ON HYPER BCC-ALGEBRAS

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We study hyper BCC-algebras which are a common generalization of BCC-algebras and hyper BCK-algebras. In particular, we investigate different types of hyper BCC-ideals and describe the relationship among them. Next, we calculate all nonisomorphic 22 hyper BCC-algebras of order 3 of which only three are not hyper BCK-algebras.

1. Introduction

The study of BCK-algebras was initiated by Iséki [7] as a generalization of the concept of set-theoretic difference and propositional calculus. Iséki posed an interesting problem; whether the class of BCK-algebras is a variety. In connection with this problem Komori introduced in [9] a notion of BCC-algebra which is a generalization of a BCK-algebra and proved that the class of all BCC-algebras is not a variety. Dudek [3–5] followed this theory and has got a lot of related results. BCC-algebras are algebraic models of BCC-logic, implicational logic whose axiom schemes are the principal-type schemes of the combinators B, I, and K, and whose inference rules are modus ponens and modus ponens 2. So, in fact, such algebras ought to have been named BIK-algebras. In this convention, a BCK-algebra is a BCC-algebra satisfying the identity $y \to (x \to z) = x \to (y \to z)$.

The hyperstructure theory (called also multialgebras) was introduced by Marty [10] at the eighth congress of Scandinavian mathematicians. Now hyperstructures of different types have many important applications (see, e.g., [1, 2]).

In this paper, we study hyper BCC-algebras which are a common generalization of BCC-algebras and hyper BCK-algebras investigated by many authors. In particular, we investigate different types of hyper BCC-ideals and describe the relationship among them. Next, we calculate all nonisomorphic 22 hyper BCC-algebras of order 3 of which only three are not hyper BCK-algebras.

2. Preliminaries

Definition 2.1. An algebra $(X,*,0)$ of type (2,0) is said to be a BCC-algebra if it satisfies the following: for all $x,y,z \in X$,

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(I) $((x*y)*(z*y))*(x*z)=0$,
(II) $x*0=x$,
(III) $x*x=0$,
(IV) $0*x=0$,
(V) $x*y=0$ and $y*x=0$ imply $x=y$.

Definition 2.2. By a *hyper* BCK-*algebra*, it is meant a nonempty set H endowed with a hyperoperation "$\circ$" and a constant "0" satisfying the following axioms:
(HK_1) $(x\circ z)\circ(y\circ z)\ll x\circ y$,
(HK_2) $(x\circ y)\circ z=(x\circ z)\circ y$,
(HK_3) $x\circ H\ll\{x\}$,
(HK_4) $x\ll y$ and $y\ll x$ imply $x=y$
for all $x,y,z\in H$, where $x\ll y$ is defined by $0\in x\circ y$ and for every $A,B\subseteq H$, $A\ll B$ is defined by for all $a\in A$, there exists $b\in B$ such that $a\ll b$. In such case, "$\ll$" is called the *hyperorder* in H.

Proposition 2.3 (see [8]). *In any hyper* BCK-*algebra H, for all $x,y,z\in H$, the following hold:*
(i) $0\circ 0=\{0\}$,
(ii) $0\circ x=\{0\}$,
(iii) $x\circ 0=\{x\}$.

Definition 2.4. Let I be a nonempty subset of a hyper BCK-algebra H and $0\in I$. Then I is said to be a *weak hyper* BCK-*ideal* of H if $x\circ y\subseteq I$ and $y\in I$ imply $x\in I$ for all $x,y\in H$, *hyper* BCK-*ideal* of H if $x\circ y\ll I$ and $y\in I$ imply $x\in I$ for all $x,y\in H$, *strong hyper* BCK-*ideal* of H if $(x\circ y)\cap I\neq\emptyset$ and $y\in I$ imply $x\in I$ for all $x,y\in H$, *hyper subalgebra* of H if $x\circ y\subseteq I$ for all $x,y\in I$.

3. Hyper BCC-algebras

Definition 3.1. By a *hyper* BCC-*algebra*, it is meant a nonempty set H endowed with a hyperoperation "$\circ$" and a constant 0 satisfying the following axioms:
(HC_1) $(x\circ z)\circ(y\circ z)\ll x\circ y$,
(HC_2) $0\circ x=\{0\}$,
(HC_3) $x\circ 0=\{x\}$,
(HC_4) $x\ll y$ and $y\ll x$ imply $x=y$
for all $x,y,z\in H$, where $x\ll y$ is defined by $0\in x\circ y$ and for every $A,B\subseteq H$, $A\ll B$ is defined by for all $a\in A$, there exists $b\in B$ such that $a\ll b$. In such case, "$\ll$" is called the *hyperorder* in H.

Example 3.2. (i) Let $(H,*,0)$ be a BCC-algebra and define a hyper operation "$\circ$" on H by $x\circ y=\{x*y\}$ for all $x,y\in H$. Then $(H,\circ)$ is a hyper BCC-algebra.

(ii) Let $H=\{0,1,2,3,\dots\}$ and hyper operation "$\circ$" on H is defined as follows:

$$x\circ y=\begin{cases}\{0,x\} & \text{if } x\leq y,\\ \{x\} & \text{if } x>y\end{cases}\tag{3.1}$$

for all $x,y\in H$. Then $(H,\circ)$ is a hyper BCC-algebra.

