LERAY-SCHAUDER RESULTS FOR MULTIVALUED NONLINEAR CONTRACTIONS DEFINED ON CLOSED SUBSETS OF A FRÉCHET SPACE

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New Leray-Schauder results are presented for multivalued contractions defined on subsets of a Fréchet space E. The proof relies on fixed point results in Banach spaces and on viewing E as the projective limit of a sequence of Banach spaces.

1. Introduction

In this paper, we present new fixed point results for nonlinear contractions (both single and multivalued) defined on subsets X (which may have empty interior) of a Fréchet space E. Some results for single-valued maps were presented in [2, 3] and the approach in these papers was based on constructing a specific map F_n (for each $n \in \mathbb{N} = \{1,2,\dots\}$) whose fixed points converge to a fixed point of the original operator F. In the approach in this paper, the maps $\{F_n\}_{n\in\mathbb{N}}$ only need to satisfy a closure property and are specified in a completely different way. The advantage of this approach is that multivalued maps can also be discussed. Our theory is based on results in Banach spaces and on viewing a Fréchet space E as a projective limit of a sequence of Banach spaces $\{E_n\}_{n\in\mathbb{N}}$.

For the remainder of this section, we present some definitions and some known facts. Let (X,d) be a metric space and S a nonempty subset of X. For $x \in X$, let $d(x,S) = \inf_{y\in S} d(x,y)$. Also $\operatorname{diam} S = \sup\{d(x,y) : x,y \in S\}$. We let $B(x,r)$ denote the open ball in X centered at x of radius r and by $B(S,r)$ we denote $\bigcup_{x\in S} B(x,r)$. For two nonempty subsets S_1 and S_2 of X, we define the generalized Hausdorff distance H to be

$$H(S_1,S_2) = \inf\{\epsilon > 0 : S_1 \subseteq B(S_2,\epsilon),\ S_2 \subseteq B(S_1,\epsilon)\}. \tag{1.1}$$

Now suppose $G : S \to 2^X$; here 2^X denotes the family of nonempty subsets of X. Then G is said to be hemicompact if each sequence $\{x_n\}_{n\in\mathbb{N}}$ in S has a convergent subsequence whenever $d(x_n, G(x_n)) \to 0$ as $n \to \infty$.

We now recall a result from the literature.

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Theorem 1.1. *Let (X,d) be a complete metric space, $\mathbb{C} \subseteq X$ closed, and $F : \mathbb{C} \to X$ with $F(\mathbb{C})$ bounded (i.e., there exists $M > 0$ with $d(z,w) \leq M$ for $z,w \in F(\mathbb{C})$). Suppose the following condition is satisfied:*

$$\begin{aligned}&\text{there exists a continuous nondecreasing function}\\&\phi : [0,\infty) \longrightarrow [0,\infty) \text{ satisfying } \phi(z) < z \text{ for } z > 0\\&\text{such that } d(Fx,Fy) \leq \phi\big(d(x,y)\big) \text{ for } x,y \in \mathbb{C}.\end{aligned} \tag{1.2}$$

Then F is hemicompact.

Now let I be a directed set with order $\leq$ and let $\{E_\alpha\}_{\alpha\in I}$ be a family of locally convex spaces. For each $\alpha \in I, \beta \in I$ for which $\alpha \leq \beta$, let $\pi_{\alpha,\beta} : E_\beta \to E_\alpha$ be a continuous map. Then the set

$$\left\{x = (x_\alpha) \in \prod_{\alpha\in I} E_\alpha : x_\alpha = \pi_{\alpha,\beta}(x_\beta)\ \forall \alpha,\beta \in I,\ \alpha \leq \beta\right\} \tag{1.3}$$

is a closed subset of $\prod_{\alpha\in I} E_\alpha$ and is called the projective limit of $\{E_\alpha\}_{\alpha\in I}$ and is denoted by $\lim_{\leftarrow} E_\alpha$ (or $\lim_{\leftarrow}\{E_\alpha,\pi_{\alpha,\beta}\}$ or the generalized intersection [5, page 439] $\bigcap_{\alpha\in I} E_\alpha$).

