PERTURBED THREE-STEP APPROXIMATION PROCESS WITH ERRORS FOR A GENERALIZED IMPLICIT NONLINEAR QUASIVARIATIONAL INCLUSIONS

M. K. AHMAD AND SALAHUDDIN

We initiate and develop some new perturbed three-step approximation process with errors for solving *generalized implicit nonlinear quasivariational inclusions*. Also, the convergence and stability of the iterative sequences with errors generated by the algorithms are presented.

1. Introduction

Variational inequality theory has become a rich source of inspiration in the pure and applied mathematics. Variational inequalities not only have stimulated new results dealing with nonlinear partial differential equations, but also have been used in a large variety of problems arising in mechanics, physics, optimization and control nonlinear programming, economics and transportation equilibrium and engineering sciences, and so forth. In recent years variational inequalities have been generalized and applied in various directions. For details we refer to [2, 5, 6, 20].

Recently Huang [10, 11] constructed some new perturbed Ishikawa and Mann iterative algorithms to approximate the solution of some *generalized implicit quasivariational inclusions* (inequalities), which includes many iterative algorithms for variational and quasivariational inequality problems as special cases.

On the other hand, Xu [19] revised the definition of Ishikawa and Mann iterative processes with errors and studied the convergence problem of Ishikawa and Mann iterative processes with errors for approximating the solutions of the generative strongly accretive operator equations.

Inspired and motivated by recent research works [4, 7, 12, 14–16], in this paper we initiate and construct some perturbed three-step approximation processes with errors for solving a class of generalized implicit nonlinear quasivariational inclusions. We also discuss the convergence and stability of the iterative sequences generated by algorithms.

2. Preliminaries

Let H be a real Hilbert space endowed with a norm $\| \cdot \|$ and an inner product $\langle \cdot, \cdot \rangle$, respectively. For a given maximal monotone mapping $A(\cdot, \cdot) : H \times H \to 2^H$ with respect to the first argument, a nonlinear mapping $N(\cdot, \cdot) : H \times H \to H$, three single-valued mappings, $V, G, g : H \to H$, find $u \in H$ such that $g(u) \in \operatorname{dom} A(\cdot, u)$ and

$$0 \in g(u) - N(Vu, Gu) + A(g(u), u), \tag{2.1}$$

where 2^H denote the power subsets of H. This variational inclusions is called the generalized implicit nonlinear quasivariational inclusions.

Remark 2.1. (1) We note that $A(g(u), u) \equiv A(g(u))$ for all $u \in H$, then the problem (2.1) is equivalent to finding $u \in H$ such that

$$0 \in g(u) - N(Vu, Gu) + A(g(u)) \tag{2.2}$$

is called *general implicit nonlinear quasivariational inclusions*.

(2) If V and G are identity mappings, then (2.2) is equivalent to finding $u \in H$ such that

$$0 \in g(u) - N(u, u) + A(g(u)). \tag{2.3}$$

(3) We note that $N(u, u) \equiv 0$, zero mapping, then (2.3) is equivalent to the finding $u \in H$ such that $g(u) \in \operatorname{dom} A$,

$$0 \in g(u) + A(g(u)) \tag{2.4}$$

is called *general variational inclusions* considered by Huang et al. [11].

(4) We again note that $g \equiv I$, an identity mapping, then (2.4) is equivalent to the classical variational inclusions, for finding $u \in H$ such that

$$0 \in u + A(u). \tag{2.5}$$

(5) If $A(\cdot, u) \equiv \partial \varphi(\cdot, u)$ is the subdifferential of $\varphi(\cdot, u)$, where $\varphi(\cdot, u) : H \times H \to R \cup \{+\infty\}$ is a proper convex lower semicontinuous functional with respect to the first argument, then the problem (2.1) is equivalent to finding $u \in H$ such that $g(u) \in \operatorname{dom} \partial \varphi(\cdot, u)$ and

$$\langle g(u) - N(Vu, Gu), v - g(u) \rangle \geq \varphi(g(u), u) - \varphi(v, u), \quad \forall v \in H, \tag{2.6}$$

is called generalized nonlinear quasivariational inclusion problems, which is the variant form of Ahmad et al. [1].

(6) If $N(Vu, Gu) = V(u) - Gu$, then (2.6) is equivalent to finding $u \in H$ such that $g(u) \in \text{dom}\,\partial\varphi(\cdot, u)$,

$$\langle g(u) - (Vu - Gu), v - g(u) \rangle \geq \varphi(g(u), u) - \varphi(v, u), \quad \forall v \in H, \qquad (2.7)$$

is called general strongly nonlinear quasivariational inclusions.

(7) If $\partial\varphi(\cdot, u) = \partial\varphi(u)$, then (2.7) is equivalent to finding $u \in H$ such that $g(u) \in \text{dom}\,A$ and

$$\langle g(u) - N(Vu - Gu), v - g(u) \rangle \geq \varphi(g(u)) - \varphi(v), \quad \forall v \in H, \qquad (2.8)$$

is called a variant form of general strongly nonlinear quasivariational inclusions, which is the variant form of that of Khan et al. [13].

