MEASURE OF NONCOMPACTNESS OF OPERATORS AND MATRICES ON THE SPACES c AND c_0

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In this note, using the Hausdorff measure of noncompactness, necessary and sufficient conditions are formulated for a linear operator and matrices between the spaces c and c_0 to be compact. Among other things, some results of Cohen and Dunford are recovered.

We will write s, c, and c_0, for the set of all complex, convergent, and null sequences, respectively. Let $A = (a_{nm})_{n,m\geq 1}$ be an infinite matrix and consider the sequence $x = (x_n)_{n\geq 1}$. We will define the product

$$Ax = (Ax)_n = \big(A_n(x)\big)_{n\geq 1} \quad \text{with } A_n(x) = \sum_{m=1}^{\infty} a_{nm}x_m,\ n = 1,2,\dots, \tag{1}$$

whenever the series are convergent for all $n \geq 1$. For any given subsets X, Y of s, we will say that the operator represented by the infinite matrix $A = (a_{nm})_{n,m\geq 1}$ maps X into Y that is $A \in (X,Y)$, if

(i) the series defined by $A_n(x) = \sum_{m=1}^{\infty} a_{nm}x_m$ are convergent for all $n \geq 1$ and for all $x \in X$;

(ii) $Ax \in Y$ for all $x \in X$.

If $c \subset c_A = \{x : Ax \in c\}$, A is *conservative*. Well-known necessary and sufficient conditions for A to be conservative are

$$\|A\| \equiv \sup_{n\geq 1} \sum_{m=1}^{\infty} |a_{nm}| < \infty, \tag{2}$$

$$a_{00} = \lim_{n\to\infty} \sum_{m=1}^{\infty} a_{nm} \quad \text{exists}, \tag{3}$$

$$a_{0m} = \lim_{n\to\infty} a_{nm} \quad \text{exists for } m = 1,2,\dots. \tag{4}$$

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In this case, (2) is the norm of operator A. If A is conservative, then $\chi(A)=\lim_n \sum_{m=1}^{\infty} a_{nm} - \sum_{m=1}^{\infty} a_{0m}$ is called the *characteristic* of A, and in the case $\chi(A)=0$, A is *conull*. If $\lim_n (Ax)_n = \lim_n x_n$ for all $x \in c$, then A is called *regular*. A conservative matrix is regular if and only if $a_{00} = 1$ and $a_{0m} = 0$ for all m [5, 6].

Let $B(c)$ be the set of all bounded linear operators on c. It is well known (see [6, Theorem 4.51-D]) that each bounded linear operator A on c into c determines and is determined by a matrix of scalars a_{nm}, $n = 1,2,\dots, m = 0,1,2,\dots, y = Ax$, is defined by

$$y_n = a_{n0}x_0 + \sum_{m=1}^{\infty} a_{nm}x_m, \quad n = 1,2,\dots, \tag{5}$$

where $x = (x_n)$ in c, and $\lim_n x_n = x_0$. In this case, the norm of A is defined by

$$\|A\| = \sup_{n\geq 1} \sum_{m=0}^{\infty} |a_{nm}|, \tag{6}$$

and for $A \in B(c)$, the additional conditions are (4) and

$$\lim_{n\to\infty} \sum_{m=0}^{\infty} a_{nm} = \alpha_{00}. \tag{7}$$

Let X, Y be Banach spaces, and let $B(X,Y)$ be the set of all linear bounded operators from X to Y. If Q is a bounded subset of X, then the *Hausdorff measure of noncompactness* of Q is denoted by $q(Q)$, and

$$q(Q) = \inf\{\epsilon > 0 : Q \text{ has a finite } \epsilon- \text{ net in } X\}. \tag{8}$$

The function q is called the *Hausdorff measure of noncompactness (ball measure of noncompactness)*; it was introduced by Gohberg et al. [4], later studied by *Goldenstein and Markus* in 1968, *Istrăţesku* in 1972, and others. Let us point out that the notation of the measure of noncompactness has proved useful results in several areas of functional analysis, operator theory, fixed point theory, differential equations, and so forth (see, e.g., [1, 2, 4]). Let us recall that $q(Q) = 0$ if and only if Q is a totally bounded set. For $A \in B(X,Y)$, the Hausdorff measure of noncompactness of A, denoted by $\|A\|_q$, is defined by $\|A\|_q = q(AB_1)$, where $B_1 = \{x \in X : \|x\| \leq 1\}$ is the unit ball in X. Hence, A is compact if and only if $\|A\|_q = 0$.

Let us recall that if X is a Banach space with a Schauder basis $\{v_1,v_2,\dots\}$, Q a bounded subset of X, $P_n : X \mapsto X$ the projector onto the linear span of $\{v_1,v_2,\dots,v_n\}$, and $\mu(Q) = \limsup_{n\to\infty}(\sup_{x\in Q} \|(I-P_n)x\|)$, then the following inequality holds:

$$\frac{1}{a}\mu(Q) \leq q(Q) \leq \inf_n \sup_{x\in Q} ||(I-P_n)x|| \leq \mu(Q), \tag{9}$$

where $a = \limsup_{n\to\infty} \|I - P_n\|$ [1, 2, 4].

Now, we can state the following main result.