Table 3.1

$\circ_1$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$	$\{2\}$
3	$\{3\}$	$\{1,3\}$	$\{0,1,3\}$	$\{0,1,3\}$

Table 3.2

$\circ_2$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{2\}$	$\{0,1\}$

(iii) Let $H_1 = \{0,1,2,3\}$, $H_2 = \{0,1,2\}$, and hyper operations "$\circ_1$" and "$\circ_2$", respectively, on H_1 and H_2 be defined as in Tables 3.1 and 3.2.

Then $(H_1,\circ_1)$ and $(H_2,\circ_2)$ are hyper BCC-algebras.

Theorem 3.3. *Any hyper* BCK-*algebra is a hyper* BCC-*algebra.*

Proof. The proof follows from (HK_1), (HK_4), and Proposition 2.3(ii) and (iii). □

The converse of Theorem 3.3 is not true in general. As it is not difficult to see in Example 3.2(iii) that H_1 and H_2 are hyper BCC-algebras but they are not hyper BCK-algebras because $(2\circ 1)\circ 2 \neq (2\circ 2)\circ 1$.

Definition 3.4. Hyper BCC-algebra H is called a *proper hyper* BCC-*algebra* if H is not a hyper BCK-algebra.

Proposition 3.5. *Let H be a hyper* BCC-*algebra. Then for all $x,y,z \in H$ and $A \subseteq H$ the following statements hold:*

(i) $0\circ 0 = \{0\}$,

(ii) $0 \ll x$,

(iii) $x \ll x$,

(iv) $x\circ y \ll \{x\}$,

(v) $A\circ 0 = A$,

(vi) $0\circ A = \{0\}$,

(vii) $x\circ y = \{0\}$ *implies* $x\circ z \ll y\circ z$.

Proof. (i) In (HC_2), let $x = 0$. Then $0\circ 0 = \{0\}$.

(ii) By (HC_2), $0 \in 0\circ x$ and so $0 \ll x$.

(iii) In (HC_1), let $y = z = 0$. Then by (i) and (HC_3), we get that $x \ll x$.

(iv) By (HC_1), we conclude that $(x\circ y)\circ(z\circ y) \ll x\circ z$. Now, let $z = 0$. Then by (HC_2) and (HC_3), we get that $x\circ y \ll \{x\}$.

(v) The proof follows by (HC_3).

(vi) The proof follows by (HC_2).

(vii) Let $x \circ y = \{0\}$. Since by (HC_1), $(x \circ z) \circ (y \circ z) \ll x \circ y = \{0\}$, then for all $a \in (x \circ z) \circ (y \circ z)$, $a \ll 0$ and so by (HC_3) and (HC_4), $a = 0$. Hence $(x \circ z) \circ (y \circ z) = \{0\}$. Now, it is easy to show that $x \circ z \ll y \circ z$. □

Theorem 3.6. *Let H be a hyper* BCC-*algebra. Then H is a hyper* BCK-*algebra if and only if $(x \circ y) \circ z = (x \circ z) \circ y$ holds for all $x, y, z \in H$.*

Proof. Of course, every hyper BCK-algebra satisfies this identity. Conversely, in a hyper BCC-algebra satisfying this identity, for all $x, y \in H$, we have

$$x \circ y \ll \{x\} \Longleftrightarrow x \circ H \ll \{x\}. \tag{3.2}$$

Proposition 3.5(iv) completes the proof. □

Theorem 3.7. *Let H be a hyper* BCC-*algebra. Then the set*

$$S(H) = \big\{x \in H : x \circ x = \{0\}\big\} \tag{3.3}$$

is a BCC-*algebra.*

Proof. Let H be a hyper BCC-algebra and $S(H) = \{x \in H : x \circ x = \{0\}\}$. We claim that for all $y, z \in S(H)$, $|y \circ z| = 1$. Let there exist $y, z \in S(H)$ such that $|y \circ z| > 1$. Hence there exist $a, b \in y \circ z$ such that $a \neq b$. Since by (HC_1) and hypothesis,

$$a \circ b,\ b \circ a \subseteq (y \circ z) \circ (y \circ z) \ll y \circ y = \{0\}. \tag{3.4}$$

Then $a \circ b \ll \{0\}$ and $b \circ a \ll \{0\}$ and so $a \ll b$ and $b \ll a$. Hence by (HC_4), $a = b$ which is a contradiction. Therefore, for all $y, z \in S(H)$, $y \circ z$ is a singleton set and so $S(H)$ is a BCC-algebra. □

Theorem 3.8. *Let H be a hyper* BCC-*algebra such that for all $x, y \in H$,*

$$x \circ (x \circ y) \ll \{y\}. \tag{3.5}$$

Then, H is a BCK-*algebra.*

Proof. By hypothesis and (HC_3), $x \circ x = x \circ (x \circ 0) \ll 0$ for all $x \in H$ and so $x \circ x = \{0\}$ for all $x \in H$. Hence by Theorem 3.7, H is a BCC-algebra. Now, since by hypothesis, $x \circ (x \circ y) \ll \{y\}$ or equivalently, $(x \circ (x \circ y)) \circ y = 0$ (since H is a BCC-algebra), then by [4, Corollary 1], H is a BCK-algebra. □

Generalizing the construction used in [4, 6], we can prove the following.