(8) If V and G are identity mappings, then (2.6) is equivalent to finding $u \in H$ such that $g(u) \in \text{dom}\,\partial\varphi(\cdot, u)$,

$$\langle g(u) - N(u, u), v - g(u) \rangle \geq \varphi(g(u), u) - \varphi(v, u), \quad \forall v \in H, \qquad (2.9)$$

is called generalized strongly nonlinear implicit quasivariational inclusions.

(9) If $g(u) - N(u, u) \equiv T(u) - A(u)$ and $\varphi(\cdot, u) = \varphi(u)$, then (2.9) reduces to the following problem of finding $u \in H$ such that $g(u) \in \text{dom}\,\partial\varphi$ and

$$\langle Tu - Au, v - g(u) \rangle \geq \varphi(g(u)) - \varphi(v), \quad \forall v \in H, \qquad (2.10)$$

which is considered by Hassouni and Moudafi [9].

(10) If K is a given closed convex subset of H and $\varphi = I_K$ is the indicator function of K, defined by

$$I_K(x) = \begin{cases} 0, & x \in K, \\ +\infty, & \text{otherwise}, \end{cases} \qquad (2.11)$$

then, problem (2.3) reduces to the following problem of finding $u \in H$ such that

$$\langle g(u) - N(Vu, Gu), v - g(u) \rangle \geq 0, \quad \forall v \in H, \qquad (2.12)$$

which is variant form of that of Verma [17].

(11) If V and G are identity mappings, then (2.12) is equivalent to finding $u \in H$ such that

$$\langle g(u) - N(u, u), v - g(u) \rangle \geq 0, \quad \forall v \in H. \qquad (2.13)$$

(12) If $g(u) - N(u, u) = u - N(u, u)$, then (2.13) is equivalent to finding $u \in H$ such that

$$\langle u - N(u, u), v - g(u) \rangle \geq 0, \quad \forall v \in H, \qquad (2.14)$$

is called generalized strongly nonlinear implicit quasivariational inequality problem, considered by Cho et al. [3].

(13) If $g \equiv I$ identity mapping, $N(u,u) = T(u)$ for all $u \in H$, and $T : H \to H$ is a single valued mapping, then (2.14) is equivalent to finding $u \in H$ such that

$$\langle u - Tu, v - u \rangle \geq 0, \quad \forall v \in H, \tag{2.15}$$

which is nonlinear variational inequality, considered by Verma [18].

(14) If $u - N(u,u) = T(u)$, then (2.13) reduces to finding $u \in H$ such that

$$\langle Tu, v - g(u) \rangle \geq 0, \quad \forall v \in H, \tag{2.16}$$

which is another classical variational inequality introduced by Hartman and Stampacchia [8].

(15) If $g \equiv I$, then (2.15) reduces to finding $u \in H$ such that

$$\langle Tu, v - u \rangle \geq 0, \quad \forall v \in H, \tag{2.17}$$

which is called classical variational inequality, considered by Hartman and Stampacchia [8].

Definition 2.2. A mapping $g : H \to H$ is said to be α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle g(u) - g(v), u - v \rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in H. \tag{2.18}$$

Definition 2.3. A mapping $g : H \to H$ is said to be β-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|g(u) - g(v)\| \leq \beta \|u - v\|, \quad \forall u, v \in H. \tag{2.19}$$

Definition 2.4. Let $V : H \to H$ and $N : H \times H \to H$ be the mappings, N is said to be the following.

   (i) σ-Lipschitz continuous in the first argument if there exists a constant $\sigma > 0$ such that

$$\|N(u,w) - N(v,w)\| \leq \sigma \|u - v\|, \quad \forall u, v, w \in H. \tag{2.20}$$

   Similarly the Lipschitz continuity of N can be defined with respect to the second argument.

  (ii) η-relaxed monotone with respect to V in the first argument if there exists a constant $\eta > 0$ such that

$$\langle N(Vu,w) - N(Vv,w), u - v \rangle \geq -\eta \|u - v\|^2, \quad \forall u, v, w \in H. \tag{2.21}$$

Definition 2.5. Let $G : H \to H$ and $N : H \times H \to H$ be the mappings. The mapping N is said to be κ-relaxed Lipschitz continuous with respect to G in the second argument if there exists a constant $\kappa \geq 0$ such that

$$\langle N(w,Gu) - N(w,Gv), u - v \rangle \leq -\kappa \|u - v\|^2, \quad \forall u, v, w \in H. \tag{2.22}$$

Definition 2.6. Let H be a Hilbert space and let $A : H \to 2^H$ be a maximal monotone mapping. For any fixed $\lambda > 0$, the mapping $J_\lambda^A : H \to H$ defined by

$$J_\lambda^A(u) = (I + \lambda A)^{-1}(u), \quad \forall u \in H, \tag{2.23}$$

is said to be the resolvent operator of A, where I is an identity mapping on H.

