3}$$

then the $\mu_{\mathbf{p}}$-dimension of E is given by

$$\begin{aligned}\dim_{\mu_{\mathbf{p}}} E &= \sup\{s \geq 0 : \mu_{\mathbf{p}}^s(E) = \infty\} \\ &= \inf\{s \geq 0 : \mu_{\mathbf{p}}^s(E) = 0\}.\end{aligned} \tag{2.4}$$

We see that if $\mu_{\mathbf{p}}(E) > 0$, then $\dim_{\mu_{\mathbf{p}}} E = 1$.

3. Proofs

Now we set to prove the results displayed in Section 1. First, let us look at some lemmas.

Lemma 3.1 [5]. *Suppose that $\mathbf{p} = (p_1, p_2, \dots, p_m)$ is a probability vector with $\Sigma_{j\in M} p_j = 1$ and $p_j > 0$. For any $q_1, \dots, q_m \in \mathbb{R}$,*

$$\Sigma_{i=1}^m p_i(-\log p_i + q_i) \leq \log \Sigma_{i=1}^m e^{q_i}. \tag{3.1}$$

The equality is attained if and only if $p_i = e^{q_i}(\Sigma_{i=1}^m e^{q_i})^{-1}$ for $i \in M$.

Lemma 3.2. *For any subset $F \subset E$, $\alpha \geq 0$, there exists a constant $c > 0$ such that $c\mathcal{H}_{\mathcal{F}}^\alpha(F) \leq \mathcal{H}^\alpha \leq \mathcal{H}_{\mathcal{F}}^\alpha(F)$.*

Proof. Denote $r = \min_{1\leq i\leq m}\{r_i\}$, $A_r^{(n)} = \{I_\omega : r^n < |I_\omega| \leq r^{n-1}\}$. For any δ-covering $\{U_i\}$ of F, for any U_i, there exists n_i such that $r^{n_i} < |U_i| \leq r^{n_i-1}$. For any $x \in U_i$, $F \cap U_i \subset \bigcup_{I_\omega \in A_r^{n_i} I_\omega \cap U_i \neq \phi} I_\omega \subset B(x, 2r^{n_i-1})$, then we have $\sharp\{I_\omega : I_\omega \in A_r^{n_i} I_\omega \cap U_i \neq \phi\} \leq 4r^{n_i-1}/r^{n_i} = 4/r$, so for this U_i, there exists a sequence $\{I_{ij}^{n_i}\}_{i=1,2,\dots}^{j=1,2,\dots,l_i} \subset A_r^{n_i}$ such that $F \cap U_i \subset \bigcup_{j=1}^{l_i} I_{ij}^{(n_i)}$, $l_i \leq 4/r$. Since $|U_i| \geq r^{n_i} \geq r|I_{ij}^{(n_i)}|$ for $j = 1, 2, \dots, l_i$, then $l_i|U_i|^\alpha \geq \Sigma_{j=1}^{l_i} r^\alpha |I_{ij}^{(n_i)}|^\alpha$, we further get $(4/r)\Sigma_{i=1}^\infty |U_i|^\alpha \geq r^\alpha \Sigma_{i=1}^\infty \Sigma_{j=1}^{l_i} |I_{ij}^{(n_i)}|^\alpha$, thus we get $\mathcal{H}^\alpha(F) \geq (r^{\alpha+1}/4)\mathcal{H}_{\mathcal{F}}^\alpha(F)$. Letting $c = r^{\alpha+1}/4$, we have $c\mathcal{H}_{\mathcal{F}}^\alpha(F) \leq \mathcal{H}^\alpha \leq \mathcal{H}_{\mathcal{F}}^\alpha(F)$. □

To prove Theorem 1.1, we need another lemma.

Lemma 3.3. *If*

$$E \subset \left\{\varphi(\omega) : \lim_{|\omega|\to\infty} \frac{\log \mu_{\mathbf{p}}(I_\omega)}{\log |I_\omega|} = \delta\right\}, \tag{3.2}$$

then $\dim E = \delta \dim_{\mu_{\mathbf{p}}} E$.

