GENERALIZED BAER RINGS

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We investigate the question whether the p.q.-Baer center of a ring R can be extended to R. We give several counterexamples to this question and consider some conditions under which the answer may be affirmative. The concept of a generalized p.q.-Baer property which is a generalization of Baer property of a ring is also introduced.

1. Introduction

In [15], Kaplansky introduced *Baer* rings as rings in which every right (left) annihilator ideal is generated by an idempotent. According to Clark [9], a ring R is called *quasi-Baer* if the right annihilator of every right ideal is generated (as a right ideal) by an idempotent. Further works on quasi-Baer rings appear in [4, 6, 17]. Recently, Birkenmeier et al. [8] called a ring R to be a *right* (resp., *left*) *principally quasi-Baer* (or simply *right* (resp., *left*) *p.q.-Baer*) ring if the right (resp., left) annihilator of a principal right (resp., left) ideal is generated by an idempotent. R is called a *p.q.-Baer* ring if it is both right and left p.q.-Baer. The class of right or left p.q.-Baer rings is a nontrivial generalization of the class of quasi-Baer rings. For example, if R is a commutative von Neumann regular ring which is not complete, then R is p.q.-Baer but not quasi-Baer. Observe that every biregular ring is also a p.q.-Baer ring.

A ring satisfying a generalization of Rickart's condition (i.e., every right annihilator of any element is generated (as a right ideal) by an idempotent) has a homological characterization as a right *PP*-ring which is also another generalization of a Baer ring. A ring R is called a *right* (resp., *left*) *PP*-ring if every principal right (resp., left) ideal is projective (equivalently, if the right (resp., left) annihilator of an element of R is generated (as a right (resp., left) ideal) by an idempotent of R). R is called a *PP*-ring (also called a *Rickart* ring [3, page 18]) if it is both right and left *PP*. Baer rings are clearly right (left) *PP*-rings, and von Neumann regular rings are also right (left) *PP*-rings by Goodearl [10, Theorem 1.1]. Note that the conditions right *PP* and right p.q.-Baer are distinct

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[8, Examples 1.3 and 1.5], but R is an abelian PP-ring if and only if R is a reduced p.q.-Baer ring [8, Corollary 1.15].

Throughout this paper, R denotes an associative ring with identity. For a nonempty subset X of R, we write $r_R(X) = \{a \in R \mid Xa = 0\}$ and $\ell_R(X) = \{a \in R \mid aX = 0\}$, which are called the right and left annihilators of X in R, respectively.

2. Principally quasi-Baer centers

As a motivation for this section, we recall the following results.

(1) The center of a Baer ring is Baer [15, Theorem 7].

(2) The center of a quasi-Baer ring is quasi-Baer [7, Proposition 1.8].

(3) The center of a right p.q.-Baer ring is PP (hence p.q.-Baer) [8, Proposition 1.12].

(4) Every reduced PI-ring with the Baer center is a Baer ring [1, Theorem D].

It is natural to ask if the p.q.-Baer center of a ring R can be extended to R. In this section, we show that this question has a negative answer, and so we investigate the class of rings with some conditions under which the answer to this question is affirmative.

Let $C(R)$ denote the center of a ring R.

Example 2.1. (1) Let K be a field. We consider the ring $R = K[X,Y,Z]$ with $XY = XZ = ZX = YX = 0$ and $YZ \neq ZY$. Then R is reduced and $C(R) = K[X]$ is Baer and so p.q-Baer. But $r_R(Y)$ has no idempotents. Thus R is not right p.q.-Baer. Note that

$$I = \{f(Y,Z) \in K[Y,Z] \mid f(0,0) = 0\} \tag{2.1}$$

is a two-sided ideal of R and $I \cap C(R) = 0$.

