IRREDUCIBLE MORPHISMS, THE GABRIEL-VALUED QUIVER AND COLOCALIZATIONS FOR COALGEBRAS

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Given a basic K-coalgebra C, we study the left Gabriel-valued quiver $({}_CQ, {}_C\mathbf{d})$ of C by means of irreducible morphisms between indecomposable injective left C-comodules and by means of the powers rad^m of the radical rad of the category C-inj of the socle-finite injective left C-comodules. Connections between the valued quiver $({}_CQ, {}_C\mathbf{d})$ of C and the valued quiver $({}_{\overline{C}}Q, {}_{\overline{C}}\mathbf{d})$ of a colocalization coalgebra quotient $f_E : C \to \overline{C}$ of C are established.

1. Introduction

Throughout this paper we fix a field K. Given a K-coalgebra C we denote by C-Comod and C-comod the categories of left C-comodules and left C-comodules of finite K-dimension, respectively. Given a left C-comodule M, we denote by $\text{soc}_C M$ the *socle* of M, that is, the sum of all simple C-subcomodules of M. We call M *socle-finite* (or finitely copresented) if $\dim_K \text{soc} M$ is finite. Following [17, page 404], a K-coalgebra C is called *basic* if the left C-comodules ${}_CC$ and $\text{soc}_C C$ have direct sum decompositions:

$$ {}_CC = \bigoplus_{j \in I_C} E(j), \qquad \text{soc}_C C = \bigoplus_{j \in I_C} S(j), \tag{1.1}$$

where I_C is a set, $E(j)$ is an indecomposable injective comodule, $S(j)$ is a simple comodule, and $E(j)$ is the injective envelope of $S(j)$, for each $j \in I_C$, $E(i) \not\cong E(j)$, and $S(i) \not\cong S(j)$, for $i \neq j$. It was shown in [22] that a K-coalgebra C is basic if and only if $\dim_K S = \dim_K \text{End}_C S$, for any simple left C-comodule S.

Throughout this paper we assume that C is a basic K-coalgebra, the decompositions (1.1) are fixed, and we set

$$F_j = \text{End}_C S(j), \tag{1.2}$$

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for each $j \in I_C$. In this case $\{S(j)\}_{j\in I_C}$ is a complete set of all pairwise nonisomorphic simple left C-comodules.

We recall from [10, Definition 4.3] and [19, Definition 8.6] that the *left Gabriel-valued quiver* of C is the valued quiver

$$({}_CQ, {}_C\mathbf{d}) = ({}_CQ_0, {}_CQ_1, {}_C\mathbf{d}), \tag{1.3}$$

where ${}_CQ_0 = I_C$ is the set of vertices, ${}_CQ_1$ is the set of valued arrows, and, given two vertices $i, j \in {}_CQ_0$, there exists a unique valued arrow

$$i \xrightarrow{(d'_{ij}, d''_{ij})} j \tag{1.4}$$

from i to j in ${}_CQ_1$ if and only if the F_j-F_i-bimodule $\operatorname{Ext}^1_C(S(i), S(j))$ is nonzero and

$$d'_{ij} = \dim \operatorname{Ext}^1_C\left(S(i), S(j)\right)_{F_i}, \qquad d''_{ij} = \dim_{F_j} \operatorname{Ext}^1_C\left(S(i), S(j)\right). \tag{1.5}$$

In other words, $({}_CQ, {}_C\mathbf{d})$ is the opposite to the left valued Ext-quiver of C (see [4, 8, 14]), which is the valued quiver $(Q^{{}_C\mathscr{E}xt}, \mathbf{d}^{{}_C\mathscr{E}xt})$ of the left Ext-species ${}_C\mathscr{E}xt$ of C; see [10].

In practice, it is useful to work with an equivalent form of the valued quiver $({}_CQ, {}_C\mathbf{d})$. We define it in Section 2, by applying the well-known concepts of the Auslander-Reiten theory for finite dimensional algebras, see [1], [2, Section 5.5], and [18, Section 11.1]. We introduce the notion of an irreducible morphism between left C-comodules, and we give an equivalent description of the quiver $({}_CQ, {}_C\mathbf{d})$ in terms of irreducible morphisms between socle-finite injective C-comodules. Then we study the valued quiver $({}_CQ, {}_C\mathbf{d})$ by means of irreducible morphisms between the indecomposable injective C-comodules $E(j)$, by means of the K-species; see (2.10),

$${}_C\mathcal{I} = \left(F_j, {}_jN_i\right)_{i,j\in I_C} \tag{1.6}$$

of F_j-F_i-bimodules ${}_jN_i = \operatorname{Irr}(E(i), E(j))$ of irreducible morphisms [10, equation (4.9)], and by means of the powers $\operatorname{rad}^m(E(i), E(j))$ of the radical rad of the full subcategory C-inj of C-Comod formed by the socle-finite injective C-comodules. In particular, we show that the existence of a valued arrow (1.4) in the quiver $({}_CQ, {}_C\mathbf{d})$ is equivalent to the existence of an irreducible morphism $E(j) \to E(i)$ in the category C-inj.

One of the main results of this paper is Theorem 2.3 of Section 2. It asserts that, for each pair of indices $i, j \in I_C$, we have

(i) $\bigcap_{m\geq 1} \operatorname{rad}^m(E(j), E(i)) = 0$;

(ii) for each noninvertible nonzero homomorphism $f \in \operatorname{Hom}_C(E(j), E(i))$, there is $m \geq 1$ such that $f \in \operatorname{rad}^m(E(j), E(i)) \setminus \operatorname{rad}^{m+1}(E(j), E(i))$;

(iii) if $\operatorname{rad}(E(j), E(i))$ is nonzero, then there exist an integer $m_{ij} \geq 1$ and a path

$$E(j) \xrightarrow{\varphi_1} E(j_1) \xrightarrow{\varphi_2} E(j_2) \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_{m_{ij}}} E(i) \tag{1.7}$$

of irreducible morphisms $\varphi_1,\dots,\varphi_{m_{ij}}$ in C-inj such that the composition $\varphi_{m_{ij}}\cdots\varphi_1$ is nonzero. If, in addition, the vector space $\mathrm{Hom}_C(E(j),E(i))$ is of finite K-dimension, then there exists a finite set $U_{ij}\subseteq I_C$ such that $\mathrm{rad}^{m_{ij}}(E(j),E(i))\neq 0$ and $\mathrm{rad}^{1+m_{ij}}(E(j),E(i))=0$, and every noninvertible nonzero C-comodule homomorphism $f:E(j)\to E(i)$ is a finite K-linear combination $f=\sum_{s=1}^{t}\lambda_s f_{sr_s}\cdots f_{s2}f_{s1}$ of compositions

$$E(j)\xrightarrow{f_{s1}}E(j_{s1})\xrightarrow{f_{s2}}E(j_{s2})\xrightarrow{f_{s3}}\cdots\xrightarrow{f_{sr_s}}E(j_{sr_s})=E(i) \tag{1.8}$$

of irreducible morphisms $f_{s1},f_{s2},f_{s3},\dots,f_{sr_s}$ in C-inj, where $\lambda_s\in K$, $\lambda_s\neq 0$, $r_s\leq m_{ij}$, for $s=1,\dots,t$, and $j_{sa}\in U_{ij}$, for all $a=1,\dots,r_s$ and $s=1,\dots,t$.