Existence in Section 2 is based on the following fixed point results in the literature [1, 6].

Theorem 1.2 [6, Theorem 3.9]. *Let U be an open subset in a Banach space $(X,\|\cdot\|)$ and $F : \overline{U} \to X$. Assume $0 \in U$ and suppose there exists a continuous nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that $\|Fx - Fy\| \leq \phi(\|x - y\|)$ for all $x,y \in \overline{U}$. In addition, assume $F(\overline{U})$ is bounded and $x \neq \lambda Fx$ for $x \in \partial U$ and $\lambda \in (0,1)$. Then F has a fixed point in $\overline{U}$.*

Theorem 1.3 [1, Theorem 2.3 (and Remark 2.1)]. *Let U be an open subset in a Banach space $(X,\|\cdot\|)$ and $F : \overline{U} \to \mathbb{C}(X)$ a closed map (i.e., has closed graph); here $\mathbb{C}(X)$ denotes the family of nonempty closed subsets of X. Assume $0 \in U$ and suppose there exists a continuous strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ satisfying $\phi(z) < z$ for $z > 0$ such that $H(Fx,Fy) \leq \phi(\|x - y\|)$ for all $x,y \in \overline{U}$. In addition, assume the following conditions hold:*

$$\Phi : [0,\infty) \longrightarrow [0,\infty),\ \text{given by } \Phi(x) = x - \phi(x),\ \text{is strictly increasing}, \tag{1.4}$$

$$\Phi^{-1}(a) + \Phi^{-1}(b) \leq \Phi^{-1}(a+b) \quad \text{for } a,b \geq 0, \tag{1.5}$$

$$\sum_{i=0}^{\infty} \phi^i(t) < \infty \quad \text{for } t > 0, \tag{1.6}$$

$$\sum_{i=1}^{\infty} \phi^i\big(x - \phi(x)\big) \leq \phi(x) \quad \text{for } x > 0, \tag{1.7}$$

$$F(\overline{U}) \text{ is bounded}, \tag{1.8}$$

$$x \notin \lambda Fx \quad \text{for } x \in \partial U,\ \lambda \in (0,1). \tag{1.9}$$

Then F has a fixed point in $\overline{U}$.

Remark 1.4. In fact, the assumption that F is closed can be removed in Theorem 1.3. In [1, Theorem 2.3], we assume a more general contractive condition and the map $G:\overline{U}\times[0,1]\to\mathbb{C}(X)$ (given by $G(x,\lambda)=\lambda Fx$ in our situation) was assumed to be closed in order to guarantee that if $\{x_n\}_1^\infty\subseteq\overline{U}$, $\{\lambda\}_1^\infty\subseteq[0,1]$ with $x_n\in G(x_n,\lambda_n)$ and $(x_n,\lambda_n)\to(x,\lambda)$, then $x\in G(x,\lambda)$. However, this is automatically true in Theorem 1.3 since the contractive condition and (1.8) guarantee that G is continuous in the Hausdorff metric and as a result,

$$\operatorname{dist}(x,G(x,\lambda))\le d(x,x_n)+H(G(x_n,\lambda_n),G(x,\lambda)). \tag{1.10}$$

Remark 1.5. If $\phi(t)=kt$, $0\le k<1$, then trivially (1.2)–(1.7) hold.