(2) Let

$$R = \left\{ \begin{pmatrix} x & y & z \\ 0 & x & u \\ 0 & 0 & v \end{pmatrix} \mid x,y,z,u,v \in \mathbb{R} \right\} \subseteq \mathrm{Mat}_3(\mathbb{R}), \tag{2.2}$$

where $\mathbb{R}$ denotes the set of real numbers. Then R is a PI-ring which is not semiprime. Then we see that

$$r_R\left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} R \right) = \left\{ \begin{pmatrix} 0 & b & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid b,c \in \mathbb{R} \right\}. \tag{2.3}$$

But this cannot be generated by an idempotent. Hence R is not right p.q.-Baer. On the other hand,

$$C(R) = \left\{ \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \mid x \in \mathbb{R} \right\} \cong \mathbb{R}. \tag{2.4}$$

Therefore $C(R)$ is Baer.

Observe that Example 2.1(2) also shows that there exists a PI-ring R with the Baer center, but R is not right p.q.-Baer.

However, we have the following results.

LEMMA 2.2 [8, Proposition 1.7]. *R is a right p.q.-Baer ring if and only if the right annihilator of any finitely generated right ideal is generated (as a right ideal) by an idempotent.*

PROPOSITION 2.3. *Let R be a ring with the p.q.-Baer center $C(R)$. If R satisfies any of the following conditions for any nonzero two-sided ideal I of R, then R is quasi-Baer (and hence right p.q.-Baer):*

(1) *$I \cap C(R)$ is a nonzero finitely generated right ideal of $C(R)$;*

(2) *$I \cap C(R) \neq 0$ and every central idempotent of R is orthogonal;*

(3) *$I \cap C(R) \neq 0$ and every right ideal of R generated by a central element contains $C(R)$.*

Proof. Let I be a nonzero two-sided ideal of R. If $r_R(I) = 0$, then we are done. Thus we assume that $r_R(I) \neq 0$.

(1) By hypothesis and Lemma 2.2, $I \cap C(R) \neq 0$ and $r_{C(R)}(I \cap C(R)) = eC(R)$ for some $e^2 = e \in C(R)$. We claim that $r_R(I) = eR$. If $Ie \neq 0$, then Ie is a nonzero two-sided ideal of R. Thus, by hypothesis, $0 \neq Ie \cap C(R) \subseteq I \cap C(R)$. Let $0 \neq x \in Ie \cap C(R)$. Then $x = ye \in I \cap C(R)$ for some $y \in I$, and so $x = xe = 0$; which is a contradiction. Hence $eR \subseteq r_R(I)$, and then $r_R(I) = R \cap r_R(I) = (eR \oplus (1-e)R) \cap r_R(I) = eR \oplus ((1-e)R \cap r_R(I))$. We show that $(1-e)R \cap r_R(I) = 0$. Suppose that $0 \neq (1-e)R \cap r_R(I)$. Then $(1-e)R \cap C(R)$ is a nonzero two-sided ideal of R. Thus, by hypothesis, $0 \neq (1-e)R \cap r_R(I) \cap C(R) = (1-e)R \cap r_{C(R)}(I) \subseteq (1-e)R \cap r_{C(R)}(I \cap C(R)) \subseteq (1-e)R \cap eC(R) \subseteq (1-e)R \cap eR = 0$; which is also a contradiction. Therefore $r_R(I) = eR$, and thus R is quasi-Baer.

(2) There exists $0 \neq a \in C(R)$ such that $a \in I$, and so $r_{C(R)}(aC(R)) = eC(R)$ for some $e^2 = e \in C(R)$ by hypothesis. Then $r_R(aR) = eR$. Since $r_R(aR) \cap C(R) = r_{C(R)}(aC(R)) = eC(R)$, $e \in r_R(aR)$, and so $eR \subseteq r_R(aR)$, and thus $r_R(aR) = eR$ by the similar method to (1). Hence $r_R(I) \subseteq eR$. Now, we claim that $eR \subseteq r_R(I)$. If not, there exists $0 \neq x \in R$ such that $x \in I \cap C(R)$ by the same arguments as in (1). Then $r_{C(R)}(xC(R)) = fC(R)$ for some $f^2 = f \in C(R)$, and so $r_R(xR) = fR$. Hence $r_R(I) \subseteq fR \cap eR = 0$; which is a contradiction. Thus $r_R(I) = eR$ for some $e^2 = e \in R$, and therefore R is a quasi-Baer ring.