Theorem 3.9. *Let* $(H_1, \circ_1)$ *be a (proper) hyper* BCC-*algebra,* $a \notin H_1$, *and* $H = H_1 \cup \{a\}$. *Define the hyper operation "*$\circ$*" on* H *as follows:*

$$x \circ y = \begin{cases} \{a\} & \text{if } x = a,\ y = 0, \\ \{x\} & \text{if } x \in H_1,\ y = a, \\ \{0, a\} & \text{if } x = y = a, \\ \{0\} & \text{if } x = a,\ y \in H_1 - \{0\}, \\ x \circ_1 y & \text{if } x, y \in H_1, \end{cases} \tag{3.6}$$

for all $x, y \in H$. *Then* $(H, \circ)$ *is a (proper) hyper* BCC-*algebra.*

Proof. The proof of (HC_2), (HC_3), and (HC_4) is clear. So, we verify only the axiom (HC_1). Let $x, y, z \in H$. We consider two cases for x.
Case 1 (($x \in H_1$)). Then $y \in H_1$ or $y = a$.
(1a) Let $y \in H_1$. If $z \in H_1$, then the proof is clear. If $z = a$, then

$$(x \circ z) \circ (y \circ z) = (x \circ a) \circ (y \circ a) = x \circ y = x \circ_1 y \ll x \circ_1 y = x \circ y. \tag{3.7}$$

(1b) Let $y = a$. If $z = 0$, then

$$(x \circ y) \circ (y \circ z) = (x \circ 0) \circ (a \circ 0) = (x \circ_1 0) \circ \{a\} = x \circ a = \{x\} \ll \{x\} = x \circ a = x \circ y. \tag{3.8}$$

If $z \in H_1 - \{0\}$, then

$$(x \circ z) \circ (y \circ z) = (x \circ z) \circ (a \circ z) = (x \circ_1 z) \circ 0 = x \circ_1 z \ll \{x\} = x \circ a = x \circ y. \tag{3.9}$$

If $z = a$, then

$$(x \circ z) \circ (y \circ z) = (x \circ a) \circ (a \circ a) = a \circ \{0, a\} = \{0, a\} \ll \{a\} = x \circ a = x \circ y. \tag{3.10}$$

Case 2 (($x = a$)). Then, as in the previous case, $y \in H_1$ or $y = a$.
(2a) Let $y \in H_1$. If $z = 0$, then

$$(x \circ z) \circ (y \circ z) = (a \circ 0) \circ (y \circ 0) = a \circ y \ll a \circ y = x \circ y. \tag{3.11}$$

If $z \in H_1 - \{0\}$, then

$$(x \circ z) \circ (y \circ z) = (a \circ z) \circ (y \circ z) = 0 \circ (y \circ z) = \{0\} \ll a \circ y = x \circ y. \tag{3.12}$$

If $z = a$, then

$$(x \circ z) \circ (y \circ z) = (a \circ a) \circ (y \circ a) = \{0, a\} \circ y = \{0\} \cup a \circ y \ll a \circ y = x \circ y. \tag{3.13}$$

(2b) Let $y = a$. If $z = 0$, then

$$(x \circ z) \circ (y \circ z) = (a \circ 0) \circ (a \circ 0) = a \circ a = \{0, a\} \ll \{0, a\} = a \circ a = x \circ y. \tag{3.14}$$

If $z \in H_1 - \{0\}$, then

$$(x \circ z) \circ (y \circ z) = (a \circ z) \circ (a \circ z) = 0 \circ 0 = \{0\} \ll \{0,a\} = a \circ a = x \circ y. \tag{3.15}$$

If $z = a$, then

$$(x \circ z) \circ (y \circ z) = (a \circ a) \circ (a \circ a) = \{0,a\} \ll \{0,a\} = a \circ a = x \circ z. \tag{3.16}$$

Hence H satisfies the axiom of (HC_1). Therefore, $(H,\circ)$ is a hyper BCC-algebra. □

Corollary 3.10. *For any $n \geq 3$, there exists at least one proper hyper* BCC-*algebra of order n.*

Proof. The proof follows from Theorem 3.9 and Example 3.2(iii). □

Theorem 3.11. *Let α be a transfinite cardinal number. Then there exists a hyper* BCC-*algebra H such that* $\mathrm{Card}(H) = \alpha$.

Proof. Since α is a transfinite cardinal number, then there exists an infinite set A such that $\mathrm{Card}(A) = \alpha$. Now, let $C = \{x_0, x_1, x_2, x_3, \ldots\}$ be an infinite countable subset of A. Let hyperoperation "$\diamond$" on C be defined as follows:

$$x_i \diamond x_j = \begin{cases} \{x_0, x_i\} & \text{if } i \leq j, \\ \{x_i\} & \text{if } i > j \end{cases} \tag{3.17}$$

for all $x_i, y_j \in C$. Similar to Example 3.2(ii), we can see that $(C, \diamond, x_0)$ is a hyper BCC-algebra. If $\mathrm{Card}(A) = \mathrm{Card}(C)$, the proof is complete. Now, let $\mathrm{Card}(C) < \mathrm{Card}(A)$. Let

$$\Omega = \{(H_i, \circ_i, x_0) \mid (H_i, \circ_i, x_0) \text{ is a hyper BCC-algebra, } C \subseteq H_i \subseteq A, \circ_i|_C = \diamond\}. \tag{3.18}$$

Since $(C, \diamond, x_0) \in \Omega$, then $\Omega \neq \emptyset$. Now, we define the relation "$\subseteq$" on Ω as follows:

$$(H_i, \circ_i, x_0) \subseteq (H_j, \circ_j, x_0) \Longleftrightarrow H_i \subseteq H_j, \quad \circ_j|_{H_i} = \circ_i. \tag{3.19}$$