LEMMA 2.7. *Let $A : H \to 2^H$ be a maximal monotone mapping. Then the resolvent operator $J_\lambda^A : H \to H$ is nonexpansive, that is,*

$$\|J_\lambda^A(u) - J_\lambda^A(v)\| \le \|u - v\|, \quad \forall u, v \in H. \tag{2.24}$$

Definition 2.8. Let $\{A^n\}$ and A be the maximal monotone mappings from H into the power set of H for $n = 0, 1, 2, \ldots$. The sequence $\{A^n\}$ is said to be graph-convergence of A (write $A^n \xrightarrow{G} A$) if the following property holds.

For every $(u, v) \in \text{Graph}(A)$, there exists a sequence $(u_n, v_n) \in \text{Graph}(A^n)$ such that $u_n \to u$ and $v_n \to v$ as $n \to \infty$.

LEMMA 2.9 [2]. *Let $\{A^n\}$ and A be the maximal monotone mappings from H into the power set of H for $n = 0, 1, 2, \ldots$. Then $A^n \xrightarrow{G} A$ if and only if*

$$J_\lambda^{A^n}(u) \to J_\lambda^A(u), \tag{2.25}$$

for every $u \in H$ and $\lambda > 0$.

LEMMA 2.10 [15]. *Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be three sequences of nonnegative numbers satisfying the following conditions: there exists a positive integer n_0 such that*

$$a_{n+1} \le (1 - t_n)a_n + b_n t_n + c_n, \quad n \ge n_0, \tag{2.26}$$

where

$$t_n \in [0, 1], \qquad \sum_{n=0}^{\infty} t_n = +\infty, \qquad \lim_{n \to \infty} b_n = 0, \qquad \sum_{n=0}^{\infty} c_n < +\infty. \tag{2.27}$$

Then $\lim_{n \to \infty} a_n = 0$.

LEMMA 2.11. *For a given $u \in H$, u is a solution of the problem (2.1) if and only if*

$$g(u) \equiv J_\lambda^{A(\cdot, u)} [g(u) - \lambda(g(u) - N(Vu, Gu))], \tag{2.28}$$

where $\lambda > 0$, a constant, $J_\lambda^{A(\cdot, u)}(w) \equiv (I + \lambda A(\cdot, u))^{-1}(w)$, where I is an identity operator, and $w \in H$.

Proof. Let $u \in H$ be a solution of problem (2.1) if and only if for given $\lambda > 0$, a constant,

$$
\begin{aligned}
& 0 \in g(u) - N(Vu, Gu) + A(g(u), u) \\
& \Longleftrightarrow 0 \in \lambda g(u) - \lambda N(Vu, Gu) + \lambda A(g(u), u) \\
& \Longleftrightarrow 0 \in -(g(u) - \lambda(g(u) - N(Vu, Gu))) + (I + \lambda A(\cdot, u))g(u) \\
& \Longleftrightarrow g(u) - \lambda(g(u) - N(Vu, Gu)) \in (I + \lambda A(\cdot, u))g(u) \\
& \Longleftrightarrow g(u) \equiv (I + \lambda A(\cdot, u))^{-1}[g(u) - \lambda(g(u) - N(Vu, Gu))] \\
& \Longleftrightarrow g(u) \equiv J_\lambda^{A(\cdot, u)}[g(u) - \lambda(g(u) - N(Vu, Gu))],
\end{aligned}
\tag{2.29}
$$

which completes the proof. □

3. The main results

THEOREM 3.1. *Let $g : H \to H$ be the α-strongly monotone and β-Lipschitz continuous mappings with constants $\alpha > 0$ and $\beta > 0$, respectively. Let $V, G : H \to H$ be the two ξ-Lipschitz and μ-Lipschitz continuous mappings with constants $\xi > 0$ and $\mu > 0$, respectively. Let bimapping $N : H \times H \to H$ be the σ-Lipschitz continuous with respect to the first argument with constant $\sigma > 0$ and δ-Lipschitz continuous with respect to the second argument with constant $\delta > 0$. Let N be the η-relaxed monotone with respect to V in the first argument with constant $\eta > 0$ and κ-relaxed Lipschitz continuous with respect to G in the second argument with constant $\kappa \geq 0$. Let $A : H \times H \to 2^H$ be the maximal monotone with respect to the first argument. It assume that for all $u, v, w \in H$,*

$$
\left\| J_\lambda^{A(\cdot, u)}(w) - J_\lambda^{A(\cdot, v)}(w) \right\| \leq \rho \|u - v\|,
\tag{3.1}
$$

where $\rho > 0$ is a constant. If

$$
\left| \lambda - \frac{\eta - p(1-k)}{\sigma^2 \xi^2 - p^2} \right| < \frac{\sqrt{(\eta - p(1-k))^2 - (\sigma^2 \xi^2 - p^2)k(2-k)}}{\sigma^2 \xi^2 - p^2},
\tag{3.2}
$$

$$
\eta > p(1-k) + \sqrt{(\sigma^2 \xi^2 - p^2)k(2-k)}, \quad k < 1, \ p < \sigma \xi, \ \eta > p,
$$

where

$$
\theta = k + \sqrt{1 - 2\lambda\eta + \lambda^2 \sigma^2 \xi^2} + \lambda p < 1,
\tag{3.3}
$$

$$
k = 2\Omega + \rho, \qquad p = \Omega + \omega, \qquad \Omega = \sqrt{1 - 2\alpha + \beta^2}, \qquad \omega = \sqrt{1 + 2\kappa + \delta^2 \mu^2}.
$$

Then the problem (2.1) has a unique solution $u \in H$.