(3) By hypothesis, there exists $0 \neq a \in I \cap C(R)$, and so $r_{C(R)}(aC(R)) = eC(R)$ for some $e^2 = e \in C(R)$. Then $r_R(aR) = eR$, and this implies that $r_R(I) \subseteq eR$ by the same method as in (2). Now, we claim that $eR \subseteq r_R(I)$. If not, there exists $0 \neq x \in Ie \cap C(R) \subseteq I \cap aR \subseteq aR$, by hypothesis. We put $x = ye \in C(R)$ for some $y \in I$. Since $r_R(x) \supseteq r_R(aR) = eR$, we obtain $x = xe = 0$; which is a contradiction. Thus $eR \subseteq r_R(I)$, and consequently $r_R(I) = eR$. Therefore R is a quasi-Baer ring. □

COROLLARY 2.4. *Let R be a semiprime PI-ring with the p.q.-Baer center $C(R)$. If either every central idempotent of R is orthogonal or every right ideal of R generated by a central element contains $C(R)$, then R is quasi-Baer.*

The proof follows from [18, Theorem 6.1.28] and Proposition 2.3.

Part (1) of the following example shows that the condition "$I \cap C(R)$ is a nonzero finitely generated right ideal of $C(R)$" and the condition "every central idempotent of R is orthogonal" in Proposition 2.3(1) and (2) are not superfluous, respectively, and parts (2) and (3) show that in Proposition 2.3, the condition (1) is not equivalent to the condition (2).

Example 2.5. (1) Let $R = \{\langle a_i\rangle \in \prod_{i=1}^{\infty} T_i \mid a_i \text{ is eventually constant}\}$, where $T_i = \mathrm{Mat}_2(F)$ for all i and F is a field. For a two-sided ideal $I = \{\langle a_i\rangle \in R \mid a_i = 0 \text{ if } i \text{ is even}\}$, $r_R(I) = \{\langle b_j\rangle \in R \mid b_j = 0 \text{ if } i \text{ is odd}\}$. Since

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots \right\rangle \notin R, \tag{2.5}$$

$r_R(I)$ cannot be generated by an idempotent of R. Thus R is not quasi-Baer. Note that

$$C(R) = \left\{ \langle a_i\rangle \in R \mid a_i = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} \text{ for some } k \in F \right\} \tag{2.6}$$

is p.q.-Baer. Now,

$$I \cap C(R) = \left\{ \langle a_i\rangle \in R \mid a_i = 0 \text{ if } i \text{ is even},\ a_i = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in \mathrm{Mat}_2(F) \text{ if } i \text{ is odd} \right\} \tag{2.7}$$

is not finitely generated. Moreover,

$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots \right\rangle, \quad \left\langle \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \dots \right\rangle \tag{2.8}$$

are idempotents, but they are not orthogonal.

(2) Let $R = F[x_1, x_2, \dots]$, where F is a field. Then R is a commutative quasi-Baer ring whose only idempotents 0 and 1 are orthogonal, but the two-sided ideal $\langle x_1^2, x_2^2, \dots\rangle$ of R is not finitely generated.

(3) Let $R = \mathbb{Z} \oplus \mathbb{Z}$. Then R is a commutative quasi-Baer ring. Since R is Noetherian, every two-sided ideal of R is finitely generated. But the central idempotents $(1,0)$ and $(1,1)$ are not orthogonal.

Related to the result of [1, Theorem D], we have the next example.