Hence we conclude, in Corollary 2.4, that the coalgebra C is a direct sum of two nonzero subcoalgebras if and only if the valued quiver $({}_CQ,{}_C\mathbf{d})$ is disconnected. In particular, this implies a new proof of [14, Corollary 2.2].

In Section 3, we study a relationship between the valued quiver $({}_CQ,{}_C\mathbf{d})$ of the coalgebra C and the valued quiver $({}_{\overline{C}}Q,{}_{\overline{C}}\mathbf{d})$ of a colocalisation coalgebra quotient

$$f_E:C\longrightarrow\overline{C}=C_E\cong e_ECe_E\cong C/\mathcal{U}_E \tag{1.9}$$

of C with respect to an injective comodule

$$E=\bigoplus_{j\in U}E(j), \tag{1.10}$$

where U is a subset of I_C; see Section 3 for details. We show in Theorem 3.2 that if E is as above and, for each $j\in U$, the comodule $E(j)/S(j)$ is E-copresented then the left Gabriel-valued quiver $({}_{\overline{C}}Q,{}_{\overline{C}}\mathbf{d})$ of the coalgebra $\overline{C}=C_E$ has ${}_{\overline{C}}Q_0=U$ and is isomorphic to the restriction of the left Gabriel-valued quiver $({}_CQ,{}_C\mathbf{d})$ of C to the subset $U\subseteq I_C={}_CQ_0$.

Throughout, we use the coalgebra representation theory notation and terminology introduced in [19–21]. In particular, given a coalgebra C and a pair of left C-comodules M and N, we denote by $\mathrm{Hom}_C(M,N)$ the vector space of all C-comodule homomorphisms $f:M\to N$, and by $\mathrm{End}_C M$ the algebra of all C-comodule endomorphisms $g:M\to M$ of M.

Given a K-coalgebra C, we denote by $C^*=\mathrm{Hom}_K(C,K)$ the K-dual algebra with respect to the convolution product (see [6, 13, 23]) viewed as a pseudocompact K-algebra (see [7, 19]). The category of pseudocompact left C^*-modules is denoted by C^*-PC.

Given a ring R with an identity element, we denote by $J(R)$ the Jacobson radical of R, and by $\mathrm{mod}(R)$ the category of finitely generated right R-modules.

The reader is referred to [3, 6, 13, 23] for the coalgebra and comodule terminology, and to [1, 2, 18] for the standard representation theory terminology and notation.

2. The Gabriel-valued quiver of a coalgebra and irreducible morphisms

Assume that K is an arbitrary field and C is a basic K-coalgebra. We fix the decompositions (1.1), and we set $F_j=\mathrm{End}_C S(j)$, for each $j\in I_C$, as in (1.2). Let $({}_CQ,{}_C\mathbf{d})$ be the Gabriel-valued quiver $({}_CQ,{}_C\mathbf{d})$ of C defined in (1.3).

In this section we present an equivalent form of the valued quiver $({}_CQ, {}_C\mathbf{d})$ in terms of irreducible morphisms between injective C-comodules. One of the applications of this new description is to compute the left Gabriel-valued quiver $({}_{C_E}Q, {}_{C_E}\mathbf{d})$ of the coalgebra $C_E = e_E C e_E$ in terms of the left Gabriel-valued quiver $({}_CQ, {}_C\mathbf{d})$ of C, given in Section 3.

We denote by C-inj the full subcategory of C-Comod formed by the socle-finite injective C-comodules. Note that a comodule E' lies in C-inj if and only if E' is isomorphic with a finite direct sum of the comodules $E(j)$, with $j \in I_C$.

Following the Auslander-Reiten theory for finite dimensional algebras, we introduce the notion of an irreducible morphism between left C-comodules as follows; see [1], [2, Section 5.5], and [18, Section 11.1].

Definition 2.1. (a) A C-comodule homomorphism $f : E' \to E''$ in C-inj is an *irreducible morphism* if f is not an isomorphism and given a factorization

$$\begin{array}{ccc} E' & \xrightarrow{\quad f \quad} & E'' \\ & {\scriptstyle f'}\searrow \quad \nearrow{\scriptstyle f''} & \\ & Z & \end{array} \tag{2.1}$$

of f with Z in C-inj, f' is a section, or f'' is a retraction, that is, f' has a left inverse or f'' has a right inverse; see [1, Section I.5]. Irreducible morphisms in any full subcategory of C-Comod are defined analogously.

(b) Given two comodules E' and E'' in C-inj, define the *radical* of $\mathrm{Hom}_C(E', E'')$ to be the K-subspace

$$\mathrm{rad}(E', E'') \subseteq \mathrm{Hom}_C(E', E'') \tag{2.2}$$

of $\mathrm{Hom}_C(E', E'')$ generated by all nonisomorphisms $\varphi : E(i) \to E(j)$ between indecomposable summands $E(i)$ of E' and $E(j)$ of E'', respectively.

(c) The square of $\mathrm{rad}(E', E'')$ is defined to be the K-subspace

$$\mathrm{rad}^2(E', E'') \subseteq \mathrm{rad}(E', E'') \subseteq \mathrm{Hom}_C(E', E'') \tag{2.3}$$

of $\mathrm{rad}(E', E'')$ generated by all composite homomorphisms of the form $E' \xrightarrow{f_j'} E(j) \xrightarrow{f_j''} E''$, where $j \in I_C$, $f_j' \in \mathrm{rad}(E', E(j))$, and $f_j'' \in \mathrm{rad}(E(j), E'')$.

(d) The mth power $\mathrm{rad}^m(E', E'')$ of $\mathrm{rad}(E', E'')$ is defined analogously, for each $m \geq 2$. Finally, set

$$\mathrm{rad}^\infty(E', E'') = \bigcap_{m=1}^{\infty} \mathrm{rad}^m(E', E''), \tag{2.4}$$

and call it the *infinite radical* of $\mathrm{Hom}_C(E', E'')$.

Then the chain of vector spaces is defined:

$$\mathrm{Hom}_C(E', E'') \supseteq \mathrm{rad}(E', E'') \supseteq \mathrm{rad}^2(E', E'') \supseteq \cdots \supseteq \mathrm{rad}^m(E', E'') \supseteq \cdots \supseteq \mathrm{rad}^\infty(E', E''). \tag{2.5}$$

The following simple lemma is very useful.

Lemma 2.2. *Assume that C is a basic K-coalgebra with fixed decompositions (1.1).*

(a) *For each $j \in I_C$, the K-algebra* $\operatorname{End}_C E(j)$ *is local, the subset*

$$J_E = \{f \in \operatorname{End}_C E(j);\ f(S(j)) = 0\} \tag{2.6}$$

of $\operatorname{End}_C E(j)$ *is the unique maximal ideal of* $\operatorname{End}_C E(j)$ *and* $F_j = \operatorname{End}_C S(j) \cong \operatorname{End}_C E(j)/J_E$.