2. Fixed point theory in Fréchet spaces

Let $E=(E,\{|\cdot|_n\}_{n\in\mathbb{N}})$ be a Fréchet space with the topology generated by a family of seminorms $\{|\cdot|_n:n\in\mathbb{N}\}$. We assume that the family of seminorms satisfies

$$|x|_1\le|x|_2\le|x|_3\le\cdots\quad\text{for every } x\in E. \tag{2.1}$$

A subset X of E is bounded if for every $n\in\mathbb{N}$ there exists $r_n>0$ such that $|x|_n\le r_n$ for all $x\in X$. To E we associate a sequence of Banach spaces $\{(\mathbf{E}_n,|\cdot|_n)\}$ described as follows. For every $n\in\mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by

$$x\sim_n y\quad\text{iff } |x-y|_n=0. \tag{2.2}$$

We denote by $\mathbf{E}^n=(E/{\sim_n},|\cdot|_n)$ the quotient space, and by $(\mathbf{E}_n,|\cdot|_n)$ the completion of $\mathbf{E}^n$ with respect to $|\cdot|_n$ (the norm on $\mathbf{E}^n$ induced by $|\cdot|_n$ and its extension to $\mathbf{E}_n$ are still denoted by $|\cdot|_n$). This construction defines a continuous map $\mu_n:E\to\mathbf{E}_n$. Now since (2.1) is satisfied, the seminorm $|\cdot|_n$ induces a seminorm on $\mathbf{E}_m$ for every $m\ge n$ (again this seminorm is denoted by $|\cdot|_n$). Also (2.2) defines an equivalence relation on $\mathbf{E}_m$ from which we obtain a continuous map $\mu_{n,m}:\mathbf{E}_m\to\mathbf{E}_n$ since $\mathbf{E}_m/{\sim_n}$ can be regarded as a subset of $\mathbf{E}_n$. We now assume the following condition holds: for each $n\in\mathbb{N}$, there exists a Banach space $(E_n,|\cdot|_n)$ and an isomorphism (between normed spaces) $j_n:\mathbf{E}_n\longrightarrow E_n$.

Remark 2.1. (i) For convenience, the norm on E_n is denoted by $|\cdot|_n$.

(ii) Usually in applications, $\mathbf{E}_n=\mathbf{E}^n$ for each $n\in\mathbb{N}$.

(iii) Note that if $x\in\mathbf{E}_n$ (or $\mathbf{E}^n$), then $x\in E$. However, if $x\in E_n$, then x is not necessarily in E and in fact, E_n is easier to use in applications (even though E_n is isomorphic to $\mathbf{E}_n$). For example, if $E=\mathbb{C}[0,\infty)$, then $\mathbf{E}^n$ consists of the class of functions in E which coincide on the interval $[0,n]$ and $E_n=\mathbb{C}[0,n]$.

Finally, we assume

$$E_1\supseteq E_2\supseteq\cdots\ \text{and for each } n\in\mathbb{N},\quad |x|_n\le|x|_{n+1}\ \forall x\in E_{n+1}. \tag{2.3}$$

Let $\lim_{\leftarrow}E_n$ (or $\bigcap_1^\infty E_n$ where $\bigcap_1^\infty$ is the generalized intersection [5]) denote the projective limit of $\{E_n\}_{n\in\mathbb{N}}$ (note that $\pi_{n,m}=j_n\mu_{n,m}j_m^{-1}:E_m\to E_n$ for $m\ge n$) and note that $\lim_{\leftarrow}E_n\cong E$, so for convenience, we write $E=\lim_{\leftarrow}E_n$.

For each $X \subseteq E$ and each $n \in \mathbb{N}$, we set $X_n = j_n\mu_n(X)$ and we let $\overline{X_n}$ and ∂X_n denote, respectively, the closure and the boundary of X_n with respect to $|\cdot|_n$ in E_n. Also the pseudo-interior of X is defined by [4]

$$\text{pseudo-intt}(X) = \{x \in X : j_n\mu_n(x) \in \overline{X_n} \backslash \partial X_n \text{ for every } n \in \mathbb{N}\}. \tag{2.4}$$

Also, here H_n and diam_n denote the Hausdorff metric and the diameter induced by $|\cdot|_n$ on E_n.