Proof. With Lemma 3.2, we can get the same dimension result if we use the basic intervals to cover E, so we first prove that if

$$E \subset \left\{\varphi(\omega) : \liminf_{|\omega|\to\infty} \frac{\log \mu_{\mathbf{p}}(I_\omega)}{\log |I_\omega|} \geq \delta\right\}, \tag{3.3}$$

then $\dim E \geq \delta \dim_{\mu_{\mathbf{p}}} E$.

If $\varphi(\omega) \in E$, then for any $\varepsilon > 0$, there exists $N > 0$ such that for any $n > N$, we have $\mu_{\mathbf{p}}(I_\omega) \leq |I_\omega|^{\delta-\varepsilon}$. Thus there are only many finite ω such that I_ω does not satisfy the above inequality. Let

$$E_k = \left\{\varphi(\omega) \in E : |I_\omega| > \frac{1}{k} \text{ or } \mu_{\mathbf{p}}(I_\omega) \le |I_\omega|^{\delta-\varepsilon}\right\}. \tag{3.4}$$

We see that E_k is increasing. On the other hand, since we have $\mu_{\mathbf{p}}(I_\omega) > 0$ for any $\omega \in \Omega$, then there exists k such that $|I_\omega| > 1/k$ for any given $\omega \in \Omega$, which implies that $\varphi(\omega) \in E_k$. Thus we have $E = \bigcup_k E_k$.

For any $(1/k >)\rho > 0$, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, there exists a sequence of intervals $\{I_\omega^j\} \subset \mathscr{F}$ such that $|I_\omega^j| \le \rho$, $I_\omega^j \cap E_k \ne \phi$, and $E_k \subset \bigcup_j I_\omega^j$. Since $\dim_{\mathscr{F}} E = \dim E$ and $E_k \subset E$, we have $\mathscr{H}_\rho^{\dim E+\varepsilon_2}(E_k) < \varepsilon_1$, that is, $\Sigma|I_\omega^j|^{\dim E+\varepsilon_2} < \varepsilon_1$, which follows $\Sigma(\mu_{\mathbf{p}}(I_\omega^j))^{(\dim E+\varepsilon_2)/(\delta-\varepsilon)} < \varepsilon_1$. Thus we have $\dim_{\mu_{\mathbf{p}}} E_k \le (\dim E + \varepsilon_2)/(\delta - \varepsilon)$. Since ε and ε_2 are arbitrary small, we have $\dim E \ge \delta \dim_{\mu_{\mathbf{p}}} E_k$, which further implies $\dim E \ge \delta \dim_{\mu_{\mathbf{p}}} E$.

With the same method we can prove that if

$$E \subset \left\{\varphi(\omega) : \limsup_{|\omega|\to\infty} \frac{\log \mu_{\mathbf{p}}(I_\omega)}{\log |I_\omega|} \le \delta\right\}, \tag{3.5}$$

then $\dim E \le \delta \dim_{\mu_{\mathbf{p}}} E$.

With the above arguments, we complete the proof. □

Proof of Theorem 1.1. For any $\omega \in \Omega$,

$$\begin{aligned}
\lim_{|\omega|\to\infty} \frac{\log \mu_{\mathbf{p}}(I_\omega)}{\log |I_\omega|} &= \lim_{|\omega|\to\infty} \frac{\Sigma_{j=1}^n \log p_{i_j}}{\Sigma_{j=1}^n \log r_{i_j}} \\
&= \lim_{|\omega|\to\infty} \frac{\Sigma_{j\in M} \delta_j(\omega,n) \log p_j}{\Sigma_{j\in M} \delta_j(\omega,n) \log r_j} \\
&= \frac{\Sigma_{j\in M} p_j \log p_j}{\Sigma_{j\in M} p_j \log r_j} = s.
\end{aligned} \tag{3.6}$$

Since $\mu_{\mathbf{p}}(B_{\mathbf{p}}) = 1$, by Lemma 3.3 we have $\dim B_{\mathbf{p}} = s$.