Proof. Define a mapping $F : H \to H$ as

$$
F(u) = u - g(u) + J_\lambda^{A(\cdot, u)}[g(u) - \lambda(g(u) - N(Vu, Gu))], \quad u \in H.
\tag{3.4}
$$

By Lemmas 2.11 and 2.7, it is enough to show that F is a contraction mapping. It follows from (3.4) that

$$
\begin{aligned}
\|F(u) - F(v)\| = &\ \|u - v - (g(u) - g(v))\| \\
&+ \left\| J_\lambda^{A(\cdot, u)} [g(u) - \lambda(g(u) - N(Vu, Gu))] \right. \\
&\quad \left. - J_\lambda^{A(\cdot, u)} [g(v) - \lambda(g(v) - N(Vv, Gv))] \right\| \\
&+ \left\| J_\lambda^{A(\cdot, u)} [g(v) - \lambda(g(v) - N(Vv, Gv))] \right. \\
&\quad \left. - J_\lambda^{A(\cdot, v)} [g(v) - \lambda(g(v) - N(Vv, Gv))] \right\| \\
\leq &\ \|u - v - (g(u) - g(v))\| \\
&+ \|g(u) - g(v) - \lambda(g(u) - N(Vu, Gu)) + \lambda(g(v) - N(Vv, Gv))\| \\
&+ \rho\|u - v\| \\
\leq &\ 2\|u - v - (g(u) - g(v))\| \\
&+ \|u - v + \lambda(N(Vu, Gu) - N(Vv, Gu))\| \\
&+ \lambda\|g(u) - g(v) - (N(Vv, Gu) - N(Vv, Gv))\| \\
&+ \rho\|u - v\| \\
\leq &\ 2\|u - v - (g(u) - g(v))\| \\
&+ \|u - v + \lambda(N(Vu, Gu) - N(Vv, Gu))\| \\
&+ \lambda\|u - v - (g(u) - g(v))\| \\
&+ \lambda\|u - v - (N(Vv, Gu) - N(Vv, Gv))\| + \rho\|u - v\|.
\end{aligned}
\tag{3.5}
$$

Since g is α-strongly monotone and β-Lipschitz continuous, we have

$$
\|u - v - (g(u) - g(v))\|^2 \leq (1 - 2\alpha + \beta^2)\|u - v\|^2.
\tag{3.6}
$$

Since N is σ-Lipschitz continuous with respect to first argument, V is ξ-Lipschitz continuous, and N is η-relaxed monotone with respect to V in the first argument with constant $\eta > 0$, we have

$$
\begin{aligned}
\|u - v &+ \lambda(N(Vu, Gu) - N(Vv, Gu))\|^2 \\
&\leq \|u - v\|^2 + 2\lambda\langle N(Vu, Gu) - N(Vv, Gu), u - v\rangle \\
&\quad + \lambda^2\|N(Vu, Gu) - N(Vv, Gu)\|^2 \\
&\leq \|u - v\|^2 - 2\lambda\eta\|u - v\|^2 + \lambda^2\sigma^2\xi^2\|u - v\|^2 \\
&\leq (1 - 2\lambda\eta + \lambda^2\sigma^2\xi^2)\|u - v\|^2.
\end{aligned}
\tag{3.7}
$$

Since N is δ-Lipschitz continuous with respect to the second argument, G is μ-Lipschitz continuous, and N is κ-relaxed Lipschitz continuous with respect to G in the second

argument with constant $\kappa \geq 0$, we get

$$
\begin{aligned}
\|u - v - &(N(Vv, Gu) - N(Vv, Gu))\|^2 \\
&\leq \|u - v\|^2 - 2\langle N(Vv, Gu) - N(Vv, Gv), u - v \rangle \\
&\quad + \|N(Vv, Gu) - N(Vv, Gv)\|^2 \\
&\leq \|u - v\|^2 + 2\kappa\|u - v\|^2 + \delta^2\mu^2\|u - v\|^2 \\
&\leq (1 + 2\kappa + \delta^2\mu^2)\|u - v\|^2.
\end{aligned}
\tag{3.8}
$$