We begin with single-valued maps and present two results. The first was motivated by Volterra type operators.

Theorem 2.2. *Let E and E_n be as described above and let $F : X \to E$ with $X \subseteq E$ and for each $n \in \mathbb{N}$ assume that $F : \overline{X_n} \to E_n$. Suppose the following conditions are satisfied:*

(a) $0 \in$ *pseudo-*$\text{intt}(X)$,

(b) *for each $n \in \mathbb{N}$, $F(\overline{X_n})$ is bounded,*

(c) *for each $n \in \mathbb{N}$, $F : \overline{X_n} \to E_n$ and there exists a continuous nondecreasing function $\phi_n : [0,\infty) \to [0,\infty)$ satisfying $\phi_n(z) < z$ for $z > 0$ such that $|Fx - Fy|_n \le \phi_n(|x - y|_n)$ for all $x, y \in \overline{X_n}$ for each $n \in \mathbb{N}$, $y \ne \lambda F y$, in E_n for all $\lambda \in (0,1)$, $y \in \partial X_n$,*

(d) *for each $n \in \{2,3,\dots\}$, if $y \in \overline{X_n}$ solves $y = Fy$ in E_n, then $y \in \overline{X_k}$ for $k \in \{1,\dots,n-1\}$.*

Then F has a fixed point in E.

Remark 2.3. If $F(X)$ is bounded, then clearly Theorem 2.2(b) holds.

Proof. Fix $n \in \mathbb{N}$. From Theorem 1.2, there exists $y_n \in \overline{X_n}$ with $y_n = Fy_n$ (note that $0 \in \overline{X_n} \backslash \partial X_n$ and $F(\overline{X_n})$ is bounded). Let us look at $\{y_n\}_{n\in\mathbb{N}}$. Notice that $y_1 \in \overline{X_1}$ and $y_k \in \overline{X_1}$ for $k \in \mathbb{N}\backslash\{1\}$ from Theorem 2.2(d). As a result, $y_n \in \overline{X_1}$ for $n \in \mathbb{N}$, $y_n = Fy_n$ in E_n together with Theorem 1.1 implies there is a subsequence $\mathbb{N}_1^\star$ of $\mathbb{N}$ and a $z_1 \in \overline{X_1}$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $\mathbb{N}_1^\star$. Let $\mathbb{N}_1 = \mathbb{N}_1^\star \backslash \{1\}$. Now $y_n \in \overline{X_2}$ for $n \in \mathbb{N}_1$ together with Theorem 1.1 guarantees that there exists a subsequence $\mathbb{N}_2^\star$ of $\mathbb{N}_1$ and a $z_2 \in \overline{X_2}$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $\mathbb{N}_2^\star$. Note from (2.3) that $z_2 = z_1$ in E_1 since $\mathbb{N}_2^\star \subseteq \mathbb{N}_1$. Let $\mathbb{N}_2 = \mathbb{N}_2^\star \backslash \{2\}$. Proceed inductively to obtain subsequences of integers

$$\begin{gathered} \mathbb{N}_1^\star \supseteq \mathbb{N}_2^\star \supseteq \cdots, \\ \mathbb{N}_k^\star \subseteq \{k, k+1, \dots\}, \end{gathered} \tag{2.5}$$

and $z_k \in \overline{X_k}$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $\mathbb{N}_k^\star$. Note that $z_{k+1} = z_k$ in E_k for $k \in \{1,2,\dots\}$. Also let $\mathbb{N}_k = \mathbb{N}_k^\star \backslash \{k\}$.