Example 2.6. (1) Let $R = \mathscr{C}[0,1]$ be the ring of all real-valued continuous functions on $[0,1]$. Then R is commutative (and so PI) and reduced. But R is not p.q.-Baer. Let

$$f : [0,1] \longrightarrow \mathbb{R} \tag{2.9}$$

be defined by

$$f(x) = \begin{cases} 0, & 0 \le x \le \dfrac{1}{2}, \\ x - \dfrac{1}{2}, & \dfrac{1}{2} < x \le 1. \end{cases} \tag{2.10}$$

Then $f \in R$, and so

$$r_R(f) = \left\{ g \in R \mid g\left(\left(\frac{1}{2}, 1\right]\right) = 0 \right\} \neq 0. \tag{2.11}$$

Suppose that $r_R(f) = eR$ for some nonzero idempotent $e \in R$. Then $e(x)^2 = e(x)$, for each $x \in [0,1]$. Thus $e(x) = 0$ or $e(x) = 1$. Since $e \in r_R(f)$, $e((1/2,1]) = \{0\}$. But e is continuous, and so $e(x) = 0$ for each $x \in [0,1]$. Hence $r_R(f) = 0$; which is a contradiction. Thus R is a reduced PI-ring which is not right p.q.-Baer.

(2) We take the ring in [12, Example 2(1)]. Let $\mathbb{Z}$ be the ring of integers and $\mathrm{Mat}_2(\mathbb{Z})$ the 2×2 full matrix ring over $\mathbb{Z}$. Let

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}) \mid a - d \equiv b \equiv c \equiv 0 (\mathrm{mod}\, 2) \right\}. \tag{2.12}$$

Then R is right p.q.-Baer, but R is neither right *PP* nor left *PP* by [12, Example 2(1)]. Moreover, it can be easily checked that R is an abelian PI-ring with the *PP* center.

3. Generalized p.q.-Baer rings

Regarding a generalization of Baer rings as well as a *PP*-ring, recall that a ring R is called a *generalized right PP*-ring if for any $x \in R$, the right ideal $x^n R$ is projective for some positive integer n, depending on n, equivalently, if for any $x \in R$, the right annihilator of x^n is generated by an idempotent for some positive integer n, depending on n. Left cases may be defined analogously. A ring R is called a *generalized PP*-ring if it is both generalized right and left *PP*-ring. Right *PP*-rings are generalized right *PP* obviously. A number of papers have been written on generalized *PP*-rings. For basic and other results on generalized *PP*-rings, see, for example, [11, 14, 16].

As a parallel definition to the generalized *PP*-property related to the p.q.-Baer property, we define the following.

Definition 3.1. A ring R is called a *generalized right p.q.-Baer* ring if for any $x \in R$, the right annihilator of $x^n R$ is generated by an idempotent for some positive integer n, depending on n. Left cases is defined analogously. A ring R is called a *generalized p.q.-Baer* ring if it is both generalized right and left p.q.-Baer ring.

We have the following connections.

Lemma 3.2 [12, Lemma 1]. *Let R be a reduced ring. The following are equivalent:*

(1) *R is right PP;*
(2) *R is PP;*
(3) *R is generalized right PP;*
(4) *R is generalized PP;*
(5) *R is right p.q.-Baer;*

(6) *R is p.q.-Baer;*
(7) *R is generalized right p.q.-Baer;*
(8) *R is generalized p.q.-Baer;*

Shin [19] defined that a ring R *satisfies* (SI) if for each $a \in R$, $r_R(a)$ is a two-sided ideal of R, and proved that R satisfies (SI) if and only if $ab = 0$ implies that $aRb = 0$ for $a, b \in R$ [19, Lemma 1.2]. The (SI) property was studied in the context of near-rings by Bell, in [2], where it is called the insertion of factors principle (IFP). It is well known that every reduced ring has the IFP, and if R has the IFP then it is abelian, but the converses do not hold, respectively.