It follows from (3.5)–(3.8) that we have

$$
\begin{aligned}
\|F(u) - F(v)\| \\
&\leq 2\sqrt{1 - 2\alpha + \beta^2}\|u - v\| + \sqrt{1 - 2\lambda\eta + \lambda^2\sigma^2\xi^2}\|u - v\| \\
&\quad + \lambda\sqrt{1 - 2\alpha + \beta^2}\|u - v\| + \lambda\sqrt{1 + 2\kappa + \delta^2\mu^2}\|u - v\| + \rho\|u - v\| \\
&\leq \Big[2\sqrt{1 - 2\alpha + \beta^2} + \rho + \sqrt{1 - 2\lambda\eta + \lambda^2\sigma^2\xi^2} + \lambda\sqrt{1 - 2\alpha + \beta^2} \\
&\quad + \lambda\sqrt{1 + 2\kappa + \delta^2\mu^2} \Big]\|u - v\| \\
&\leq \Big[2\Omega + \rho + \sqrt{1 - 2\lambda\eta + \lambda^2\sigma^2\xi^2} + \lambda\Omega + \lambda\omega \Big]\|u - v\| \\
&\leq \Big[k + \sqrt{1 - 2\lambda\eta + \lambda^2\sigma^2\xi^2} + \lambda(\Omega + \omega) \Big]\|u - v\| \\
&\leq \Big[k + \sqrt{1 - 2\lambda\eta + \lambda^2\sigma^2\xi^2} + \lambda p \Big]\|u - v\| \\
&\leq \theta\|u - v\|,
\end{aligned}
\tag{3.9}
$$

where

$$
\theta = k + \sqrt{1 - 2\lambda\eta + \lambda^2\sigma^2\xi^2} + \lambda p,
$$
$$
k = 2\Omega + \rho, \qquad p = \Omega + \omega, \qquad \Omega = \sqrt{1 - 2\alpha + \beta^2}, \qquad \omega = \sqrt{1 + 2\kappa + \delta^2\mu^2}.
\tag{3.10}
$$

It is easy to verify that (3.3) means $0 < \theta < 1$. Hence F is a contraction mapping and has a fixed point $u \in H$. It follows from Lemma 2.11 that u is a unique solution of problem (2.1). This completes the proof. □

Now we suggest the following perturbed iterative approximation process with errors for solving (2.1).

Algorithm 3.2. For any given $u_0 \in H$, compute the approximate solution $\{u_n\}$ by the perturbed iterative process with errors:

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n[v_n - g(v_n) + J_\lambda^{A_n(\cdot,v_n)}\{g(v_n) - \lambda(g(v_n) - N(Vv_n, Gv_n))\}] + \alpha_n e_n + l_n,$$

$$v_n = (1-\beta_n)u_n + \beta_n[w_n - g(w_n) + J_\lambda^{A_n(\cdot,w_n)}\{g(w_n) - \lambda(g(w_n) - N(Vw_n, Gw_n))\}] + f_n,$$

$$w_n = (1-\gamma_n)u_n + \gamma_n[u_n - g(u_n) + J_\lambda^{A_n(\cdot,u_n)}\{g(u_n) - \lambda(g(u_n) - N(Vu_n, Gu_n))\}] + h_n,$$

$$(3.11)$$

for all $n \geq 0$, where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in $[0,1]$; and $\{e_n\}$, $\{f_n\}$, $\{l_n\}$, and $\{h_n\}$ are bounded sequences in H, satisfying suitable conditions:

$$\sum_{n=0}^{n} \alpha_n = +\infty, \qquad \sum_{n=0}^{n} ||l_n|| < +\infty, \qquad \lim_{n\to\infty} ||e_n|| = \lim_{n\to\infty} ||f_n|| = \lim_{n\to\infty} ||h_n|| = 0, \qquad (3.12)$$

and $\lambda > 0$ is a constant.

If we remark that $\gamma_n = 0$ and $h_n = 0$, for $n \geq 0$, then Algorithm 3.2 reduces to the following.

Algorithm 3.3. For any given $u_0 \in H$, compute the approximate solution $\{u_n\}$ by the perturbed Ishikawa iterative process with errors:

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n[v_n - g(v_n) + J_\lambda^{A_n(\cdot,v_n)}\{g(v_n) - \lambda(g(v_n) - N(Vv_n, Gv_n))\}] + e_n\alpha_n + l_n,$$

$$v_n = (1-\beta_n)u_n + \beta_n[u_n - g(u_n) + J_\lambda^{A_n(\cdot,u_n)}\{g(u_n) - \lambda(g(u_n) - N(Vu_n, Gu_n))\}] + f_n,$$

$$(3.13)$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{e_n\}$, $\{f_n\}$, and $\{l_n\}$ are the same as in Algorithm 3.2.

If $\beta_n = 0$ and $f_n = 0$ for $n \geq 0$, then Algorithm 3.3 reduces to the following.

Algorithm 3.4. For any given $u_0 \in H$, compute the approximate solution $\{u_n\}$ by the perturbed Mann iterative process with errors:

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n[u_n - g(u_n) + J_\lambda^{A_n(\cdot,u_n)}\{g(u_n) - \lambda(g(u_n) - N(Vu_n, Gu_n))\}]$$

$$+ e_n\alpha_n + l_n,$$

$$(3.14)$$

where $\{\alpha_n\}$, $\{e_n\}$, and $\{l_n\}$ are the same as in Algorithm 3.2.