Then $(\Omega, \subseteq)$ is a partially ordered set. Now, let $\{(H_k, \circ_k, x_0)\}_{k \in I}$ be a totally ordered set in Ω and $H' = \bigcup_{k \in I} H_k$. For all $x, y \in H'$, there exists $H_{t_{xy}} \in \{(H_k, \circ_k, x_0)\}_{k \in I}$ such that $x, y \in H_{t_{xy}}$ and so we can define hyperoperation "$\circ$" on H' by $x \circ y = x \circ_{t_{xy}} y$. It is easy to prove that $(H', \circ, x_0)$ is a hyper BCC-algebra and so $H' \in \Omega$. Hence by Zoren's lemma, Ω has a maximal element. Let $(H, \circ')$ be a maximal element of Ω. If $H = A$, the proof is complete. Let $H \neq A$. Hence, there exists an element $a \in A$ such that $a \notin H$. Now, hyperoperation "$\circ$" on $H \cup \{a\}$ is defined as follows:

$$x \circ y = \begin{cases} \{a\} & \text{if } x = a,\ y = x_0, \\ \{x\} & \text{if } x \in H,\ y = a, \\ \{x_0, a\} & \text{if } x = y = a, \\ \{x_0\} & \text{if } x = a,\ y \in H - \{x_0\}, \\ x \circ' y & \text{if } x, y \in H \end{cases} \tag{3.20}$$

for all $x, y \in H$. Then by Theorem 3.9, $(H \cup \{a\}, \circ, x_0)$ is a hyper BCC-algebra and so belongs to Ω, which is a contradiction, because H is maximal in Ω. Hence, $H = A$ and so $\text{Card}(H) = \alpha$. □

Corollary 3.12. *For all transfinite cardinal number α, there are infinite hyper* BCC-*algebras of order α.*

Proof. By Theorem 3.11, there exists a hyper BCC-algebra H such that $\text{Card}(H) = \alpha$. Now, let $A = \{a_1, a_2, a_3, \ldots\}$ be an infinite set such that $H \cap A = \emptyset$. By Theorem 3.9, $H_1 = H \cup \{a_1\}$ is a hyper BCC-algebra of order $\alpha + 1$. Since α is a transfinite cardinal number, then $\alpha + 1 = \alpha$. Hence, $\text{Card}(H_1) = \alpha$. Similarly, $H_2 = H_1 \cup \{a_2\}$ is a hyper BCC-algebra of order $\alpha + 1 = \alpha$. By continuing this process, we get infinite hyper BCC-algebras of order α. □

4. Hyper BCC-ideals

Definition 4.1. A subset I of a hyper BCC-algebra H such that $0 \in I$ is called the following:

(i) a *hyper* BCC-*ideal of type* 1, if

$$(x \circ y) \circ z \ll I, \quad y \in I \Longrightarrow x \circ z \subseteq I, \tag{4.1}$$

(ii) a *hyper* BCC-*ideal of type* 2, if

$$(x \circ y) \circ z \subseteq I, \quad y \in I \Longrightarrow x \circ z \subseteq I, \tag{4.2}$$

(iii) a *hyper* BCC-*ideal of type* 3, if

$$(x \circ y) \circ z \ll I, \quad y \in I \Longrightarrow x \circ z \ll I, \tag{4.3}$$

(iv) a *hyper* BCC-*ideal of type* 4, if

$$(x \circ y) \circ z \subseteq I, \quad y \in I \Longrightarrow x \circ z \ll I. \tag{4.4}$$

Theorem 4.2. *In any hyper* BCC-*algebra, the following statements are valid.*

(i) *Any hyper* BCC-*ideal of type* 1 *is a hyper* BCC-*ideal of types* 2 *and* 3.
(ii) *Any hyper* BCC-*ideal of type* 2 *is a hyper* BCC-*ideal of type* 4.
(iii) *Any hyper* BCC-*ideal of type* 3 *is a hyper* BCC-*ideal of type* 4.
(iv) *Any hyper* BCC-*ideal of type* 1 *is a hyper* BCK-*ideal.*
(v) *Any hyper* BCC-*ideal of type* 2 *is a weak hyper* BCK-*ideal.*

Proof. The statements (i), (ii), and (iii) are clear. So, we prove only (iv) and (v).

(iv) Let I be a hyper BCC-ideal of type 1, $x \circ y \ll I$, and $y \in I$. Hence, by Proposition 3.5(v), we obtain $(x \circ y) \circ 0 = x \circ y \ll I$. But $y \in I$, so, applying the hypothesis and (HC_3), we get $\{x\} = x \circ 0 \subseteq I$. This shows that I is a hyper BCK-ideal of H.

(v) The proof of (iv) is analogous. □

Table 4.1

$\circ$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$

Table 4.2

$\circ_1$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{b\}$	$\{a,b\}$	$\{0,a\}$

Table 4.3

$\circ_2$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$	$\{0\}$
c	$\{c\}$	$\{b\}$	$\{b\}$	$\{0,a\}$

Table 4.4

$\circ_3$	0	a	b	c
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$	$\{0,b\}$
c	$\{c\}$	$\{b\}$	$\{a,b\}$	$\{0,a,b\}$

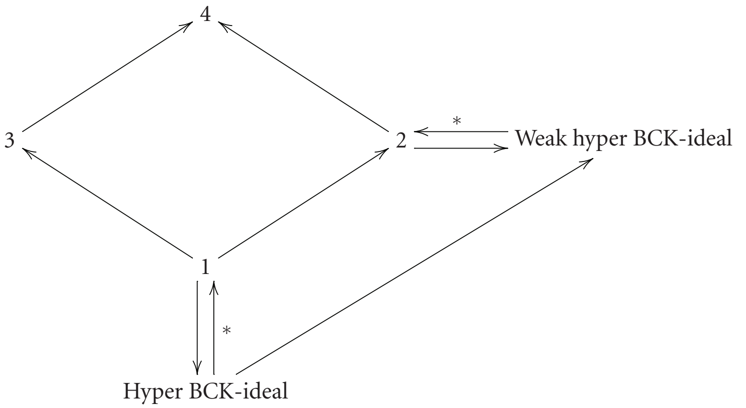
Example 4.3. (i) In a hyper BCC-algebra H defined in Example 3.2(i), every BCC-ideal I of a BCC-algebra $(H,*,0)$ is a hyper BCC-ideal of types 1, 2, 3, and 4 of a hyper BCC-algebra $(H,\circ,0)$.