(b) *Given $i, j \in I_C$, the radical* $\operatorname{rad}(E(i),E(j))$ *consists of all nonisomorphisms* $f \in \operatorname{Hom}_C(E(i),E(j))$. *Moreover, a C-comodule homomorphism $f : E(i) \to E(j)$ is an irreducible morphism in* C-inj *if and only if* $f \in \operatorname{rad}(E(i),E(j)) \setminus \operatorname{rad}^2(E(i),E(j))$.

Proof. (a) It is clear that J_E is a two-sided ideal of the algebra $\operatorname{End}_C E(j)$, and that $f \in \operatorname{End}_C E(j)$ is noninvertible if and only if $f(S(j)) = 0$, that is, if and only if $f \in J_E$. This shows that $\operatorname{End}_C E(j)$ is a local algebra and J_E is its unique maximal ideal. Since the map $\operatorname{End}_C E(j) \to \operatorname{End}_C S(j) = F_j$ that associates to $f \in \operatorname{End}_C E(j)$ the restriction of f to $S(j)$ is a K-algebra homomorphism with kernel J_E, then (a) follows.

(b) The first statement is an immediate consequence of definition. To prove the second one, we apply the standard Auslander-Reiten theory arguments; see [18, page 174]. For the convenience of the reader we present a proof.

Assume, to the contrary, that $f \in \operatorname{Hom}_C(E(i),E(j))$ is irreducible and $f \in \operatorname{rad}^2(E(i), E(j))$. Then $f = \sum_{s=1}^{t} f_s'' f_s'$, where $f_s' \in \operatorname{rad}(E(i),Z_s)$, $f_s'' \in \operatorname{rad}(Z_s,E(j))$, and $Z_1,\dots,Z_t$ are indecomposable in C-inj. Obviously, f has a factorization $f = f'' f'$, where

$$E(i) \xrightarrow{f'} Z_1 \oplus Z_2 \oplus \cdots \oplus Z_t \xrightarrow{f''} E(j) \tag{$*$}$$

are the homomorphisms $f' = (f_1',\dots,f_t')$ and $f'' = (f_1'',\dots,f_t'')$. Since f is irreducible, then f' is a section or f'' is a retraction.

First, assume that f' is a section. Then there exists a C-comodule homomorphism

$$r = (r_1,\dots,r_t) : Z_1 \oplus Z_2 \oplus \cdots \oplus Z_t \longrightarrow E(i) \tag{2.7}$$

such that $1_{E(i)} = r f' = r_1 f_1' + \cdots + r_t f_t'$. Since $f_s' \in \operatorname{rad}(E(i),Z_s)$ and Z_s is indecomposable, then f_s' is not an isomorphism and, hence, $r_s f_s'$ is noninvertible, for any $s \in \{1,\dots,t\}$. Since, by (a), the algebra $\operatorname{End}_C E(i)$ is local, then $r_s f_s' \in J_{E(i)} = J \operatorname{End}_C E(i)$, for each $s \in \{1,\dots,t\}$, and we get $1_{E(i)} = r_1 f_1' + \cdots + r_t f_t' \in J \operatorname{End}_C E(i)$; a contradiction. Similarly, if f'' is a retraction, we also get a contradiction. Consequently, if $f \in \operatorname{Hom}_C(E(i),E(j))$, then $f \in \operatorname{rad}(E(i),E(j)) \setminus \operatorname{rad}^2(E(i),E(j))$.

Conversely, assume that $f \in \operatorname{rad}(E(i),E(j)) \setminus \operatorname{rad}^2(E(i),E(j))$. To prove that f is irreducible, assume that f has a factorization

$$f = f'' f' = \sum_{s=1}^{t} f_s'' f_s' \tag{2.8}$$

in C-inj, where $f' = (f_1',\dots,f_t')$ and $f'' = (f_1'',\dots,f_t'')$ are as in $(*)$ and the comodules $Z_1,\dots,Z_t$ are indecomposable. Since $f \notin \operatorname{rad}^2(E(i),E(j))$, then, according to (a), there is an index $a \in \{1,\dots,t\}$ such that the map f_a' is bijective or there is an index $b \in \{1,\dots,t\}$

such that the map f''_b is bijective. It follows that f' is a section or f'' is a retraction. This shows that f is an irreducible morphism in C-inj and finishes the proof of the lemma. □

Following the finite-dimensional algebras terminology; see [1], [2, Section 5.5], [10], and [18, Section 11.1], we call the K-vector space

$$\operatorname{Irr}(E(j),E(i)) = \operatorname{rad}(E(j),E(i))/\operatorname{rad}^2(E(j),E(i)) \tag{2.9}$$

the *bimodule of irreducible morphisms*, for each pair $i,j \in I_C$; see [10, 20].

It is easy to see that the K-vector space ${}_iN_j = \operatorname{Irr}(E(j),E(i))$ is an F_i-F_j-bimodule, where $F_i = \operatorname{End}_C S(i)$ and $F_j = \operatorname{End}_C S(j)$ are division K-algebras. Following [10, Definition 4.9], we consider the K-species

$$_C\mathcal{I} = \left(F_j, {}_jN_i\right)_{i,j\in I_C} \tag{2.10}$$

of irreducible morphisms of C, where ${}_jN_i=\operatorname{Irr}(E(i),E(j))$ is viewed as an F_j-F_i-bimodule.

One of the main results of this paper is the following useful theorem.

Theorem 2.3. *Assume that C is a basic K-coalgebra with fixed decompositions (1.1), and set $F_j = \operatorname{End}_C S(j)$, for each $j \in I_C$. Let $({}_CQ, {}_C\mathbf{d})$ be the left Gabriel-valued quiver (1.3) of C.*

(a) *There exists a unique valued arrow $i \xrightarrow{(d'_{ij},d''_{ij})} j$ (1.4) in $({}_CQ,{}_C\mathbf{d})$ if and only if the F_i-F_j-bimodule $\operatorname{Irr}(E(j),E(i))$ is nonzero and the numbers (1.5) have the forms*

$$d'_{ij} = \dim \operatorname{Irr}(E(j),E(i))_{F_j}, \qquad d''_{ij} = \dim_{F_i} \operatorname{Irr}(E(j),E(i)). \tag{2.11}$$

(b) $\operatorname{rad}^\infty(E(j),E(i)) = \bigcap_{m=1}^\infty \operatorname{rad}^m(E(j),E(i)) = 0$, *for each $i,j \in I_C$.*

(c) *For each $i,j \in I_C$ and any noninvertible nonzero homomorphism $f \in \operatorname{Hom}_C(E(j), E(i))$, there is an integer $m \geq 1$ such that*

$$f \in \operatorname{rad}^m(E(j),E(i)) \setminus \operatorname{rad}^{m+1}(E(j),E(i)). \tag{2.12}$$

In this case there is a path

$$E(j) \xrightarrow{\varphi_1} E(j_1) \xrightarrow{\varphi_2} E(j_2) \xrightarrow{\varphi_3} \cdots \xrightarrow{\varphi_m} E(i) \tag{2.13}$$

of irreducible morphisms $\varphi_1,\dots,\varphi_m$ in C-inj such that the composition $\varphi_m \cdots \varphi_2\varphi_1$ is nonzero.