Recall from [8, Corollary 1.15] that R is an abelian PP-ring if and only if R is a reduced p.q.-Baer ring. Similarly, we have the following.

Proposition 3.3. *Let a ring R have the IFP. Then R is a generalized right PP-ring if and only if R is a generalized right p.q.-Baer ring.*

Proof. For any $x \in R$ and positive integer n, $r_R(x^n) = r_R(x^n R)$ since R has the IFP. □

Every right p.q.-Baer rings is a generalized right p.q.-Baer, but the converse does not hold, by the next example.

Given a ring R and an (R,R)-bimodule M, the *trivial extension* of R by M is the ring $T(R,M) = R \oplus M$ with the usual addition and the following multiplication:

$$(a_1, m_1)(a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2). \tag{3.1}$$

This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} a & m \\ 0 & a \end{smallmatrix}\right)$, where $a \in R$ and $m \in M$ and the usual matrix operations are used.

Example 3.4 [14, Example 2]. Let D be a domain and let $R = T(D,D)$ be the trivial extension of D. Then R has the IFP and R is a generalized right PP-ring, but it is not a right PP-ring. Thus R is a generalized right p.q.-Baer ring by Proposition 3.3, but it is not right p.q.-Baer by [8, Proposition 1.14].

Recall from [5] that an idempotent $e \in R$ is called *left* (resp., *right*) *semicentral* if $xe = exe$ (resp., $ex = exe$) for all $x \in R$. The set of left (resp., right) semicentral idempotents of R is denoted by $S_\ell(R)$ (resp., $S_r(R)$). Note that $S_\ell(R) \cap S_r(R) = \mathbf{B}(R)$, where $\mathbf{B}(R)$ is the set of all central idempotents of R, and if R is semiprime then $S_\ell(R) = S_r(R) = \mathbf{B}(R)$. Some of the basic properties of these idempotents are indicated in the following.

Lemma 3.5 [7, Lemma 1.1]. *For an idempotent $e \in R$, the following are equivalent:*
(1) $e \in S_\ell(R)$;
(2) $1 - e \in S_r(R)$;
(3) $(1 - e)Re = 0$;
(4) *eR is a two-sided ideal of R;*
(5) *$R(1 - e)$ is a two-sided ideal of R.*

The following example shows that the condition "R has the IFP" in Proposition 3.3 cannot be dropped.

Example 3.6 [8, Example 1.6]. For a field F, take $F_n = F$ for $n = 1, 2, \ldots$, and let

$$R = \begin{pmatrix} \prod_{n=1}^{\infty} F_n & \bigoplus_{n=1}^{\infty} F_n \\ \bigoplus_{n=1}^{\infty} F_n & \left\langle \bigoplus_{n=1}^{\infty} F_n, 1 \right\rangle \end{pmatrix}, \tag{3.2}$$

which is a subring of the 2×2 matrix ring over the ring $\prod_{n=1}^{\infty} F_n$, where $< \bigoplus_{n=1}^{\infty} F_n, 1 >$ is the F-algebra generated by $\oplus_{n=1}^{\infty} F_n$ and 1. Then R is a regular ring by [10, Lemma 1.6], and so R is a generalized *PP*-ring.

Let $a \in (a_n) \in \prod_{n=1}^{\infty} F_n$ such that $a_n = 1$ if n is odd and $a_n = 0$ if n is even, and let $\alpha = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) \in R$. Now we assume that there exists an idempotent $e \in R$ such that $r_R(\alpha^k R) = eR$ for a positive integer k. Then e is left semicentral, and so e is central since R is semiprime, but this is impossible. Thus R is not generalized right p.q.-Baer. Similarly R is not generalized left p.q.-Baer.