Put $q_i = \log r_i^\alpha$ in Lemma 3.1, we get $s \le \alpha$. Furthermore, if we put

$$p_i = e^{\log r_i^\alpha} \left(\Sigma_{i=1}^m e^{\log r_i^\alpha}\right)^{-1} = r_i^\alpha, \tag{3.7}$$

we have $s = \alpha$. □

Lemma 3.4 (law of iterated logarithm (Hartman-Winter)). *Suppose that $\{X_n : n \ge 1\}$ is a sequence of independent, identically distributed random variables (i.i.d.r.v.) satisfying $\mathbf{E}(X_n) = 0$ and $\mathbf{E}(X_n^2) = \sigma^2 \in (0, +\infty)$, then*

$$\begin{aligned}
\limsup_{n\to\infty} \frac{\Sigma_{i=1}^n X_i}{\sqrt{2n\sigma^2 \log\log n\sigma^2}} &= 1, \quad p \text{ a.e.}, \\
\liminf_{n\to\infty} \frac{\Sigma_{i=1}^n X_i}{\sqrt{2n\sigma^2 \log\log n\sigma^2}} &= -1 \quad p \text{ a.e.}
\end{aligned} \tag{3.8}$$

Remark 3.5. We suppose that $\mathbf{E}$ denotes the mathematical expectation and $\mathbf{V}$ denotes the mean variance here and in the following context.

Proof of Theorem 1.2. Put $p = \mu_{\mathbf{p}} \circ \varphi$, $X_i = \log p_i - s \log r_i$, $i \in M$, then $\{X_i\}$ is a sequence of i.i.d.r.v. with respect to p. We see that $\mathbf{E}(X_i) = \Sigma_{i=1}^m p_i(\log p_i - s \log r_i) = 0$. Suppose that $\mathbf{p} \neq (r_1^\alpha, \dots, r_m^\alpha)$, then

$$0 < \mathbf{E}(X_i^2) = \Sigma_{i=1}^m p_i \left(\log p_i - s \log r_i\right)^2 =: \sigma^2 < \infty. \tag{3.9}$$

Set

$$B' = \left\{ \varphi(\omega) : \limsup_{n \to \infty} \frac{\Sigma_{i=1}^n X_i}{\sqrt{2n\sigma^2 \log\log n\sigma^2}} = 1 \right\}, \tag{3.10}$$

then $\mu_{\mathbf{p}}(B') = 1$ by Lemma 3.4. Since $\mu_{\mathbf{p}}(B_{\mathbf{p}}) = 1$. Let $B = B' \cap B_{\mathbf{p}}$, we get $\mu_{\mathbf{p}}(B) = 1$.

With the above arguments, we see that $\Sigma_{j=1}^\infty X_{i_j} = \infty$ a.e., that is,

$$\log \frac{\Pi_{j=1}^n p_{i_j}}{\Pi_{j=1}^n r_{i_j}^s} \longrightarrow \infty, \quad \text{as } n \longrightarrow \infty, \tag{3.11}$$

from which we also have

$$\frac{\mu_{\mathbf{p}}(I_\omega)}{\left| I_\omega \right|^s} \longrightarrow \infty, \quad \text{as } |\omega| \longrightarrow \infty. \tag{3.12}$$

Let $\mathcal{J}_n^k = \{I_\omega : \mu_{\mathbf{p}}(I_\omega)/|I_\omega|^s > k, \text{ for } \omega \in \Omega_n\}$, $\mathcal{J}^k = \bigcup_{n=1}^\infty \mathcal{J}_n^k$. For any $\delta > 0$, $k > 0$, choose $\{I_j^k\} \subset \mathcal{J}^k$ such that $B \subset \bigcup_j I_j^k$, $|I_j^k| \leq \delta$, and $I_i^k \cap I_j^k = \phi (i \neq j)$. By the definition of $\mathcal{J}^k$, we have

$$\frac{\mu_{\mathbf{p}}(I_j^k)}{\left| I_j^k \right|^s} > k, \quad \forall j. \tag{3.13}$$

Thus

$$\mathcal{H}_\delta^s(B) \leq \Sigma \left| I_j^k \right|^s \leq \frac{1}{k} \Sigma \mu_{\mathbf{p}}(I_j^k) \leq \frac{1}{k}. \tag{3.14}$$

Letting $k \to \infty$, we have $\mathcal{H}_\delta^s(B) = 0$, from which it follows that $\mathcal{H}^s(B) = 0$. □

LEMMA 3.6 [6]. *Suppose that μ is a finite Borel measure such that*

$$0 < \limsup_{|\omega| \to \infty} \frac{\mu(I_\omega)}{\left| I_\omega \right|^\alpha} < \infty, \quad 0 < \liminf_{|\omega| \to \infty} \frac{\mu(I_\omega)}{\left| I_\omega \right|^\alpha} < \infty, \quad \textit{for any } \omega \in \Omega, \tag{3.15}$$

then $\mu \sim \mathcal{H}^\alpha \sim \mathcal{P}^\alpha$ on E.