(d) *If $\dim_K \operatorname{Hom}_C(E(j),E(i))$ is finite and $\operatorname{rad}(E(j),E(i)) \neq 0$, then there exist an integer $m_{ij} \geq 1$ and a finite subset U_{ij} of I_C such that*

$$\operatorname{rad}^{m_{ij}}(E(j),E(i)) \neq 0, \qquad \operatorname{rad}^{1+m_{ij}}(E(j),E(i)) = 0, \tag{2.14}$$

and every noninvertible nonzero homomorphism $f \in \operatorname{Hom}_C(E(j),E(i))$ is a finite K-linear combination $f = \sum_{s=1}^t \lambda_s f_{sr_s} \cdots f_{s2} f_{s1}$ of compositions

$$E(j) \xrightarrow{f_{s1}} E(j_{s1}) \xrightarrow{f_{s2}} E(j_{s2}) \xrightarrow{f_{s3}} \cdots \xrightarrow{f_{sr_s}} E(j_{sr_s}) = E(i) \tag{2.15}$$

of irreducible morphisms $f_{s1}, f_{s2}, f_{s3},\dots, f_{sr_s}$ in C-inj, where $\lambda_s \in K$, $\lambda_s \neq 0$, $r_s \leq m_{ij}$, for $s = 1,\dots,t$, and $j_{sa} \in U_{ij}$, for all $a = 1,\dots,r_s$ and $s = 1,\dots,t$.

Proof. The statement (a) follows from [10, Proposition 4.10(b)].

(b) We recall that the Yoneda map $\varphi \mapsto \varepsilon \circ \varphi$ defines an isomorphism $\Lambda_C = \mathrm{End}_C C \cong C^*$ of pseudocompact algebras; see also [19, Sections 3 and 4]. Note that, given $j \in I_C$, the direct summand projection ${}_C C \to E(j)$ of left C-comodules induces a direct summand injection

$$E(j)^* \cong \mathrm{Hom}_C(E(j), C) \hookrightarrow \Lambda_C \cong C^* \tag{2.16}$$

of left pseudocompact Λ_C-modules and an isomorphism $E(j)^* \cong C^* e_j$, where e_j is the primitive idempotent of Λ_C defined by the direct summand injection $\mathrm{Hom}_C(E(j), C) \hookrightarrow \Lambda_C$.

Then, in view of the duality $C\text{-Comod} \cong (C^*\text{-PC})^{\mathrm{op}}$ given by $M \mapsto M^*$ (see [19, Theorem 4.5]), for each pair $i, j \in I_C$, we get F_j-F_i-bimodule dualities

$$\mathrm{rad}(E(i), E(j)) \cong \mathrm{rad}_{\Lambda_C}(E(j)^*, E(i)^*) \cong \mathrm{rad}_{\Lambda_C}(C^* e_j, C^* e_i) \cong e_j J(\Lambda_C) e_i, \tag{2.17}$$

and for each $m \geq 2$, we get F_j-F_i-bimodule dualities

$$\mathrm{rad}^m(E(i), E(j)) \cong \mathrm{rad}^m_{\Lambda_C}(E(j)^*, E(i)^*) \cong \mathrm{rad}^m_{\Lambda_C}(C^* e_j, C^* e_i) \cong e_j J(\Lambda_C)^m e_i. \tag{2.18}$$

Since $C = \bigcup_{m=0}^{\infty} C_m$, where $\{C_m\}_m^{\infty}$ is the coradical filtration of C, then

$$\bigcap_{m=0}^{\infty} J(\Lambda_C)^m = 0; \tag{2.19}$$

see [11, 12] and [9, Section 4]. It then follows that $\bigcap_{m=0}^{\infty} e_j J(\Lambda_C)^m e_i = 0$ and consequently $\mathrm{rad}^{\infty}(E(j), E(i)) = 0$.

(c) Assume that $f \in \mathrm{Hom}_C(E(j), E(i))$ is a noninvertible nonzero homomorphism. Then $f \in \mathrm{rad}(E(j), E(i))$ and, since $\mathrm{rad}^{\infty}(E(j), E(i)) = 0$, then there is $m \geq 1$ such that

$$f \in \mathrm{rad}^m(E(j), E(i)) \setminus \mathrm{rad}^{m+1}(E(j), E(i)). \tag{2.20}$$

It follows that f is a finite sum $f = \sum_{s=1}^{t} \lambda_s f_{sm} \cdots f_{s2} f_{s1}$ of nonzero composite homomorphisms $\lambda_s f_{sm} \cdots f_{s2} f_{s1}$ of the form (2.15), with $\lambda_s \in K$, $r_1 = \cdots = r_s = m$ and $f_{sq} \in \mathrm{rad}(E(j), E(i))$, for any s and $q = 1, \ldots, m$.

Since $f \notin \mathrm{rad}^{m+1}(E(j), E(i))$, then there is a nonzero summand $\lambda_s f_{sm} \cdots f_{s2} f_{s1}$ such that $f_{sa} \notin \mathrm{rad}^2(E(j), E(i))$, for any s and $a = 1, \ldots, m$. It follows from Lemma 2.2 that $f_{sm}, \ldots, f_{s2}, f_{s1}$ are irreducible morphisms, and (c) follows.

(d) Assume that $\dim_K \mathrm{Hom}_C(E(j), E(i))$ is finite and $\mathrm{rad}(E(j), E(i)) \neq 0$. Since, by (b), $\mathrm{rad}^{\infty}(E(j), E(i)) = 0$, then there exists a minimal integer $m_{ij} \geq 1$ such that $\mathrm{rad}^{m_{ij}}(E(j), E(i)) \neq 0$, and $\mathrm{rad}^m(E(j), E(i)) = 0$, for $m \geq 1 + m_{ij}$.

Assume that $f \in \mathrm{Hom}_C(E(j), E(i))$ is a nonzero non-isomorphism. Then $f \in \mathrm{rad}(E(j), E(i))$ and there is an integer $q = q(f) \in \{1, \ldots, m_{ij}\}$ such that

$$f \in \mathrm{rad}^q(E(j), E(i)) \setminus \mathrm{rad}^{q+1}(E(j), E(i)). \tag{2.21}$$

We show, by induction on $m_{ij} - q(f) \geq 0$ that f is a K-linear combination of compositions of irreducible morphisms between indecomposable injective comodules.