If $e_n = f_n = h_n = l_n = 0$, $n \geq 0$, then Algorithm 3.2 reduces to the following.

Algorithm 3.5. For any given $u_0 \in H$, compute the approximate solution $\{u_n\}$ by the iterative process:

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n[v_n - g(v_n) + J_\lambda^{A_n(\cdot, v_n)}\{g(v_n) - \lambda(g(v_n) - N(Vv_n, Gv_n))\}],$$

$$v_n = (1 - \beta_n)u_n + \beta_n[w_n - g(w_n) + J_\lambda^{A_n(\cdot, w_n)}\{g(w_n) - \lambda(g(w_n) - N(Vw_n, Gw_n))\}],$$

$$w_n = (1 - \gamma_n)u_n + \gamma_n[u_n - g(u_n) + J_\lambda^{A_n(\cdot, u_n)}\{g(u_n) - \lambda(g(u_n) - N(Vu_n, Gu_n))\}],$$

$$(3.15)$$

where α_n, β_n, and γ_n are the same as in Algorithm 3.2.

Now we discuss the convergence and stability of the iterative sequences with errors generated by Algorithm 3.2; we first give some concepts.

Let T be a self map of H, $x_0 \in H$, and $x_{n+1} = f(T, x_n)$, define an iterative procedure which yields a sequence of points $\{x_n\}$ in H. Suppose that $\{x \in H : Tx = x\} = \varnothing$ and $\{x_n\}$ converge to a fixed point $x^\star \in H$. Let $\{y_n\} \subset H$ and $\epsilon_n = \|y_{n+1} - f(T, y_n)\|$. If $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} y_n = x^\star$, then the iterative procedure $\{x_n\}$ defined by $x_{n+1} = f(T, x_n)$ is said to be T-stable or stable with respect to T. If $\sum_{n=0}^{\infty} \epsilon_n < +\infty$ implies that $\lim_{n\to\infty} y_n = x^\star$, then the iterative procedure $\{x_n\}$ is said to almost T-stable.

Remark 3.6. An iterative procedure $\{x_n\}$ which is T-stable is almost T-stable, and an iterative procedure $\{x_n\}$ which is almost T-stable need not be T-stable, see [5].

THEOREM 3.7. *Let g, V, G, and N be the same as in Theorem 3.1, and the conditions (3.1) and (3.3) hold. Let $\{A_n\}$ and A be maximal monotone mappings from H into power set of H such that $A_n \xrightarrow{G} A$. Let $\{x_n\}$ be a sequence in H and define a sequence $\{\epsilon_n\}$ of real numbers as follows:*

$$\epsilon_n = \|x_{n+1} = \{(1 - \alpha_n)x_n$$

$$+ \alpha_n[y_n - g(y_n) + J_\lambda^{A_n(\cdot, y_n)}(g(y_n) - \lambda(g(y_n) - N(Vy_n, Gy_n)))]$$

$$+ \alpha_n e_n + l_n\}\|,$$

$$y_n = (1 - \beta_n)x_n + \beta_n[z_n - g(z_n) + J_\lambda^{A_n(\cdot, z_n)}(g(z_n) - \lambda(g(z_n) - N(Vz_n, Gz_n)))] + f_n,$$

$$z_n = (1 - \gamma_n)x_n + \gamma_n[x_n - g(x_n) + J_\lambda^{A_n(\cdot, x_n)}(g(x_n) - \lambda(g(x_n) - N(Vx_n, Gx_n)))] + h_n.$$

$$(3.16)$$

Then the following conditions hold.

(i) *The sequence $\{u_n\}$ generated by Algorithm 3.2 converges strongly to the unique solution $u^\star$ of the problem (2.1).*

(ii) *If $\epsilon_n = \alpha_n D_n + q_n$ with $\sum_{n=0}^{\infty} q_n < +\infty$ and $\lim_{n\to\infty} D_n = 0$, then*

$$\lim_{n\to\infty} x_n = u^\star. \tag{3.17}$$

(iii) *$\lim_{n\to\infty} x_n = u^\star$ implies that*

$$\lim_{n\to\infty} \epsilon_n = 0. \tag{3.18}$$

Proof. By Theorem 3.1, we know that problem (2.1) has a unique solution $u^\star \in H$. It is easy to see that conclusion (i) follows from (ii). Now we prove (ii). It follows from the Lemma 2.11 that

$$
\begin{aligned}
u^\star &= (1-\alpha_n)u^\star + \alpha_n[u^\star - g(u^\star) + J_\lambda^{A(\cdot,u^\star)}\{g(u^\star) - \lambda(g(u^\star) - N(Vu^\star, Gu^\star))\}] \\
&= (1-\beta_n)u^\star + \beta_n[u^\star - g(u^\star) + J_\lambda^{A(\cdot,u^\star)}\{g(u^\star) - \lambda(g(u^\star) - N(Vu^\star, Gu^\star))\}] \\
&= (1-\gamma_n)u^\star + \gamma_n[u^\star - g(u^\star) + J_\lambda^{A(\cdot,u^\star)}\{g(u^\star) - \lambda(g(u^\star) - N(Vu^\star, Gu^\star))\}].
\end{aligned}
\tag{3.19}
$$