Fix $k \in \mathbb{N}$. Let $y = z_k$ in E_k. Notice that y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n = Fy_n$ in E_n for $n \in \mathbb{N}_k$ and $y_n \to y$ in E_k as $n \to \infty$ in $\mathbb{N}_k$ (since $y = z_k$ in E_k) together with the fact that $F : \overline{X_k} \to E_k$ is continuous (note that $y_n \in \overline{X_k}$ for $n \in \mathbb{N}_k$) implies $y = Fy$ in E_k. We can do this for each $k \in \mathbb{N}$, so $y = Fy$ in E. □

Our next result was motivated by contractions considered in [3]. In this case, the map F_n will be related to F by the closure property Theorem 2.4(f).

THEOREM 2.4. *Let E and E_n be as described in the beginning of Section 2 and let $F : X \to E$ with $X \subseteq E$. Also for each $n \in \mathbb{N}$ assume there exists $F_n : \overline{X_n} \to E_n$. Suppose the following conditions are satisfied:*

(a) $0 \in$ *pseudo-* $\text{intt}(X)$,

(b) $\overline{X_1} \supseteq \overline{X_2} \supseteq \cdots$,

(c) *for each $n \in \mathbb{N}$, $F_n(\overline{X_n})$ is bounded, for each $n \in \mathbb{N}$, $F_n : \overline{X_n} \to E_n$ and there exists a continuous nondecreasing function $\phi_n : [0,\infty) \to [0,\infty)$ satisfying $\phi_n(z) < z$ for $z > 0$ such that $|F_n x - F_n y|_n \le \phi_n(|x-y|_n)$ for all $x, y \in \overline{X_n}$ for each $n \in \mathbb{N}$, $y \neq \lambda F_n y$ in E_n for all $\lambda \in (0,1)$, $y \in \partial X_n$,*

(d) *for each $n \in \mathbb{N}$, the map $\mathcal{K}_n : \overline{X_n} \to 2^{E_n}$ given by $\mathcal{K}_n(y) = \bigcup_{m=n}^{\infty} F_m(y)$ (see Remark 2.5) satisfies $H_n(\mathcal{K}_n(x), \mathcal{K}_n(y)) \le \psi_n(|x-y|_n)$ for all $x, y \in \overline{X_n}$; here $\psi_n : [0,\infty) \to [0,\infty)$ is continuous, $\psi_n(z) < z$ for $z > 0$ with the map $\Psi_n : [0,\infty) \to [0,\infty)$, defined by $\Psi_n(x) = x - \psi_n(x)$, strictly increasing,*

(e) *for each $k \in \mathbb{N}$, for every $\epsilon > 0$, and sequence $\{x_n\}_{n\in S}$, $S = \{k, k+1, k+2, \dots\}$, with $x_n \in \overline{X_n}$ and $x_n \in \mathcal{K}_n x_n$ in E_n, there exists $n_k \in S$ such that $\text{diam}_k(\mathcal{K}_k x_n) < \epsilon$ for each $n \in S$ with $n \ge n_k$,*

(f) *if there exists $w \in E$ and a sequence $\{y_n\}_{n\in\mathbb{N}}$ with $y_n \in \overline{X_n}$ and $y_n = F_n y_n$ in E_n such that for every $k \in \mathbb{N}$ with $y_n \to w$ in E_k as $n \to \infty$ in $S = \{k+1, k+2, \dots\}$, then $w = Fw$ in E.*

Then F has a fixed point in E.

Remark 2.5. The definition of $\mathcal{K}_n$ in Theorem 2.4(d) is as follows. If $y \in \overline{X_n}$ and $y \notin \overline{X_{n+1}}$, then $\mathcal{K}_n(y) = F_n(y)$, whereas if $y \in \overline{X_{n+1}}$ and $y \notin \overline{X_{n+2}}$, then $\mathcal{K}_n(y) = F_n(y) \cup F_{n+1}(y)$, and so on.