COROLLARY 3.7. *$\mu_{\mathbf{p}} \sim \mathcal{H}^\alpha \sim \mathcal{P}^\alpha$ on E in case of $\mathbf{p} = (r_1^\alpha, r_2^\alpha, \dots, r_m^\alpha)$.*

Proof. Since $\mu_{\mathbf{p}}(I_\omega) = \Pi_{j=1}^{n} p_{i_j}$, for $\omega = i_1 i_2 \cdots i_n \in \Omega_n$, then

$$\frac{\mu_{\mathbf{p}}(I_\omega)}{|I_\omega|^\alpha} = \frac{\Pi_{j=1}^{n} p_{i_j}}{\Pi_{j=1}^{n} r_{i_j}^\alpha} = 1, \quad \text{when } \mathbf{p} = (r_1^\alpha, r_2^\alpha, \ldots, r_m^\alpha). \tag{3.16}$$

So by Lemma 3.6, we have $\mu_{\mathbf{p}} \sim \mathcal{H}^\alpha \sim \mathcal{P}^\alpha$ on E in case of $\mathbf{p} = (r_1^\alpha, r_2^\alpha, \ldots, r_m^\alpha)$. □

Corollary 3.8. *$\mathcal{H}^\alpha(E \setminus B_{\mathbf{p}}) = \mathcal{P}^\alpha(E \setminus B_{\mathbf{p}}) = 0$, $\mathcal{H}^\alpha(E) = \mathcal{H}^\alpha(B_{\mathbf{p}})$, and $\mathcal{P}^\alpha(E) = \mathcal{P}^\alpha(B_{\mathbf{p}})$ in case of $\mathbf{p} = (r_1^\alpha, r_2^\alpha, \ldots, r_m^\alpha)$.*

Proof. By the definition of equivalence, $\mu_{\mathbf{p}} \sim \mathcal{H}^\alpha$ on $E \Leftrightarrow$ for any $F \subset E$, $\mu_{\mathbf{p}}(F) = 0 \Leftrightarrow \mathcal{H}^\alpha(F) = 0$. It is clear that $B_{\mathbf{p}}$ is the subset of E with $\mu_{\mathbf{p}}(E) = \mu_{\mathbf{p}}(B_{\mathbf{p}}) = 1$, then we get immediately $\mu_{\mathbf{p}}(E \setminus B_{\mathbf{p}}) = 0$, which further implies $\mathcal{H}^\alpha(E \setminus B_{\mathbf{p}}) = 0$ by Corollary 3.7. Furthermore, $\mathcal{H}^\alpha(B_{\mathbf{p}}) \le \mathcal{H}^\alpha(E) \le \mathcal{H}^\alpha(B_{\mathbf{p}}) + \mathcal{H}^\alpha(E \setminus B_{\mathbf{p}}) = \mathcal{H}^\alpha(B_{\mathbf{p}})$, we have $\mathcal{H}^\alpha(E) = \mathcal{H}^\alpha(B_{\mathbf{p}})$.

Similar arguments will prove that $\mathcal{P}^\alpha(E \setminus B_{\mathbf{p}}) = 0$ and $\mathcal{P}^\alpha(E) = \mathcal{P}^\alpha(B_{\mathbf{p}})$. □

For the simplicity of proof of the following results, we would like to make a hypothesis here.

We say that the Cantor-type set E satisfies the gap condition if there exists a constant $\delta > 0$ such that

$$\operatorname{dist}(I_{i_1 i_2 \cdots i_n}, I_{i_1 i_2 \cdots i_n'}) \ge \delta \max\{|I_{i_1 i_2 \cdots i_n}|, |I_{i_1 i_2 \cdots i_n'}|\} \tag{3.17}$$

for any $n \in \mathbb{N}$ and $i_n \ne i_n'$.