(ii) Let $H=\{0,a,b\}$. Consider Table 4.1.

Then H is a proper hyper BCC-algebra (since $(b\circ a)\circ b\neq(b\circ b)\circ a$). Moreover, $I_1=\{0,a\}$ is a hyper BCC-ideal of types 1, 2, 3 and 4 while $I_2=\{0,b\}$ is a hyper BCC-ideal of types 3 and 4 but it is not a hyper BCC-ideal of types 1 and 2 (since $(a\circ b)\circ 0\subseteq I_2$ and $b\in I_2$ but $a\circ 0\nsubseteq I_2$, then I_2 is not a hyper BCC-ideal of type 2 and so by Theorem 4.2(i) it is not a hyper BCC-ideal of type 1).

(iii) Define on $H=\{0,a,b,c\}$ the following three hyperoperations, Tables 4.2, 4.3, and 4.4.

Then, as it is not difficult to verify, $(H,\circ_1)$, $(H,\circ_2)$, and $(H,\circ_3)$ are hyper BCC-algebras. These algebras are proper BCC-algebras because $(c\circ a)\circ b\neq(c\circ b)\circ a$. In $(H,\circ_1)$, $I=\{0,b\}$ is a hyper BCC-ideal of type 4 but it is not a hyper BCC-ideal of type 3 (since



$*$: if H is a hyper BCK-algebra.

Figure 4.1

$(c \circ b) \circ 0 \ll I$ and $b \in I$ but $c \circ 0 \not\ll I$). In $(H, \circ_2)$, $I = \{0, a\}$ is a hyper BCK-ideal of H but it is not a hyper BCC-ideal of type 1 (since $(c \circ a) \circ b \ll I$ and $a \in I$ but $c \circ b \nsubseteq I$). In $(H, \circ_3)$, $I = \{0, a, c\}$ is a hyper BCC-ideal of type 2 but it is not a hyper BCC-ideal of type 1 (since $(b \circ a) \circ a \ll I$ and $a \in I$ but $b \circ a \nsubseteq I$).

Open problem 4.4. Is there a proper hyper BCC-algebra in which the concepts of a hyper BCC-ideal of type 2 and a weak hyper BCK-ideal are different? We think that in hyper BCC-algebras of orders 3 and 4, these concepts are equivalent(?).

THEOREM 4.5. *Let H be a hyper* BCK-*algebra and let I be a nonempty subset of H. Then,*
(i) *I is a hyper* BCC-*ideal of type* 1 *if and only if I is a hyper* BCK-*ideal of H,*
(ii) *I is a hyper* BCC-*ideal of type* 2 *if and only if I is a weak hyper* BCK-*ideal of H.*

Proof. (i) "If" part. By Theorem 4.2(iv).
"Only if" part. Let I be a hyper BCK-ideal of H, $(x \circ y) \circ z \ll I$ and $y \in I$. By (HK_2), $(x \circ z) \circ y = (x \circ y) \circ z \ll I$ and so for each $a \in x \circ z$, $a \circ z \ll I$. Since $z \in I$ and I is a hyper BCK-ideal of H, then $a \in I$ and so $x \circ y \subseteq I$. Hence I is a hyper BCC-ideal of type 1.
(ii) The proof is similar to the proof of (i). □

Now, we summarize Theorems 4.2 and 4.5 in Figure 4.1.

Definition 4.6. A nonempty subset I of a hyper BCC-algebra H satisfies the *closed condition* if $x \ll y$ and $y \in I$ imply $x \in I$.

THEOREM 4.7. *A nonempty subset I of a hyper* BCC-*algebra H satisfying the closed condition is a hyper* BCC-*ideal of type i, for $1 \leq i \leq 4$ if and only if I is a hyper* BCC-*ideal of type j, for $1 \leq j \leq 4$.*

Proof. Let I satisfy the closed condition. It is easy to prove that for any subset A of H if $A \ll I$, then $A \subseteq I$. Hence the proof is clear. □

Theorem 4.8. *Any strong hyper* BCK-*ideal of a hyper* BCC (BCK)-*algebra is its hyper subalgebra.*

Proof. Let I be a strong hyper BCK-ideal of H and $x,y \in I$. By Proposition 3.5(iv), $x \circ y \ll \{x\}$ and so for each $a \in x \circ y$, $a \ll x$ and so $0 \in a \circ x$. Since $0 \in I$, then $a \circ x \cap I \neq \emptyset$. Now, since $x \in I$ and I is a strong hyper BCK-ideal of H, then $a \in I$ and so $x \circ y \subseteq I$. Hence I is a hyper subalgebra of H. □

5. Classification of hyper BCC-algebras of order 3

Based on the results of the previous section, we are to able to calculate of all nonisomorphic hyper BCC-algebras of order 3. For simplicity, let in this section H be a hyper BCC-algebra and let $H = \{0,a,b\}$.