From (3.16) and (3.19), we have

$$
\begin{aligned}
\|x_{n+1} &- u^\star\| \\
&\leq \|x_{n+1} - \{(1-\alpha_n)x_n \\
&\qquad + \alpha_n[y_n - g(y_n) + J_\lambda^{A_n(\cdot,y_n)}(g(y_n) - \lambda(g(y_n) - N(Vy_n, Gy_n)))] \\
&\qquad + \alpha_n e_n + l_n\}\| \\
&\quad + \|(1-\alpha_n)(x_n - u^\star) + \alpha_n[y_n - u^\star - (g(y_n) - g(u^\star))] \\
&\qquad + \alpha_n[J_\lambda^{A_n(\cdot,y_n)}(g(y_n) - \lambda(g(y_n) - N(Vy_n, Gy_n))) \\
&\qquad\quad - J_\lambda^{A(\cdot,u^\star)}(g(u^\star) - \lambda(g(u^\star) - N(Vu^\star, Gu^\star)))]\| \\
&\quad + \alpha_n\|e_n\| + \|l_n\| \\
&\leq (1-\alpha_n)\|x_n - u^\star\| + \alpha_n\|y_n - u^\star - (g(y_n) - g(u^\star))\| \\
&\quad + \alpha_n\|J_\lambda^{A_n(\cdot,y_n)}(g(y_n) - \lambda(g(y_n) - N(Vy_n, Gy_n))) \\
&\qquad - J_\lambda^{A_n(\cdot,y_n)}(g(u^\star) - \lambda(g(u^\star) - N(Vu^\star, Gu^\star)))\| \\
&\quad + \alpha_n\|J_\lambda^{A_n(\cdot,y_n)}(g(u^\star)\lambda(g(u^\star) - N(Vu^\star, Gu^\star))) \\
&\qquad - J_\lambda^{A(\cdot,u^\star)}(g(u^\star) - \lambda(g(u^\star) - N(Vu^\star, Gu^\star)))\| \\
&\quad + \alpha_n\|J_\lambda^{A_n(\cdot,u^\star)}(g(u^\star) - \lambda(g(u^\star) - N(Vu^\star, Gu^\star))) \\
&\qquad - J_\lambda^{A(\cdot,u^\star)}(g(u^\star) - \lambda(g(u^\star) - N(Vu^\star, Gu^\star)))\| + \epsilon_n + \alpha_n\|e_n\| + \|l_n\| \\
&\leq (1-\alpha_n)\|x_n - u^\star\| + 2\alpha_n\|y_n - u^\star - (g(y_n) - g(u^\star))\| \\
&\quad + \alpha_n\|y_n - u^\star + \lambda(N(Vy_n, Gy_n) - N(Vu^\star, Gy_n))\| \\
&\quad + \alpha_n\lambda\|y_n - u^\star - (g(y_n) - g(u^\star))\| \\
&\quad + \alpha_n\lambda\|y_n - u^\star - (N(Vu^\star, Gy_n) - N(Vu^\star, Gu^\star))\| \\
&\quad + \alpha_n\rho\|y_n - u^\star\| + \epsilon_n + \alpha_n P_n + \alpha_n\|e_n\| + \|l_n\|,
\end{aligned}
\tag{3.20}
$$

where

$$P_n = ||J_\lambda^{A_n(\cdot, u^\star)}(g(u^\star) - \lambda(g(u^\star) - N(Vu^\star, Gu^\star)))$$
$$- J_\lambda^{A(\cdot, u^\star)}(g(u^\star) - \lambda(g(u^\star) - N(Vu^\star, Gu^\star)))|| \longrightarrow 0. \tag{3.21}$$

As the proof of (3.6)–(3.8), we have

$$||y_n - u^\star - (g(y_n) - g(u^\star))|| \le \sqrt{1 - 2\alpha + \beta^2}||y_n - u^\star||,$$

$$||y_n - u^\star + \lambda(N(Vy_n, Gy_n) - N(Vu^\star, Gy_n))|| \le \sqrt{1 - 2\lambda\eta + \lambda^2\sigma^2\xi^2}||y_n - u^\star||, \tag{3.22}$$

$$||y_n - u^\star - (N(Vu^\star, Gy_n) - N(Vu^\star, Gu^\star))|| \le \sqrt{1 + 2\kappa + \delta^2\mu^2}||y_n - u^\star||.$$