Proof. Fix $n \in \mathbb{N}$. From Theorem 1.2 there exists $y_n \in \overline{X_n}$ with $y_n = F_n y_n$ in E_n. Let us look at $\{y_n\}_{n\in\mathbb{N}}$. From Theorem 2.4(b) we know that $y_n \in \overline{X_1}$ for $n \in \mathbb{N}$. Note as well that $y_n \in \mathcal{K}_1 y_n$ for $n \in \mathbb{N}$ since $|x|_1 \le |x|_n$ for all $x \in E_n$ and $y_n = F_n y_n$ in E_n. We claim

$$\exists z_1 \in E_1 \quad \text{with } y_n \longrightarrow z_1 \text{ in } E_1, \quad n \longrightarrow \infty \text{ in } \mathbb{N}. \tag{2.6}$$

To see this, let $\epsilon > 0$ be given. Let $m, n \in \mathbb{N}$. It is easy to see, since $y_n \in \mathcal{K}_1 y_n$ and $y_m \in \mathcal{K}_1 y_m$, that

$$|y_n - y_m|_1 \le H_1(\mathcal{K}_1 y_n, \mathcal{K}_1 y_m) + \text{diam}_1(\mathcal{K}_1 y_n) + \text{diam}_1(\mathcal{K}_1 y_m), \tag{2.7}$$

so Theorem 2.4(d) yields

$$|y_n - y_m|_1 \le \Psi_1^{-1}(\text{diam}_1(\mathcal{K}_1 y_n) + \text{diam}_1(\mathcal{K}_1 y_m)). \tag{2.8}$$

Now Theorem 2.4(e) guarantees that there exists $n_1 \in \mathbb{N}$ such that

$$|y_n - y_m|_1 \le \Psi_1^{-1}(2\epsilon) \quad \text{for } m, n \ge n_1. \tag{2.9}$$

Consequently, $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy, so (2.6) holds. Let $\mathbb{N}_1 = \mathbb{N}\backslash\{1\}$.

Now $y_n \in \mathcal{K}_2 y_n$ for $n \in \mathbb{N}_1$. Let $m,n \in \mathbb{N}_1$ and since $y_n \in \mathcal{K}_2 y_n$ and $y_m \in \mathcal{K}_2 y_m$ we have

$$|y_n - y_m|_2 \leq \Psi_2^{-1}\left(\operatorname{diam}_2(\mathcal{K}_2 y_n) + \operatorname{diam}_2(\mathcal{K}_2 y_m)\right). \tag{2.10}$$

This together with Theorem 2.4(e) guarantees that $\{y_n\}_{n\in\mathbb{N}_1}$ is Cauchy, so there exists a $z_2 \in E_2$ with $y_n \to z_2$ in E_2 as $n \to \infty$ in $\mathbb{N}_1$. Note that $z_2 = z_1$ in E_1 since $\mathbb{N}_1 \subseteq \mathbb{N}$. Let $\mathbb{N}_2 = \mathbb{N}_1 \backslash \{2\}$. Proceed inductively to obtain $z_k \in E_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $\mathbb{N}_{k-1} = \{k, k+1, \dots\}$. Note that $z_{k+1} = z_k$ in E_k for $k \in \mathbb{N}$. Also let $\mathbb{N}_k = \mathbb{N}_{k-1} \backslash \{k\}$.

Fix $k \in \mathbb{N}$. Let $y = z_k$ in E_k. Notice that y is well defined and $y \in \lim_{\leftarrow} E_n = E$. Now $y_n = F_n y_n$ in E_n for $n \in \mathbb{N}_k$ and $y_n \to y$ in E_k as $n \to \infty$ in $\mathbb{N}_k$ (since $y = z_k$ in E_k) together with Theorem 2.4(f) implies $y = Fy$ in E. □

Our next two results are for multivalued maps.