Lemma 3.9 [6]. *If the gap condition holds, then for any $\omega \in \Omega$ and finite Borel measure μ, the following inequalities hold:*

$$\begin{aligned} (a\delta)^\alpha \liminf_{|\omega| \to \infty} \frac{|I_\omega|^\alpha}{\mu(I_\omega)} &\le \liminf_{r \to 0} \frac{r^\alpha}{\mu(B(\varphi(\omega), r))} \le a^{-\alpha} \liminf_{|\omega| \to \infty} \frac{|I_\omega|^\alpha}{\mu(I_\omega)}, \\ (a\delta)^\alpha \limsup_{|\omega| \to \infty} \frac{|I_\omega|^\alpha}{\mu(I_\omega)} &\le \limsup_{r \to 0} \frac{r^\alpha}{\mu(B(\varphi(\omega), r))} \le a^{-\alpha} \limsup_{|\omega| \to \infty} \frac{|I_\omega|^\alpha}{\mu(I_\omega)}, \end{aligned} \tag{3.18}$$

where $a = \min_{i \in M}\{r_i\}$.

Lemma 3.10 [8]. *For finite Borel measure μ, there exists constants c_1, c_2, c_3, c_4 such that*

$$\begin{aligned} c_1 \mu(E) \inf_{x \in E} \liminf_{r \to 0} \frac{r^\alpha}{\mu(B(x, r))} &\le \mathcal{H}^\alpha(E) \le c_2 \mu(\mathbb{R}) \sup_{x \in E} \liminf_{r \to 0} \frac{r^\alpha}{\mu(B(x, r))}, \\ c_3 \mu(E) \inf_{x \in E} \limsup_{r \to 0} \frac{r^\alpha}{\mu(B(x, r))} &\le \mathcal{P}^\alpha(E) \le c_4 \mu(\mathbb{R}) \sup_{x \in E} \limsup_{r \to 0} \frac{r^\alpha}{\mu(B(x, r))}. \end{aligned} \tag{3.19}$$

Corollary 3.11. *If E is the Cantor-type set satisfying gap condition and $B_{\mathbf{p}}$ is the Besicovitch set defined in Section 1, then $0 < \mathcal{H}^\alpha(E) = \mathcal{H}^\alpha(B_{\mathbf{p}}) < \infty$ and $0 < \mathcal{P}^\alpha(E) = \mathcal{P}^\alpha(B_{\mathbf{p}}) < \infty$, when $\mathbf{p} = (r_1^\alpha, r_2^\alpha, \ldots, r_m^\alpha)$.*

Proof. From Lemma 3.10, we see that the density of balls is comparable with that of intervals. So we have

$$c_1'\mu(E)\inf_{x\in E}\liminf_{|\omega|\to\infty}\frac{|I_\omega|^\alpha}{\mu(I_\omega)} \le \mathscr{H}^\alpha(E) \le c_2'\mu(\mathbb{R})\sup_{x\in E}\liminf_{|\omega|\to\infty}\frac{|I_\omega|^\alpha}{\mu(I_\omega)} \tag{3.20}$$

together with Lemmas 3.9 and 3.10.

Since

$$\frac{\mu_{\mathbf{p}}(I_\omega)}{|I_\omega|^\alpha} = 1, \quad \text{in case of } \mathbf{p} = (r_1^\alpha, r_2^\alpha, \dots, r_m^\alpha), \tag{3.21}$$

so we get $c_1' \le \mathscr{H}^\alpha(E) \le c_2'$ by substituting μ with $\mu_{\mathbf{p}}$ in (3.20), which implies that $0 < \mathscr{H}^\alpha(E) < \infty$. Since we have proved in Corollary 3.8 that $\mathscr{H}^\alpha(E) = \mathscr{H}^\alpha(B_{\mathbf{p}})$, we have $0 < \mathscr{H}^\alpha(E) = \mathscr{H}^\alpha(B_{\mathbf{p}}) < \infty$.

Analogously, we can prove $0 < \mathscr{P}^\alpha(E) = \mathscr{P}^\alpha(B_{\mathbf{p}}) < \infty$. □

Remark 3.12. Corollary 3.11 reinforces the assertion in Theorem 1.1.

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Fang-Xiong Zhen: Department of Mathematics, Xianning College, Xianning, Hubei 437005, China
E-mail address: cookie4993@sina.com