It follows from (3.20)–(3.22) that

$$||x_{n+1} - u^\star|| \le (1 - \alpha_n)||x_n - u^\star|| + \theta\alpha_n||y_n - u^\star|| + \epsilon_n + \alpha_n P_n + \alpha_n||e_n|| + ||l_n||, \tag{3.23}$$

where

$$\theta = k + \sqrt{1 - 2\lambda\eta + \lambda^2\sigma^2\xi^2} + \lambda p < 1,$$
$$k = 2\Omega + \rho, \qquad p = \Omega + \omega, \qquad \Omega = \sqrt{1 - 2\alpha + \beta^2}, \qquad \omega = \sqrt{1 + 2\kappa + \delta^2\mu^2}. \tag{3.24}$$

Similarly, we have

$$||y_n - u^\star|| \le (1 - \beta_n)||x_n - u^\star|| + \beta_n\theta||z_n - u^\star|| + \beta_n P_n + ||f_n||, \tag{3.25}$$

$$||z_n - u^\star|| \le (1 - \gamma_n)||x_n - u^\star|| + \gamma_n\theta||x_n - u^\star|| + \gamma_n P_n + ||h_n||$$
$$\le (1 - \gamma_n(1 - \theta))||x_n - u^\star|| + \gamma_n P_n + ||h_n|| \tag{3.26}$$
$$\le ||x_n - u^\star|| + \gamma_n P_n + ||h_n||,$$

where

$$(1 - \gamma_n(1 - \theta)) \le 1. \tag{3.27}$$

Substituting (3.26) into (3.25), we have

$$||y_n - u^\star|| \le (1 - \beta_n)||x_n - u^\star|| + \beta_n\theta||x_n - u^\star|| + \beta_n\gamma_n P_n\theta + \beta_n\theta||h_n|| + \beta_n P_n + ||f_n||$$
$$\le (1 - \beta_n(1 - \theta))||x_n - u^\star|| + \beta_n\gamma_n P_n\theta + \beta_n\theta||h_n|| + \beta_n P_n + ||f_n||$$
$$\le ||x_n - u^\star|| + 2P_n + ||h_n|| + ||f_n||. \tag{3.28}$$

From (3.23) and (3.28),

$$\|x_{n+1} - u^\star\| \le (1 - \alpha_n)\|x_n - u^\star\| + \alpha_n\theta\|x_n - u^\star\| + \alpha_n\theta(2P_n + \|h_n\| + \|f_n\|)$$

$$+ \alpha_n\Delta_n + q_n + \alpha_n P_n + \alpha_n\|e_n\| + \|l_n\|$$

$$\le (1 - \alpha_n(1 - \theta))\|x_n - u^\star\| \tag{3.29}$$

$$+ \alpha_n(1 - \theta)\frac{1}{1 - \theta}(3P_n + \|h_n\| + \|f_n\| + \Delta_n + \|e_n\|) + (q_n + \|l_n\|).$$

Set $a_n = x_n - u^\star$, $b_n = (1/(1 - \theta))(3P_n + \|h_n\| + \|f_n\| + \Delta_n + \|e_n\|)$ and $t_n = \alpha_n(1 - \theta)$, $c_n = q_n + \|l_n\|$.
Then we can rewrite (3.29) as follows:

$$a_{n+1} \le (1 - t_n)a_n + b_n t_n + c_n. \tag{3.30}$$

From the assumption, we know that $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{t_n\}$ satisfy the conditions of Lemma 2.10. It follows from Lemma 2.10 that $a_n \to 0$ and so $x_n \to u^\star$ as $n \to \infty$.

Next, we prove (iii). From (3.11) and (3.28), we know that $y_n \to u^\star$ as $n \to \infty$. It follows from (3.16) that

$$\epsilon_n \le \|x_n - u^\star\|$$

$$+ \|(1 - \alpha_n)x_n + \alpha_n[y_n - g(y_n) + J_\lambda^{A_n(\cdot, y_n)}(g(y_n) - \lambda(g(y_n) - N(Vy_n, Gy_n)))] - u^\star\|$$

$$+ \alpha_n\|e_n\| + \|l_n\|. \tag{3.31}$$

As the proof of (3.23), we have

$$\|(1 - \alpha_n)x_n + \alpha_n[y_n - g(y_n) + J_\lambda^{A_n(\cdot, y_n)}(g(y_n) - \lambda(g(y_n) - N(Vy_n, Gy_n)))] - u^\star\|$$

$$\le (1 - \alpha_n)\|x_n - u^\star\| + \alpha_n\theta\|y_n - u^\star\| + \alpha_n P_n. \tag{3.32}$$

Substituting (3.32) and (3.31), we get

$$\epsilon_n \le \|x_{n+1} - u^\star\| + (1 - \alpha_n)\|x_n - u^\star\| + \alpha_n\theta\|y_n - u^\star\| + \alpha_n P_n. \tag{3.33}$$

Since $x_n \to u^\star$, $y_n \to u^\star$ and $P_n \to 0$ as $n \to \infty$, it follows that $\epsilon_n \to 0$. This completes the proof. $\qquad\square$

Acknowledgment

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M. K. Ahmad: Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
E-mail address: mmt13mka@amu.ac.in

Salahuddin: Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
E-mail address: salahuddin12@mailcity.com