First, assume that $m_{ij} - q(f) = 0$, that is, $q(f) = m_{ij}$ and $f \in \operatorname{rad}_C^{m_{ij}}(E(j), E(i))$. Then f is a finite sum:

$$f = \sum_{s=1}^{t} \lambda_s f_{sq} \cdots f_{s2} f_{s1}, \tag{$*'$}$$

of nonzero composite homomorphisms

$$E(j) \xrightarrow{f_{s1}} E(j_{s1}) \xrightarrow{f_{s2}} E(j_{s2}) \longrightarrow \cdots \xrightarrow{f_{sq}} E(i) \tag{2.22}$$

of length $q = m_{ij}$ and with $\lambda_s \in K$ and $f_{sa} \in \operatorname{rad}(E(j), E(i))$, for any s and $a = 1, \ldots, q$. Since $\operatorname{rad}^{q+1}(E(j), E(i)) = 0$, then each f_{sa} is an irreducible morphism, by Lemma 2.2, and we are done.

Next, assume that $m = m_{ij} - q(f) \geq 1$ and (d) is proved, for all $h \in \operatorname{rad}(E(j), E(i))$ such that $m_{ij} - q(h) \leq m - 1$. Let f'' be the sum of all nonzero summands $\lambda_s f_{sq} \cdots f_{s2} f_{s1}$ in $(*')$ such that $f_{sa} \in \operatorname{rad}^2(E(j), E(i))$, for some $1 \leq a \leq q$. Then $f' = f - f''$ is the sum of all nonzero summands $\lambda_s f_{sq} \cdots f_{s2} f_{s1}$ in $(*')$ such that

$$f_{sa} \in \operatorname{rad}(E(j), E(i)) \setminus \operatorname{rad}^2(E(j), E(i)), \tag{2.23}$$

for all $1 \leq a \leq q$. It follows that each such a homomorphism f_{sa} is an irreducible morphism. By the choice of f'', we get $f'' \in \operatorname{rad}^{1+q(f)}(E(j), E(i))$ and therefore $q(f'') \geq q(f) + 1$. Since, by induction hypothesis, f'' is a K-linear combination of compositions of irreducible morphisms, then so is $f = f' + f''$, and we are done.

Let $\psi_1, \ldots, \psi_b$ be a K basis of $\operatorname{rad}(E(j), E(i))$. By applying the above to each of the basis element $\psi_1, \ldots, \psi_b$, we find a finite subset U_{ij} of I_C such that each $f \in \{\psi_1, \ldots, \psi_b\}$ is a finite K-linear combination $f = \sum_{s=1}^{t} \lambda_s f_{sr_s} \cdots f_{s2} f_{s1}$ $(*')$ of compositions (2.15) of irreducible morphisms $f_{sr_s}, \ldots, f_{s2}, f_{s1}$ in C-inj, where $\lambda_s \in K$ and $r_s \leq m_{ij}$, for $s = 1, \ldots, t$, and $j_{sa} \in U_{ij}$, for all $a = 1 \ldots, r_s$ and $s = 1, \ldots, t$. It follows that the same holds, for any nonzero element $f \in \operatorname{rad}(E(j), E(i))$. This finishes the proof of the theorem. □

As a consequence of the properties of irreducible morphisms proved in Lemma 2.2 and Theorem 2.3, we get the following important corollary. We note that the second part of it was proved in [19, Corollary 8.7], by applying the Ext-quiver of C and [14, Corollary 2.2].

Corollary 2.4. *Assume that C is a basic K-coalgebra with fixed decompositions (1.1). Let $({}_CQ, {}_C\mathbf{d})$ be the left Gabriel-valued quiver (1.4) of C.*

(a) *There exists a valued arrow $i \xrightarrow{(d'_{ij}, d''_{ij})} j$ from i to j in ${}_CQ_1$ if and only if there is an irreducible morphism $E(j) \to E(i)$ in C-inj.*

(b) *The coalgebra C is indecomposable (i.e., C is not a direct sum of two nonzero subcoalgebras) if and only if the Gabriel-valued quiver $({}_CQ, {}_C\mathbf{d})$ is connected.*

Proof. Statement (a) is an immediate consequence of Lemma 2.2 and Theorem 2.3(a).

(b) Assume that C is basic and $C = \bigoplus_{j\in I_C} E(j)$ is the decomposition (1.1). In view of the coalgebra isomorphism

$$(\mathrm{End}_C C)^{\circ} \cong (C^*)^{\circ} \cong C; \tag{2.24}$$

see [17, page 404] and [19, Lemma 4.9 and Theorem 3.6], there are nonzero subcoalgebras C' and C'' of C such that $C = C' \oplus C''$ if and only if there are nonzero injective left subcomodules E' and E'' of C such that

(i) ${}_CC = E' \oplus E''$,
(ii) $\mathrm{Hom}_C(E', E'') = 0$, and
(iii) $\mathrm{Hom}_C(E'', E') = 0$.

By the unique decomposition property, the former statement is equivalent to the existence of two disjoint nonempty subsets I' and I'' of I_C such that

(i) $I_C = I' \cup I''$,
(ii) $\mathrm{Hom}_C(E(a), E(b)) = 0$, for all $a \in I'$ and $b \in I''$, and
(iii) $\mathrm{Hom}_C(E(b), E(a)) = 0$, for all $a \in I'$ and $b \in I''$.

It follows that C is not a direct sum of two nonzero subcoalgebras if and only if, for each pair $i, j \in I_C$, there exists a path

$$E(j) \;\text{———}\; E(j_1) \;\text{———}\; E(j_2) \;\text{———}\; \cdots \;\text{———}\; E(i)\ , \tag{2.25}$$

where $E(j_s) \text{ — } E(j_{s+1})$ means $\mathrm{Hom}_C(E(j_s), E(j_{s+1})) \neq 0$ or $\mathrm{Hom}_C(E(j_{s+1}), E(j_s)) \neq 0$. In view of (a), the former condition is satisfied, if the valued quiver $({}_CQ, {}_C\mathbf{d})$ is connected.

The converse implication follows from the fact that $\mathrm{Hom}_C(E(a), E(b)) \neq 0$ implies the existence of a path $E(a) \xrightarrow{\varphi_1} E(j_1) \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_m} E(b)$ of irreducible morphisms $\varphi_1, \dots, \varphi_m$ in C-inj; see Theorem 2.3(c). The proof is complete. □

Corollary 2.5. *Assume that C is a basic K-coalgebra with fixed decompositions (1.1), and assume that C is left computable, that is,* $\dim_K \mathrm{Hom}_C(E(j), E(i))$ *is finite, for all $i, j \in I_C$; see [22].*

(a) *For each pair $i, j \in I_C$, there exists a minimal integer $m_{ij} \geq 0$ such that* $\mathrm{rad}^{1+m_{ij}}(E(j), E(i)) = 0$.