Proposition 3.7. *Let R be a ring. The following are equivalent:*

(1) *R is generalized right p.q.-Baer;*

(2) *for any principal ideal I of the form Ra^nR of R, where n is a positive integer, there exists $e \in S_r(R)$ such that $I \subseteq Re$ and $r_R(I) \cap Re = (1-e)Re$.*

Proof. The proof is an adaptation from [8, Proposition 1.9]. (1)⇒(2). Assume (1) holds. Then $r_R(I) = r_R(Ra^nR) = r_R(a^nR) = fR$ with $f \in S_\ell(R)$. So $I \subseteq \ell_R(r_R(I)) = R(1-f)$. Let $e = 1 - f$, then $e \in S_r(R)$. Hence $r_R(I) \cap Re = (1-e)R \cap Re = (1-e)Re$.

(2)⇒(1). Assume (2) holds. Clearly $(1-e)R \subseteq r_R(I)$ for any ideal I of the form Ra^nR. Let $\alpha \in r_R(I)$, then $\alpha e = e\alpha e + (1-e)\alpha e \in r_R(I) \cap Re = (1-e)Re$. So $e\alpha = e\alpha e = 0$. Hence $\alpha = (1-e)\alpha \in (1-e)R$. Thus $r_R(I) = (1-e)R$, and therefore R is generalized right p.q.-Baer. □

Corollary 3.8. *Let R be a generalized right p.q.-Baer ring. If I is a principal ideal of the form Ra^nR of R, then there exists $e \in S_r(R)$ such that $I \subseteq Re$, $(1-e)Re$ is an ideal of R, and $I + (1-e)Re$ is left essential in Re.*

As a parallel result to [8, Proposition 1.12], we have the following whose proof is also an adaptation from [8].

Proposition 3.9. *If R is a generalized right p.q.-Baer ring, then the center $C(R)$ of R is a generalized PP-ring.*

Proof. Let $a \in C(R)$. For any positive integer n, there exists $e \in S_\ell(R)$ such that $\ell_R(a^n) = \ell_R(Ra^n) = r_R(a^n) = r_R(a^nR) = eR$. Observe that $\ell_R(Ra^n) = \ell_R r_R \ell_R(Ra^n) = \ell_R r_R(eR)$. Let $r_R(eR) = r_R(e^nR) = fR$ with $f \in S_\ell(R)$, then $1 - f \in S_r(R)$. Hence $eR = \ell_R(Ra^n) = \ell_R r_R(eR) = \ell_R(fR) = R(1-f)$. So there exists $x \in R$ such that $e = x(1-f)$, and hence $ef = x(1-f)f = 0$. Now $fe = efe = 0$ because $e \in S_\ell(R)$, and so $ef = fe = 0$. Since $eR = R(1-f)$, there is $y \in R$ such that $1 - f = ey$, and so $e = e(1-f) = ey = 1 - f$. Thus $e \in S_\ell(R) \cap S_r(R) = \mathbf{B}(R)$. Consequently, $r_{C(R)}(a^n) = r_R(a^n) \cap C(R) = eR \cap C(R) = eC(R)$. Therefore the center $C(R)$ of R is a generalized *PP*-ring. □

The following example shows that there exists a semiprime ring $\mathcal{R}$ whose center is a generalized PP, but $\mathcal{R}$ is not a generalized right p.q.-Baer.

Example 3.10. Let $\mathcal{R} = R \oplus \mathrm{Mat}_2(\mathbb{Z}[x])$, where

$$R = \begin{pmatrix} \prod_{n=1}^{\infty} F_n & \bigoplus_{n=1}^{\infty} F_n \\ \bigoplus_{n=1}^{\infty} F_n & \left\langle \bigoplus_{n=1}^{\infty} F_n, 1 \right\rangle \end{pmatrix}, \tag{3.3}$$

in Example 3.6. Then the center of $\mathcal{R}$ is generalized PP. Since R is not a generalized right p.q.-Baer by Example 3.6, $\mathcal{R}$ is not a generalized right p.q.-Baer either. Furthermore, due to [14, Example 4], $\mathrm{Mat}_2(\mathbb{Z}[x])$ is not a generalized right PP. Thus $\mathcal{R}$ is not generalized right PP.