We will say that this hyper BCC-algebra is *linear* if all its elements are comparable, that is, if $0 \ll a \ll b$ or $0 \ll b \ll a$. Any other hyper BCC-algebra H will be called *nonlinear*. Of course, in any case, we have $0 \ll a$ and $0 \ll b$.

Theorem 5.1. *There are only three nonisomorphic nonlinear hyper* BCC-*algebras of order 3, which are not proper hyper* BCC-*algebras.*

Proof. In a nonlinear hyper BCC-algebra H, we have the following:

(1) $x \notin y \circ x$ for all $x,y \in H$ such that $x \neq y$,

(2) $a \circ b = \{a\}$ and $b \circ a = \{b\}$,

(3) $a \circ a = \{0\}$ or $\{0,a\}$ and $b \circ b = \{0\}$ or $\{0,b\}$.

Indeed, if $x \neq y$ and $x \in y \circ x$, then $x \neq 0$, because $x = 0$ implies $y \neq 0$ and $0 \in y \circ 0 = \{y\}$, whence $y = 0$, which is impossible. By Proposition 3.5(iv), $y \circ x \ll \{y\}$. Therefore, $x \ll y$. But $x \neq 0$, so, $a \ll b$ or $b \ll a$, which is impossible by the assumption. This proves (1).

(2) From $a \not\ll b$, it follows that $0 \notin a \circ b$. So, $a \circ b$ cannot be equal to $\{0\}$, $\{0,a\}$, $\{0,b\}$, or $\{0,a,b\}$. Because by (1), we have also $b \notin a \circ b$, then $a \circ b$ cannot be equal to $\{b\}$ or $\{a,b\}$. Thus $a \circ b = \{a\}$. Similarly, we can prove that $b \circ a = \{b\}$.

(3) For $a \circ a$, the following cases are possible: $\{0\}$, $\{0,a\}$, $\{0,b\}$, or $\{0,a,b\}$. In the case $a \circ a = \{0,b\}$ or $a \circ a = \{0,a,b\}$, by Proposition 3.5(iv), we get $a \circ a \ll a$. But by the assumption, $b \in a \circ a$, whence $b \ll a$, which is impossible. So, $a \circ a \neq \{0,a,b\}$ and $a \circ a \neq \{0,b\}$. This means that $a \circ b = \{0\}$ or $a \circ b = \{0,a\}$. Similarly, we can show that $b \circ b = \{0\}$ or $b \circ b = \{0\},\{0,b\}$.

Therefore, by (1), (2), and (3), we conclude that there are four nonlinear hyper BCC-algebras containing three elements. Their hyperoperations are given by Tables 5.1, 5.2, 5.3, and 5.4.

But $(H,\circ_2)$ and $(H,\diamond)$ are isomorphic. The isomorphism $f:(H,\circ) \to (H,\circ_2)$ is defined by $f(0) = 0$, $f(a) = b$ and $f(b) = a$. Other nonlinear hyper BCC-algebras are not isomorphic. It is clear that they are not proper hyper BCC-algebras. □

Theorem 5.2. *There are no linear hyper* BCC-*algebras of order 3 containing two proper hyper* BCK-*ideals.*

Proof. Suppose that there is a linear hyper BCC-algebra H of order 3 that has two proper hyper BCK-ideals. Then $I_1 = \{0,a\}$ and $I_2 = \{0,b\}$ must be proper hyper BCK-ideals. Let

Table 5.1

$\circ_1$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

Table 5.2

$\circ_2$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$

Table 5.3

$\circ_3$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$

Table 5.4

$\diamond$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

$a \ll b$. Then $a \circ b$ is equal to $\{0\}$, $\{0,a\}$, $\{0,b\}$, or $\{0,a,b\}$. In all these cases, $a \circ b \ll I_2$. Since I_2 is a hyper BCK-ideal of H and $b \in I_2$, so, $a \in I_2$, which is impossible. For $b \ll a$, we obtain a similar contradiction. □

Theorem 5.3. *There are* 12 *nonisomorphic linear hyper* BCC-*algebras of order 3 containing only one proper hyper* BCK-*ideal and this ideal is strong too. Only two such hyper* BCC-*algebras are proper.*

Proof. Without loss of generality, let $I_1 = \{0,a\}$ be the proper strong hyper BCK-ideal of a linear hyper BCC-algebra H. Then $a \ll b$ or $b \ll a$. For $b \ll a$, we have $0 \in b \circ a$, which gives $b \circ a \cap I_1 \neq \emptyset$. But I_1 is a strong hyper BCK ideal, so, $a \in I_1$ implies $b \in I_1$, which is impossible. Hence, $a \ll b$ and $b \not\ll a$. Thus $b \circ a$ is equal to $\{a\}$, $\{b\}$, or to $\{a,b\}$. If $b \circ a = \{a\}$, then $b \circ a \ll I_1$, whence $b \in I_1$, which is not true. Therefore, $b \circ a \neq \{a\}$. In the case $b \circ a = \{a,b\}$, we have $b \circ a \cap I_1 \neq \emptyset$, consequently, $b \in I_1$, because I_1 is strong. This also is impossible. Thus it must be $b \circ a = \{b\}$. Moreover, $a \ll b$ implies $0 \in a \circ b$. Thus $a \circ b$ is equal to one of the sets $\{0\}, \{0,a\}, \{0,b\}$, $\{0,a,b\}$. If $a \circ b = \{0,b\}$, then, by Proposition 3.5(iv), $\{0,b\} = a \circ b \ll a$, whence $b \ll a$ which is not true. For $a \circ b =$