(b) *If* $\mathrm{rad}(E(j), E(i)) \neq 0$, *then* $m_{ij} \geq 1$, $\mathrm{rad}^{m_{ij}}(E(j), E(i)) \neq 0$, *and* $\mathrm{rad}(E(j), E(i))$ *has a K-linear basis $f_1, \dots, f_{b_{ij}}$, where each f_s is the composed homomorphism $f_s = f_{sr_s} \cdots f_{s2} f_{s1}$ of the form (2.15) and $f_{sr_s}, \dots, f_{s2}, f_{s1}$ are irreducible morphisms in C-inj.*

(c) *Every nonzero non-isomorphism $f : E(j) \to E(i)$ between the indecomposable injective comodules $E(j)$ and $E(i)$ is a K-linear combination of the compositions $f_s = f_{sr_s} \cdots f_{s2} f_{s1}$ of irreducible morphisms $f_{sr_s}, \dots, f_{s2}, f_{s1}$ in C-inj.*

Proof. Since $\dim_K \mathrm{Hom}_C(E(j), E(i))$ is finite, for all $i, j \in I_C$, then Theorem 2.3 applies and the corollary follows. □

3. The Gabriel-valued quiver of a colocalization coalgebra C_E

One of the main aims of this section is to compute the left Gabriel-valued quiver $({}_{C_E}Q, {}_{C_E}\mathbf{d})$ of the colocalisation coalgebra $C_E = e_E C e_E$ (defined below) in terms of the left Gabriel-valued quiver $({}_CQ, {}_C\mathbf{d})$ of C.

Assume that C is a basic K-coalgebra with fixed decompositions (1.1), and that E is an injective left C-comodule of the form

$$E = \bigoplus_{j \in I_E} E(j), \tag{3.1}$$

where I_E is a subset of I_C. In [22], we associate to E the coalgebra surjection

$$f_E : C \longrightarrow C_E, \tag{3.2}$$

where C_E is the topological K-dual coalgebra to the pseudocompact K-algebra $\operatorname{End}_C E$, called the colocalisation coalgebra quotient of C at E. More precisely, the topological K-dual vector space

$$C_E = \left(\operatorname{End}_C E\right)^{\circ} \tag{3.3}$$

of $\operatorname{End}_C E$ is equipped with a natural coalgebra structure induced by the pseudocompact K-algebra structure of $\operatorname{End}_C E$; see [19, 22] for details, compare with [16]. It is shown in [22] that there is a coalgebra isomorphism $C_E \cong e_E C e_E$, and the kernel of the coalgebra surjection $f_E : C \to C_E$ is the coideal

$$\mathcal{U}_E = (1 - e_E)C + C(1 - e_E) \tag{3.4}$$

of C, where e_E is the idempotent of C^* defined by the direct summand embedding $E \hookrightarrow C$. We know from [5, 22, 24] that the restriction functor

$$\operatorname{res}_E : C\text{-Comod} \longrightarrow C_E\text{-Comod}, \tag{3.5}$$

given by $M \mapsto Me_E$, is exact and has a right adjoint

$$\square_E : C_E\text{-Comod} \longrightarrow C\text{-Comod} \tag{3.6}$$

and the kernel $\operatorname{Ker}\square_E$ of $\square_E$ is a localizing subcategory of C-Comod in the sense of Gabriel [7]. Conversely, every localizing subcategory of C-Comod is of this form; see [15, 24].

Now we show that, under a suitable assumption on E, the left Gabriel-valued quiver $({}_CQ, {}_C\mathbf{d})$ of the coalgebra C is the restriction to the subset $I_E = {}_{\overline{C}}Q_0$ of $I_C = {}_CQ_0$ of the valued quiver $({}_{\overline{C}}Q, {}_{\overline{C}}\mathbf{d})$ of the colocalisation coalgebra quotient homomorphism

$$f_E : C \longrightarrow \overline{C} = C_E \cong e_E C e_E \cong C/\mathcal{U}_E \tag{3.7}$$

of C at the injective comodule $E = \bigoplus_{j \in I_E} E(j)$.

To formulate the main result of this section we need some notation. We define a comodule M in C-Comod to be *E-copresented*, if M admits an *E-injective copresentation*, that is, that there is a short exact sequence

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1, \tag{3.8}$$

where E_0 and E_1 are direct sums of direct summands of the comodule E.

If, in addition, the comodules E_0 and E_1 are socle-finite, we say that the sequence

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \tag{3.9}$$

is a *socle-finite E-injective copresentation*, and then M is called a *finitely E-copresented comodule.*

We denote by

$$C\text{-Comod}_E \supseteq C\text{-Comod}_E^f \tag{3.10}$$

the full subcategories of C-Comod consisting of the E-copresented comodules and the finitely E-copresented comodules, respectively. We set

$$C\text{-comod}_E = C\text{-comod} \bigcap \left(C\text{-Comod}_E^f\right). \tag{3.11}$$

Definition 3.1. A valued subquiver $(U,\mathbf{d})$ of $({}_CQ,{}_C\mathbf{d})$ is defined to be a *full convex* valued subquiver if $(U,\mathbf{d})$ is connected and the following two conditions are satisfied.

(a) For each pair of points $a,b \in U$, the valued arrows from a to b in $({}_CQ,{}_C\mathbf{d})$ and in $(U,\mathbf{d})$ coincide.

(b) If $a,b \in U$, then any valued path

$$a \xrightarrow{(d_1',d_1'')} a_1 \xrightarrow{(d_2',d_2'')} a_2 \longrightarrow \cdots \longrightarrow a_{m-1} \xrightarrow{(d_m',d_m'')} b \tag{3.12}$$

in $({}_CQ,{}_C\mathbf{d})$ belongs to $(U,\mathbf{d})$.

THEOREM 3.2. *Assume that C is a basic K-coalgebra with fixed decompositions (1.1), and let E be an injective left C-comodule of the form (3.1), where I_E is a subset of I_C.*

(a) *If, for each $j \in I_E$, the comodule $E(j)/S(j)$ is E-copresented, then the left Gabriel-valued quiver $({}_{\overline{C}}Q,{}_{\overline{C}}\mathbf{d})$ of the coalgebra $\overline{C} = C_E$ has ${}_{\overline{C}}Q_0 = I_E$ and is isomorphic to the restriction of the valued quiver $({}_CQ,{}_C\mathbf{d})$ (1.4) of C to the subset $I_E \subseteq I_C = {}_CQ_0$.*

(b) *Let $(U,\mathbf{d})$ be a finite full convex valued subquiver of the valued quiver $({}_CQ,{}_C\mathbf{d})$ of the coalgebra C. Given $j \in U$, denote by $e_j \in R_E$ the primitive idempotent defined by the left ideal* $\operatorname{Hom}_C(E(j),E) \subseteq R_E$*, which is a direct summand of R_E. If the algebra*

$$R_E = \operatorname{End}_C E, \quad \textit{with } E = E_U = \bigoplus_{j\in U} E(j), \tag{3.13}$$

has finite K-dimension, then

(b1) *the equivalence of categories $H_E : C\text{-Comod}_E^f \xrightarrow{\simeq} \operatorname{mod} R_E$ [22, equation (3.4)] defined by the formula $H_E N = \operatorname{Hom}_C(N,E)^*$, for N in C-Comod_E^f, carries the indecomposable* *injective left comodule* $E(j)$ *to the indecomposable injective right* R_E*-module* $\check{E}(j) = H_E E(j)$, *induces a division ring isomorphism*

$$F_j \cong e_j R_E e_j / e_i J(R_E) e_j, \tag{3.14}$$

and induces F_i*-*F_j*-bimodule isomorphisms*

$$\operatorname{Irr}(E(j), E(i)) \cong \operatorname{Irr}(\check{E}(j), \check{E}(i)) \cong e_i[J(R_E)/J^2(R_E)]e_j, \tag{3.15}$$

for all $i, j \in U$, *and*

(b2) *the right Gabriel-valued quiver* $(Q_{R_E}, \mathbf{d})$ *of the algebra* R_E *is isomorphic with* $(U, \mathbf{d})$.