Theorem 2.6. *Let E and E_n be as described above and let $F : X \to 2^E$ with $X \subseteq E$ and for each $n \in \mathbb{N}$, assume $F : \overline{X_n} \to \mathbb{C}(E_n)$. Suppose the following conditions are satisfied:*

(a) $0 \in$ *pseudo-*$\operatorname{intt}(X)$,

(b) *for each $n \in \mathbb{N}$, $F(\overline{X_n})$ is bounded,*

(c) *for each $n \in \mathbb{N}$, $F : \overline{X_n} \to \mathbb{C}(E_n)$, and there exists a continuous strictly increasing function $\phi_n : [0,\infty) \to [0,\infty)$ satisfying $\phi_n(z) < z$ for $z > 0$ such that $H_n(Fx, Fy) \leq \phi_n(|x-y|_n)$ for all $x, y \in \overline{X_n}$,*

(d) *for each $n \in \mathbb{N}$, the map $\Phi_n : [0,\infty) \to [0,\infty)$ given by $\Phi_n(x) = x - \phi_n(x)$ is strictly increasing, $\Phi_n^{-1}(a) + \Phi_n^{-1}(b) \leq \Phi_n^{-1}(a+b)$ for $a, b \geq 0$, with $\sum_{i=0}^{\infty} \phi_n^i(t) < \infty$ for $t > 0$ and $\sum_{i=1}^{\infty} \phi_n^i(x - \phi(x)) \leq \phi_n(x)$ for $x > 0$,*

(e) *for each $n \in \mathbb{N}$, $y \notin \lambda F y$ in E_n for all $\lambda \in (0,1)$, $y \in \partial X_n$,*

(f) *for each $n \in \{2,3,\dots\}$, if $y \in \overline{X_n}$ solves $y \in Fy$ in E_n, then $y \in \overline{X_k}$ for $k \in \{1,\dots,n-1\}$,*

(g) *for each $k \in \mathbb{N}$, for every $\epsilon > 0$ and sequence $\{x_n\}_{n\in S}$, $S = \{k, k+1, k+2, \dots\}$, with $x_n \in \overline{X_n}$ and $x_n \in F x_n$ in E_n there exists $n_k \in S$ such that $\operatorname{diam}_k(F x_n) < \epsilon$ for each $n \in S$ with $n \geq n_k$.*

Then F has a fixed point in E.

Proof. Fix $n \in \mathbb{N}$. From Theorem 1.3 (and Remark 1.4) there exists $y_n \in \overline{X_n}$ with $y_n \in F y_n$ in E_n. Let us look at $\{y_n\}_{n\in\mathbb{N}}$. Notice that $y_n \in \overline{X_1}$ for $n \in \mathbb{N}$ from Theorem 2.6(f). Let $\epsilon > 0$ be given and $m, n \in \mathbb{N}$. Now since $y_n \in F y_n$ and $y_m \in F y_m$, we have

$$|y_n - y_m|_1 \leq H_1(F y_n, F y_m) + \operatorname{diam}_1(F y_n) + \operatorname{diam}_1(F y_m) \tag{2.11}$$

so

$$|y_n - y_m|_1 \leq \Phi_1^{-1}\left(\operatorname{diam}_1(F y_n) + \operatorname{diam}_1(F y_m)\right). \tag{2.12}$$

This, together with Theorem 2.6(g), guarantees that $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy, so there exists a $z_1 \in E_1$ with $y_n \to z_1$ in E_1 as $n \to \infty$ in $\mathbb{N}$. Let $\mathbb{N}_1 = \mathbb{N} \backslash \{1\}$. Proceed inductively to obtain $z_k \in E_k$ with $y_n \to z_k$ in E_k as $n \to \infty$ in $\mathbb{N}_{k-1} = \{k, k+1, \dots\}$. Note that $z_{k+1} = z_k$ in E_k for $k \in \mathbb{N}$. Also let $\mathbb{N}_k = \mathbb{N}_{k-1} \backslash \{k\}$.