Table 5.5

$\diamond_1$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

Table 5.6

$\diamond_2$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a\}$

Table 5.7

$\diamond_3$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$

Table 5.8

$\diamond_4$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$

$\{0,a,b\}$, we obtain the similar contradiction. Thus $a \circ b = \{0\}$ or $\{0,a\}$. To compute $a \circ a$, observe that $a \circ a$ can be equal to one of the sets $\{0\}$, $\{0,a\}$, $\{0,b\}$, or $\{0,a,b\}$. If $a \circ a = \{0,b\}$ or $\{0,a,b\}$, then $a \circ a \ll a$ implies $b \ll a$, which is impossible. So, $a \circ a = \{0\}$ or $\{0,a\}$. Furthermore, $b \circ b$ is equal to $\{0\}$, $\{0,a\}$, $\{0,b\}$, or $\{0,a,b\}$. This means that totally we have $2 \times 2 \times 4 = 16$ different cases for H.

Consider Tables 5.5, 5.6, 5.7, and 5.8.

In $(H, \diamond_i), 1 \leq i \leq 4$, we have $(a \circ b) \circ (a \circ b) \not\ll a \circ a$. Thus $(H, \diamond_i)$ are not hyper BCC-algebras.

The other 12 cases are presented in Tables 5.9, 5.10, 5.11, 5.12, 5.13, 5.14, 5.15, 5.16, 5.17, 5.18, 5.19, and 5.20.

It is not difficult to verify that hyperstructures defined by these tables are hyper BCC-algebras. Because the nonidentity homomorphism f from $(H, \circ_i)$ onto $(H, \circ_j)$, $i \neq j$, and $4 \leq i, j \leq 15$, is defined by $f(0) = 0$, $f(a) = b$, $f(b) = a$, then these 12 hyper BCC-algebras are not isomorphic. Only hyper BCC-algebras $(H, \circ_4)$ and $(H, \circ_5)$ are proper. In these hyper BCC-algebras, we have $(b \circ a) \circ b \neq (b \circ b) \circ a$. □

Table 5.9

$\circ_4$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0,a\}$

Table 5.10

$\circ_5$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$

Table 5.11

$\circ_6$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

Table 5.12

$\circ_7$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$

Table 5.13

$\circ_8$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

Table 5.14

$\circ_9$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0\}$

Table 5.15

$\circ_{10}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0,a\}$

Table 5.16

$\circ_{11}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$

Table 5.17

$\circ_{12}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{a\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$

Table 5.18

$\circ_{13}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,b\}$

Table 5.19

$\circ_{14}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a\}$

THEOREM 5.4. *There are four nonisomorphic linear hyper* BCK-*algebras of order 3 containing only one proper hyper* BCK-*ideal and this ideal is not strong. One of them is a proper* BCC-*algebra.*

Proof. Without loss of generality, let $I_1 = \{0,a\}$ be the proper hyper BCK-ideal of H which is not strong. Since H is linear, then $b \ll a$ or $a \ll b$. If $b \ll a$, then $0 \in b \circ a$ and so $b \circ a = \{0\}$ or $\{0,a\}$ or $\{0,b\}$, or $\{0,a,b\}$. Since $b \ll a$, then in any of the above cases, $b \circ a \ll I_1$. Now, since $a \in I_1$ and I_1 is a hyper BCK-ideal of H, then $b \in I_1$ which is impossible. Hence $a \ll b$ and so $0 \notin b \circ a$. Therefore, $b \circ a$ is equal to $\{a\}$, $\{b\}$, or $\{a,b\}$. If $b \circ a = \{a\}$, then $b \circ a \ll I_1 = \{0,a\}$. Since $a \in I_1$ and I_1 is a hyper BCK-ideal, we get $b \in I_1$, which is not true. In the case $b \circ a = \{b\}$, I_1 is a strong hyper BCK-ideal, because $b \circ a \cap I_1 = b \circ 0 \cap I_1 = \varnothing$. This is a contradiction. Thus it must be $b \circ a = \{a,b\}$.

Table 5.20

$\circ_{15}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$

Table 5.21

$\circ_{16}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,b}

Table 5.22

$\circ_{17}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0,a}	{0,a}
b	{b}	{a,b}	{0,a,b}

Next, similarly as in the proof of Theorem 5.3, we can show that $a \circ a$ is equal to $\{0\}$ or $\{0,a\}$, $a \circ b$ is equal to $\{0\}$ or $\{0,a\}$, and $b \circ b$ is equal to $\{0\}$, $\{0,a\}$, $\{0,b\}$, or $\{0,a,b\}$. Thus there are $2 \times 2 \times 4 = 16$ different cases for H.

(i) Let $a \circ a = \{0\}$. If $a \circ b = \{0\}$, $b \circ b = \{0\}$, or $b \circ b = \{0,a\}$, then

$$(b \circ a) \circ (b \circ a) \not\ll b \circ b. \tag{5.1}$$

If $a \circ b = \{0,a\}$, then we have

$$(a \circ b) \circ (a \circ b) \not\ll a \circ a. \tag{5.2}$$

This means that in these six cases, obtained hyperstructures are not hyper BCC-algebras.