THEOREM 1. *Let $A \in B(c)$, let α_{00} be as in (7), and let a_{0n}, $n = 1,2,\ldots$, be as in (4). Then*

$$\frac{1}{2}\limsup_{n\to\infty}\left(\left|a_{n0}-\alpha_{00}+\sum_{m=1}^{\infty}a_{0m}\right|+\sum_{m=1}^{\infty}\left|a_{nm}-a_{0m}\right|\right) \leq \|A\|_q \leq \limsup_{n\to\infty}\left(\left|a_{n0}-\alpha_{00}+\sum_{m=1}^{\infty}a_{0m}\right|+\sum_{m=1}^{\infty}\left|a_{nm}-a_{0m}\right|\right). \tag{10}$$

Proof. Suppose that $x \in c$, $\lim_m x_m = x_0$ and $y = Ax$. Now $y_n = a_{n0}x_0 + \sum_{m=1}^{\infty} a_{nm}x_m$, $n = 1,2,\ldots$, and $\lim_n y_n = y_0$. By [6, page 219, (4.51-11)], (5) and (7),

$$y_0 = x_0\alpha_{00} + \sum_{m=1}^{\infty}(x_m - x_0)a_{0m} = x_0\left(\alpha_{00} - \sum_{m=1}^{\infty}a_{0m}\right) + \sum_{m=1}^{\infty}x_m a_{0m}. \tag{11}$$

The elements $e = (1,1,1,\ldots)$ and $e_i = \{\delta_{ij}\}$, $i = 1,2,\ldots$ form the Schauder basis in c. Let $P_n : c \mapsto c$ be the projector defined by

$$P_n(x) = x_0 e + \sum_{i=1}^{n}(x_i - x_0)e_i, \quad n = 1,2,\ldots. \tag{12}$$

It is easy that $\|I - P_n\| = 2$, and by (9) we have

$$\begin{aligned}\sup_{\|x\|\leq 1}\left\|(I-P_n)Ax\right\| &= \sup_{\|x\|\leq 1}\sup_{k\geq n+1}\left|y_k - y_0\right| \\ &= \sup_{\|x\|\leq 1}\sup_{k\geq n+1}\left|x_0\left(a_{k0}-\alpha_{00}+\sum_{m=1}^{\infty}a_{0m}\right)+\sum_{m=1}^{\infty}(a_{km}-a_{0m})x_m\right| \\ &= \sup_{k\geq n+1}\left(\left|a_{k0}-\alpha_{00}+\sum_{m=1}^{\infty}a_{0m}\right|+\sum_{m=1}^{\infty}\left|a_{km}-a_{0m}\right|\right).\end{aligned} \tag{13}$$

Now, by (9) and (13), we obtain (10). □

COROLLARY 2. *Let $A \in B(c)$. Then A is compact if and only if*

$$\lim_{n\to\infty}\left(\left|a_{n0}-\alpha_{00}+\sum_{m=1}^{\infty}a_{0m}\right|+\sum_{m=1}^{\infty}\left|a_{nm}-a_{0m}\right|\right) = 0. \tag{14}$$

Let us recall that if $A \in B(c)$, $y = Ax$, then $y_0 = x_0$ for every choice of x if and only if $\alpha_{00} = 1$ and $a_{01} = a_{02} = \cdots = 0$ (see, e.g., [6]); in this case A is called *regular*. Now, by Corollary 2, we have the next well-known result of Cohen and Dunford [3, Corollary 3].

COROLLARY 3. *Let $A \in B(c)$ be regular transformation. Then A is compact if and only if*

$$\lim_{n\to\infty}\left(\left|a_{n0}-1\right|+\sum_{m=1}^{\infty}\left|a_{nm}\right|\right) = 0. \tag{15}$$

Corollary 4. *Let $A \in (c,c)$, let a_{00} be as in (3), and let a_{0m}, $n = 1,2,\ldots$, be as in (4). Then*

$$\frac{1}{2}\limsup_{n\to\infty}\left(\left|a_{00}-\sum_{m=1}^{\infty}a_{0m}\right|+\sum_{m=1}^{\infty}\left|a_{nm}-a_{0m}\right|\right) \leq \|A\|_q \leq \limsup_{n\to\infty}\left(\left|a_{00}-\sum_{m=1}^{\infty}a_{0m}\right|+\sum_{m=1}^{\infty}\left|a_{nm}-a_{0m}\right|\right), \tag{16}$$

and A is compact if and only if

$$\lim_{n\to\infty}\left(\left|a_{00}-\sum_{m=1}^{\infty}a_{0m}\right|+\sum_{m=1}^{\infty}\left|a_{nm}-a_{0m}\right|\right)=0. \tag{17}$$

Let us remark that Corollary 4 implies that compact conservative matrix is conull.

If we recall the characterizations of the sets (c,c_0) and (c_0,c_0) [5, 6], and remark that in this case the projector $P_n(x) = (x_1,x_2,\ldots,x_n,0,\ldots)$ maps c_0 into c_0, and $\|I - P_n\| = 1$, then by the proof of Theorem 1, we have the next result.

Corollary 5. *If $A \in (c,c_0)$ or $A \in (c_0,c_0)$, then*

$$\|A\|_q = \limsup_{n\to\infty}\sum_{m=1}^{\infty}\left|a_{nm}\right|, \tag{18}$$

and A is compact if and only if

$$\lim_{n\to\infty}\sum_{m=1}^{\infty}\left|a_{nm}\right| = 0. \tag{19}$$

Corollary 6. *If $A \in (c_0,c)$, then*

$$\frac{1}{2}\limsup_{n\to\infty}\sum_{m=1}^{\infty}\left|a_{nm}-a_{0m}\right| \leq \|A\|_q \leq \limsup_{n\to\infty}\sum_{m=1}^{\infty}\left|a_{nm}-a_{0m}\right|, \tag{20}$$

and A is compact if and only if

$$\lim_{n\to\infty}\sum_{m=1}^{\infty}\left|a_{nm}-a_{0m}\right| = 0. \tag{21}$$

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