Fix $k \in \mathbb{N}$. Let $y = z_k$ in E_k. Notice that $y_n \in Fy_n$ in E_n for $n \in \mathbb{N}_k$ and $y_n \to y$ in E_k as $n \to \infty$ in $\mathbb{N}_k$ together with Remark 1.4 (note that $F : \overline{X_k} \to \mathbb{C}(E_k))$ implies $y \in Fy$ in E_k. We can do this for each $k \in \mathbb{N}$, so $y \in Fy$ in E. □

Theorem 2.7. *Let E and E_n be as described in the beginning of Section 2 and let $F : X \to 2^E$ with $X \subseteq E$. Also for each $n \in \mathbb{N}$ assume there exists $F_n : \overline{X_n} \to \mathbb{C}(E_n)$. Suppose the following conditions are satisfied:*

(a) $0 \in$ *pseudo-* $\operatorname{intt}(X)$,

(b) $\overline{X_1} \supseteq \overline{X_2} \supseteq \cdots$,

(c) *for each $n \in \mathbb{N}$, $F_n(\overline{X_n})$ is bounded,*

(d) *for each $n \in \mathbb{N}$, $F_n : \overline{X_n} \to \mathbb{C}(E_n)$ and there exists a continuous strictly increasing function $\phi_n : [0,\infty) \to [0,\infty)$ satisfying $\phi_n(z) < z$ for $z > 0$ such that $H_n(F_n x, F_n y) \le \phi_n(|x-y|_n)$ for all $x, y \in \overline{X_n}$,*

(e) *for each $n \in \mathbb{N}$, the map $\Phi_n : [0,\infty) \to [0,\infty)$ given by $\Phi_n(x) = x - \phi_n(x)$ is strictly increasing, $\Phi_n^{-1}(a) + \Phi_n^{-1}(b) \le \Phi_n^{-1}(a+b)$ for $a, b \ge 0$, with $\sum_{i=0}^{\infty} \phi_n^i(t) < \infty$ for $t > 0$ and $\sum_{i=1}^{\infty} \phi_n^i(x - \phi(x)) \le \phi_n(x)$ for $x > 0$,*

(f) *for each $n \in \mathbb{N}$, $y \notin \lambda F_n y$ in E_n for all $\lambda \in (0,1)$ and $y \in \partial X_n$,*

(g) *for each $n \in \mathbb{N}$, the map $\mathcal{K}_n : \overline{X_n} \to 2^{E_n}$ given by $\mathcal{K}_n(y) = \bigcup_{m=n}^{\infty} F_m(y)$ satisfies $H_n(\mathcal{K}_n(x), \mathcal{K}_n(y)) \le \psi_n(|x-y|_n)$ for all $x, y \in \overline{X_n}$; here $\psi_n : [0,\infty) \to [0,\infty)$ is continuous, $\psi_n(z) < z$ for $z > 0$ with the map $\Psi_n : [0,\infty) \to [0,\infty)$ defined by $\Psi_n(x) = x - \psi_n(x)$ is strictly increasing,*

(h) *for each $k \in \mathbb{N}$, for every $\epsilon > 0$ and sequence $\{x_n\}_{n\in S}$, $S = \{k, k+1, k+2, \ldots\}$, with $x_n \in \overline{X_n}$ and $x_n \in \mathcal{K}_n x_n$ in E_n there exists $n_k \in S$ such that $\operatorname{diam}_k(\mathcal{K}_k x_n) < \epsilon$ for each $n \in S$ with $n \ge n_k$,*

(i) *if there exists a $w \in E$ and a sequence $\{y_n\}_{n\in\mathbb{N}}$ with $y_n \in \overline{X_n}$ and $y_n \in F_n y_n$ in E_n such that for every $k \in \mathbb{N}$ with $y_n \to w$ in E_k as $n \to \infty$ in $S = \{k+1, k+2, \ldots\}$, then $w \in Fw$ in E.*

Then F has a fixed point in E.

Proof. The proof is essentially the same as in Theorem 2.4 (except that here we use Theorem 1.3 (and Remark 1.4) instead of Theorem 1.2). □

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