(ii) Let $a \circ a = \{0,a\}$. If $a \circ b = \{0\}$, then

$$(a \circ a) \circ (b \circ a) \not\ll a \circ b. \tag{5.3}$$

If $a \circ b = \{a\}$ and $b \circ b = \{0\}$, or $b \circ b = \{0,a\}$, then

$$(b \circ a) \circ (b \circ a) \not\ll b \circ b. \tag{5.4}$$

So, in these cases, we do not obtain a hyper BCC-algebra too. In the remaining four cases, we obtain hyperstructures in Tables 5.21, 5.22, 5.23, and 5.24.

Direct computations show that these four hyperstructures are hyper BCC-algebras. It is clear that these hyper BCC-algebras are not isomorphic. Only the last hyperstructure is not a hyper BCK-algebra. Indeed, in $(H, \circ_{19})$, we have $(b \circ a) \circ b \neq (b \circ b) \circ a$. □

Table 5.23

$\circ_{18}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,a,b}

Table 5.24

$\circ_{19}$	0	a	b
0	{0}	{0}	{0}
a	{a}	{0}	{0}
b	{b}	{a,b}	{0,b}

THEOREM 5.5. *There are three nonisomorphic and nonproper linear hyper* BCC-*algebras of order 3 without proper hyper* BCK-*ideals.*

Proof. Without loss of generality, we let $a \ll b$. Then $0 \in a \circ b$ and so $a \circ b$ is equal to one of the sets $\{0\}$, $\{0,a\}$, $\{0,b\}$, or $\{0,a,b\}$. If $a \circ b = \{0,b\}$, then by Proposition 2.3, $\{0,b\} = a \circ b \ll \{a\}$, thus $b \ll a$ which is not true. If $a \circ b = \{0,a,b\}$, then similarly, we obtain a contradiction $b \ll a$. Thus $a \circ b = \{0\}$ or $\{0,a\}$. Clearly, $a \circ a$ can be equal to one of the sets $\{0\}$, $\{0,a\}$, $\{0,b\}$, or $\{0,a,b\}$. If $a \circ a = \{0,b\}$ or $\{0,a,b\}$, then since $a \circ a \ll a$, we get that $b \ll a$ which is impossible. Thus $a \circ a = \{0\}$ or $\{0,a\}$. Also $b \circ b$ is equal to one of the sets $\{0\}$, $\{0,a\}$, $\{0,b\}$, or $\{0,a,b\}$. Furthermore, $b \circ a$ is equal to one of the sets $\{a\}$, $\{b\}$, or $\{a,b\}$. If $b \circ a = \{b\}$, then similar to the proof of Theorem 5.4, we conclude that I_1 is strong, which is a contradiction. If $b \circ a = \{a,b\}$, then all of hyper BCK-algebras which are obtained are the same as Theorem 5.4. But each of these kinds of hyper BCK-algebras has a proper BCK-ideal $I_1 = \{0,a\}$, while by hypothesis, the involved hyper BCK-algebras have not a proper BCK-ideal. So $b \circ a \neq \{a,b\}$ and we must have $b \circ a = \{a\}$. Thus we have $2 \times 2 \times 4 = 16$ different possibilities.

(i) Let $a \circ a = \{0\}$. If $a \circ b = \{0\}$ and $b \circ b = \{0,b\}$, or $b \circ b = \{0,a,b\}$, then

$$(b \circ b) \circ (a \circ b) \not\ll b \circ a. \tag{5.5}$$

If $a \circ b = \{0,a\}$, then

$$(a \circ b) \circ (a \circ b) \not\ll a \circ a. \tag{5.6}$$

Therefore, in the above six cases, we do not obtain a hyper BCC-algebra.

(ii) Let $a \circ a = \{0,a\}$. If $a \circ b = \{0\}$, then

$$(a \circ a) \circ (b \circ a) \not\ll a \circ b. \tag{5.7}$$

If $a \circ b = \{0,a\}$ and $b \circ b$ is one of the sets $\{0,b\}$ or $\{0,a,b\}$, then

$$(b \circ b) \circ (a \circ b) \not\ll b \circ a. \tag{5.8}$$

Table 5.25

$\circ_{20}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a\}$	$\{0\}$

Table 5.26

$\circ_{21}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a\}$	$\{0,a\}$

Table 5.27

$\circ_{22}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0,a\}$	$\{0,a\}$
b	$\{b\}$	$\{a\}$	$\{0,a\}$

Table 5.28

$\circ_4$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0,a\}$

If $a \circ b = \{0,a\}$ and $b \circ b = \{0\}$, then

$$(b \circ a) \circ (b \circ a) \not\ll b \circ b. \tag{5.9}$$

Hence, in the above seven cases, we do not obtain a hyper BCC-algebra too. In the remaining three cases, we obtain the following hyperstructures in Tables 5.25, 5.26, and 5.27.

It is not difficult to check that these hyperstructures are hyper BCC-algebras which are not proper hyper BCC-algebras. Obviously, these hyper BCC-algebras are not isomorphic. □

Summarizing our calculations, we obtain the following corollaries.

Corollary 5.6. *There are 22 nonisomorphic hyper* BCC-*algebras of order 3.*

Corollary 5.7. *There are only three proper nonisomorphic hyper* BCC-*algebras of order 3 in Tables 5.28, 5.29, and 5.30.*

Table 5.29

$\circ_5$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{b\}$	$\{0,a,b\}$

Table 5.30

$\circ_{19}$	0	a	b
0	$\{0\}$	$\{0\}$	$\{0\}$
a	$\{a\}$	$\{0\}$	$\{0\}$
b	$\{b\}$	$\{a,b\}$	$\{0,b\}$

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