Proof. (a) Assume that, for each $j \in I_E$, the comodule $E(j)/S(j)$ is E-copresented, and consider the pair of adjoint functors

$$C_E\text{-Comod} \underset{\operatorname{res}_E}{\overset{E}{\rightleftarrows}} C\text{-Comod}, \tag{3.16}$$

defined by the formulae $\operatorname{res}_E(-) = (-)e_E$ and ${}_E(-) = e_E C_{C_E}(-)$. By [22, Proposition 2.7 and Theorem 2.10], the K-coalgebra C_E is basic, ${}_E$ is a full and faithful K-linear functor such that $\operatorname{res}_E \circ {}_E \cong \mathrm{id}$. The functor ${}_E$ is right adjoint to res_E, the functor res_E is exact, and ${}_E$ is left exact and restricts to the functor ${}_E : C_E\text{-comod} \to C\text{-comod}$. Moreover, for each $j \in I_E$, the left C_E-comodule $\check{S}(j) = \operatorname{res}_E S(j)$ is simple, $\operatorname{soc} C_E \cong \bigoplus_{j \in I_E} \check{S}(j)$, and $\check{S}(i) \not\cong \check{S}(j)$, for $i \neq j$, $i, j \in I_E$.

Since, for each $j \in I_E$, the comodule $E(j)/S(j)$ is E-copresented, then the left C-comodule $\operatorname{soc} E(j)/S(j)$ is (up to isomorphism) a direct sum of copies of simple comodules $S(i)$, with $i \in I_E$. Then, according to [22, Corollary 2.14], for each $j \in I_C$, there are isomorphisms $S(j) \cong \square_E \operatorname{res}_E S(j) \cong \check{S}(j)$, the simple C-comodule $S(j)$ lies in $C\text{-Comod}_E$, and the minimal injective three-term copresentation

$$0 \longrightarrow S(j) \longrightarrow E_0 \xrightarrow{\varphi_1} E_1 \xrightarrow{\varphi_2} E_2 \tag{3.17}$$

of $S(j)$ lies in the category $C\text{-Comod}_E$, where $E_0 = E(j)$. By [22, Theorem 2.10], for each $m \leq 2$, the C_E-comodule $\check{E}_m = \operatorname{res}_E E_m$ is injective and

$$0 \longrightarrow \check{S}(j) \longrightarrow \check{E}_0 \xrightarrow{\check{\varphi}_1} \check{E}_1 \xrightarrow{\check{\varphi}_2} \check{E}_2 \tag{3.18}$$

is an injective copresentation of $\check{S}(j) = \operatorname{res}_E S(j)$, because the functor res_E is exact and [22, Proposition 2.7] applies. Moreover, since $\operatorname{res}_E : C\text{-Comod}_E \to C_E\text{-Comod}$ is an equivalence of categories, by [22, Theorem 2.10], then res_E induces an isomorphism of the complexes

$$\begin{aligned} &0 \longrightarrow \operatorname{Hom}_C(S(i), E(j)) \xrightarrow{\widetilde{\varphi}_1} \operatorname{Hom}_C(S(i), E_1) \xrightarrow{\widetilde{\varphi}_2} \operatorname{Hom}_C(S(i), E_2) \longrightarrow 0, \\ &0 \longrightarrow \operatorname{Hom}_{C_E}(\check{S}(i), \check{E}(j)) \xrightarrow{\widetilde{\check{\varphi}}_1} \operatorname{Hom}_{C_E}(\check{S}(i), \check{E}_1) \xrightarrow{\widetilde{\check{\varphi}}_2} \operatorname{Hom}_{C_E}(\check{S}(i), \check{E}_2) \longrightarrow 0, \end{aligned} \tag{3.19}$$

and an isomorphism of their first homology groups. Hence we get an F_j-F_i-bimodule isomorphism

$$\mathrm{Ext}^1_{C_E}(\check{S}(i),\check{S}(j)) \cong \mathrm{Ext}^1_C(S(i),S(j)). \tag{3.20}$$

Since ${}_CQ = I_C$ and ${}_{\overline{C}}Q = I_E$, it follows from the definition (1.4) of the Gabriel-valued quiver of a colagebra that there exists a unique valued arrow $i \xrightarrow{(d'_{ij},d''_{ij})} j$ from i to j in the left Gabriel-valued quiver $({}_CQ,{}_C\mathbf{d})$ of C if and only if there is a unique valued arrow $i \xrightarrow{(d'_{ij},d''_{ij})} j$ in the left Gabriel-valued quiver $({}_{\overline{C}}Q,{}_{\overline{C}}\mathbf{d})$ of $\overline{C}$. This finishes the proof of (a).

(b) Assume that $\dim_K R_E$ is finite. We recall from Lemma 2.2 that $\mathrm{rad}(E(j),E(i))$ consists of all nonisomorphisms $f : E(j) \to E(i)$. Hence

(i) $\mathrm{rad}(E(i),E(i)) = J\,\mathrm{End}_C E(i)$, and

(ii) $\mathrm{rad}(E(j),E(i)) = \mathrm{Hom}_C(E(j),E(i))$, for $i \neq j$.

It follows that the equivalence of categories

$$H_E : C\text{-}\mathrm{Comod}^f_E \xrightarrow{\simeq} \mathrm{mod}\, R_E \tag{3.21}$$

[22, (3.4)] defined by the formula $H_E N = \mathrm{Hom}_C(N,E)^*$, for N in $C\text{-}\mathrm{Comod}^f_E$, induces isomorphisms

$$\begin{aligned} \mathrm{rad}\,(E(j),E(i)) &\cong \mathrm{rad}_{R_E}(H_E E(j),H_E E(i)) = \mathrm{rad}_{R_E}(\check{E}(j),\check{E}(i)),\\ \mathrm{rad}^2\,(E(j),E(i)) &\cong \mathrm{rad}^2_{R_E}(H_E E(j),H_E E(i)) = \mathrm{rad}^2_{R_E}(\check{E}(j),\check{E}(i)) \end{aligned} \tag{3.22}$$

of $\mathrm{End}_C E(i)$-$\mathrm{End}_C E(j)$-bimodules. The first isomorphism is obvious, but the second one follows from the convexity of $(U,\mathbf{d})$ in $({}_CQ,{}_C\mathbf{d})$. To show it, take a nonzero homomorphism $f \in \mathrm{rad}^2(E(j),E(i))$. Then