Note that given a reduced ring R, the trivial extension of R (by R) has the IFP by simple computations. However, the trivial extension of a ring R which has the IFP does not have the IFP by [13, Example 11]. We give examples of generalized right p.q.-Baer rings, which are extensions of the trivial extension, as in the following.

Lemma 3.11. *Let S be a ring and for $n \geq 2$,*

$$R_n = \left\{ \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in S \right\}. \tag{3.4}$$

If S has the IFP, then for any $A \in R_n$ and any $E^2 = E \in R_n$, $AE = \mathbf{0}$ implies that $AR_nE = \mathbf{0}$, where $\mathbf{0}$ is the zero matrix in R_n.

Proof. Note that every idempotent E in R_n is of the form

$$\begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{pmatrix} \tag{3.5}$$

with $e^2 = e \in S$ by [14, Lemma 2]. Suppose that $AE = \mathbf{0}$ for any

$$A = \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in R_n. \tag{3.6}$$

Then we have the following: $ae = 0$ and $a_{ij}e = 0$ for $i < j$, $1 \le i$ and $2 \le j$. Since S has the IFP, $aSe = 0$ and $a_{ij}Se = 0$ for $i < j$, $1 \le i$ and $2 \le j$. These imply that $AR_nE = \mathbf{0}$. □

Proposition 3.12. *Let a ring S have the IFP and let R_n for $n \ge 2$ be the ring in Lemma 3.11. Then the following are equivalent:*

(1) *S is generalized right p.q.-Baer;*

(2) *R_n is generalized right PP;*

(2) *R_n is generalized right p.q.-Baer.*

Proof. (1)⇒(2). Suppose that S is generalized right p.q.-Baer. By Proposition 3.3, S is a generalized right *PP*. Hence R_n is also a generalized right *PP* by [14, Proposition 3].

(2)⇒(3). Suppose that R_n is a generalized right *PP*. Then for any

$$A = \begin{pmatrix} a & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in R_n \tag{3.7}$$

and a positive integer k, there exists an idempotent

$$E = \begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{pmatrix} \in R_n \tag{3.8}$$

with $e^2 = e \in S$ such that $r_{R_n}(A^k) = ER_n$. Note that $r_{R_n}(A^kR_n) \subseteq ER_n$. From $r_{R_n}(A^k) = ER_n$, $A^kE = \mathbf{0}$, and so $A^kR_nE = \mathbf{0}$ by Lemma 3.11. Thus we have $E \in r_{R_n}(A^kR_n)$, and so $ER_n \subseteq r_{R_n}(A^kR_n)$. Consequently, $r_{R_n}(A^kR_n) = ER_n$, and therefore R_n is generalized right p.q.-Baer.

(3)⇒(1). Suppose that R_n is a generalized right p.q.-Baer. Let $a \in S$ and consider

$$A = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 0 & a & 0 & \cdots & 0 \\ 0 & 0 & a & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a \end{pmatrix} \in R_n. \tag{3.9}$$

Since R_n is a generalized right p.q.-Baer, $r_{R_n}(A^kR_n) = ER_n$ for some $E^2 = E \in R_n$ and a positive integer k. Then by [14, Lemma 2], there is $e^2 = e \in S$ such that

$$E = \begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & e & 0 & \cdots & 0 \\ 0 & 0 & e & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e \end{pmatrix} \in R_n. \tag{3.10}$$

Hence $eS \subseteq r_S(a^kS)$. Let $b \in r_S(a^kS)$, then

$$\begin{pmatrix} b & 0 & 0 & \cdots & 0 \\ 0 & b & 0 & \cdots & 0 \\ 0 & 0 & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & b \end{pmatrix} \in R_n \tag{3.11}$$

is contained in $r_{R_n}(A^kR_n) = ER_n$, so $b \in eS$. Thus S is also a generalized right p.q.-Baer ring. □

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