$$f = f''_1 f'_2 + \cdots + f''_t f'_t, \tag{3.23}$$

where $E(j) \xrightarrow{f'_s} E(r_s) \xrightarrow{f''_s} E(i)$, $r_s \in I_C$, and $f'_s \in \mathrm{rad}(E(j),E(r_s))$, $f''_s \in \mathrm{rad}(E(r_s),E(i))$ are nonzero homomorphisms. It follows from Theorem 2.3(c) that, for each $s \in \{1,\dots,t\}$, there is a path

$$E(j) \xrightarrow{\varphi_1} E(j_1) \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_m} E(r_s) \xrightarrow{\varphi'_1} E(j'_1) \xrightarrow{\varphi'_2} \cdots \xrightarrow{\varphi'_n} E(i) \tag{3.24}$$

of irreducible morphisms $\varphi_1,\dots,\varphi_m,\varphi'_1,\dots,\varphi'_n$ in C-inj. Since $(U,\mathbf{d})$ is a full convex valued subquiver of $({}_CQ,{}_C\mathbf{d})$ and $i,j \in U$, then the vertices $j_1,\dots,j_m = r_s, j'_1,\dots,j'_{n-1}$ belong to U. It follows that the homomorphisms $E(j) \xrightarrow{f'_s} E(r_s) \xrightarrow{f''_s} E(i)$ are in $C\text{-}\mathrm{Comod}^f_E$ and the image

$$H_E(f) = H_E(f''_1)H_E(f'_2) + \cdots + H_E(f''_t)H_E(f'_t), \tag{3.25}$$

of f under H_E belongs to $\mathrm{rad}^2_{R_E}(H_E E(j),H_E E(i))$. Consequently, the functor H_E induces an isomorphism

$$\mathrm{rad}^2\,(E(j),E(i)) \cong \mathrm{rad}^2_{R_E}(H_E E(j),H_E E(i)) = \mathrm{rad}^2_{R_E}(\check{E}(j),\check{E}(i)), \tag{3.26}$$

and division ring isomorphisms

$$F_j \cong \operatorname{End}_C E(j)/J \operatorname{End}_C E(j) \cong \operatorname{End}_{C_E} \check{E}(j)/J \operatorname{End}_{C_E} \check{E}(j) \cong e_j R_E e_j / e_j J(R_E) e_j. \tag{3.27}$$

Here we apply the Nakayama functor $\nu : \operatorname{mod} R_E \to \operatorname{mod} R_E$; see [1, Chapter III, Definition 2.8, Lemma 2.9, and Proposition 2.10]. Note also that, given $j \in U$, the idempotent e_j of $R_E = \operatorname{End}_C E$ is the composite endomorphism $E \xrightarrow{\pi_j} E(j) \hookrightarrow E$, where π_j is the direct summand projection.

Hence we derive F_i-F_j-bimodule isomorphisms:

$$\begin{aligned}
\operatorname{Irr}(E(j), E(i)) &= \operatorname{rad}(E(j), E(i)) / \operatorname{rad}^2(E(j), E(i)) \\
&\cong \operatorname{Hom}_{R_E}(H_E E(j), H_E E(i)) / \operatorname{rad}^2_{R_E}(H_E E(j), H_E E(i)) \\
&\cong \operatorname{Hom}_{R_E}(\check{E}(j), (\check{E}(i)) / \operatorname{rad}^2_{R_E}(\check{E}(j), \check{E}(i)) \\
&= \operatorname{Irr}(\check{E}(j), (\check{E}(i)).
\end{aligned} \tag{3.28}$$

Note also that, in view of [1, Chapter III, Proposition 2.10], the Nakayama functor $\nu : \operatorname{mod} R_E \to \operatorname{mod} R_E$ carries the injective right R_E-module $\check{E}(j)$ to projective right R_E-module $e_j R_E$ and induces the isomorphisms

$$\begin{aligned}
\operatorname{rad}_{R_E}(\check{E}(j), \check{E}(i)) &\cong e_i J(R_E) e_j, \\
\operatorname{rad}^2_{R_E}(\check{E}(j), \check{E}(i)) &\cong e_i J^2(R_E) e_j
\end{aligned} \tag{3.29}$$

of $\operatorname{End}_{C_E} \check{E}(i)$-$\operatorname{End}_{C_E} \check{E}(j)$-bimodules, compare with the proof of [1, Lemmas IV.2.12 and VII.1.6]. Hence we derive F_i-F_j-bimodule isomorphisms:

$$\begin{aligned}
\operatorname{Irr}(E(j), E(i)) &\cong \operatorname{Hom}_{R_E}(\check{E}(j), (\check{E}(i)) / \operatorname{rad}^2_{R_E}(\check{E}(j), \check{E}(i)) \\
&\cong e_i J(R_E) e_j / e_i J^2(R_E) e_j \\
&\cong e_i [J(R_E)/J^2(R_E)] e_j,
\end{aligned} \tag{3.30}$$

for all $i, j \in U$. This finishes the proof of (b1).

This also shows that the right Gabriel-valued quiver of the K-algebra R_E (see (3.31) below, [1, Section II.3], and [2]) is isomorphic with the valued quiver $(U, \mathbf{d})$, because the algebra R_E is basic, the modules $\check{E}(j) = H_E E(j)$, with $j \in U$, form a complete set of pairwise non-isomorphic indecomposable injective right R_E-modules, and hence $\{e_j\}_{j \in U}$ is a complete set of primitive orthogonal idempotents of R_E. This proves the statement (b2) and completes the proof of the theorem. □

For the convenience of the reader, we recall that the *right Gabriel-valued quiver*

$$(Q_B, \mathbf{d}) = ((Q_B)_0, (Q_B)_1, \mathbf{d}) \tag{3.31}$$

of a basic-finite dimensional algebra B is defined as follows. Fix a complete set $\{e_j\}_{j\in U}$ of primitive orthogonal idempotents of B such that $B = \bigoplus_{j\in U} e_j B$. Note that, given $j \in U$, the algebra $D_j = e_j B e_j / e_j J(B) e_j$ is a division ring. Moreover, given $i, j \in U$, the vector space

$$e_i J(R_E) e_j / e_i J^2(R_E) e_j \cong e_i [J(R_E)/J^2(R_E)] e_j \tag{3.32}$$

is a D_i-D_j-bimodule.

The set $(Q_B)_0$ of vertices of B is defined to be the set $(Q_B)_0 = U$. Given two vertices $i, j \in (Q_B)_0 = U$ of Q_B, there exists a unique valued arrow

$$i \xrightarrow{(d'_{ij}, d''_{ij})} j \tag{3.33}$$

from i to j in $(Q_B)_1$ if and only if $e_i[J(R_E)/J^2(R_E)]e_j \neq 0$ and

$$d'_{ij} = \dim_{D_i}(e_i[J(R_E)/J^2(R_E)]e_j), \qquad d''_{ij} = \dim(e_i[J(R_E)/J^2(R_E)]e_j)_{D_j}; \tag{3.34}$$

compare with [1, Chapter II, Definition 3.1].

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