# ON SOME EXPONENTIAL MEANS. PART II

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We prove some new inequalities involving an exponential mean, its complementary, and some means derived from known means by applying the exp-log method.



## 1. Introduction

All the means that appear in this paper are functions $M:\mathbb{R}_+^2 \to \mathbb{R}_+$ with the property that

$$\min(a,b) \le M(a,b) \le \max(a,b) \quad \forall\, a,b > 0. \tag{1.1}$$

Of course $M(a,a) = a$, for all $a > 0$. As usual $A$, $G$, $L$, $I$, $A_p$ denote the arithmetic, geometric, logarithmic, identric, respectively, power means of two positive numbers, defined by

$$\begin{gathered} A = A(a,b) = \frac{a+b}{2}, \qquad G = G(a,b) = \sqrt{ab}, \\ L = L(a,b) = \frac{b-a}{\log b - \log a}, \qquad I = I(a,b) = \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{1/(b-a)}, \\ A_p = A_p(a,b) = \left(\frac{a^p + b^p}{2}\right)^{1/p}, \quad p \ne 0. \end{gathered} \tag{1.2}$$

In [16], the first part of this paper, we have studied the exponential mean

$$E = E(a,b) = \frac{be^b - ae^a}{e^b - e^a} - 1 \tag{1.3}$$

introduced in [23]. Another exponential mean was defined in [19] by

$$\overline{E} = \overline{E}(a,b) = \frac{ae^b - be^a}{e^b - e^a} + 1. \tag{1.4}$$

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It is the complementary of $E$, according to a definition from [4], that is,

$$\overline{E} = 2A - E. \tag{1.5}$$

A basic inequality proved in [23] is

$$E > A, \tag{1.6}$$

which gives the new inequality

$$\overline{E} < A. \tag{1.7}$$

More general means have been studied in [14, 17, 19]. For example, letting $f(x) = e^x$ in [14, Formula (5)], we recapture (1.6). We note that by selecting $f(x) = \log x$ in [14, Formula (8)], and then $f(x) = 1/x$, we get the standard inequalities

$$G < L < I < A \tag{1.8}$$

(for history, see, e.g., [7]).

In what follows, for any mean $M$, we will denote by $\mathcal{M}$ the new mean given by

$$\mathcal{M}(x,y) = \log M(e^x, e^y), \quad x, y > 0. \tag{1.9}$$

As we put $a = e^x$, $b = e^y$ and then take logarithms, we call this procedure the exp-log method. The method will be applied also to some inequalities for deriving new inequalities. For example, in [16] we proved that

$$E = \mathcal{I}, \tag{1.10}$$

and so (1.8) becomes

$$A < \mathcal{L} < E < \mathcal{A}. \tag{1.11}$$

In [16], it was also shown that

$$\begin{gathered} A + \mathcal{A} - \mathcal{L} < E < 2\mathcal{L} - A, \\ \mathcal{A}_{2/3} < E < \mathcal{A}_{\log 2} \end{gathered} \tag{1.12}$$

(see also [6, 22]). In [9], the first author improved the inequality (1.6) by

$$E > \frac{A + 2\mathcal{A}}{3} > A. \tag{1.13}$$

This is based on the following identity proved there:

$$(E - A)(a,b) = \frac{A(e^a, e^b)}{L(e^a, e^b)} - 1. \tag{1.14}$$

We get the same result using the known result

$$I > \frac{2A+G}{3} > \left(A^2G\right)^{1/3} \tag{1.15}$$

and the exp-log method.

The aim of this paper is to obtain other inequalities related to the above means.

## 2. Main results

(1) After some computations, the inequality (1.6) becomes

$$\frac{e^b - e^a}{b-a} < \frac{e^a + e^b}{2}. \tag{2.1}$$

This follows at once from the Hadamard inequality

$$\frac{1}{b-a}\int_a^b f(t)dt < \frac{f(a)+f(b)}{2}, \tag{2.2}$$

applied to the strictly convex function $f(t) = e^t$. We note that by the second Hadamard inequality, namely

$$\frac{1}{b-a}\int_a^b f(t)dt > f\left(\frac{a+b}{2}\right), \tag{2.3}$$

for the same function, one obtains

$$\frac{e^b - e^a}{b-a} > e^{(a+b)/2}, \tag{2.4}$$

which has been proposed as a problem in [3].

The relation (1.11) improves the inequality (2.1), which means that $\mathcal{A} > \mathcal{L}$, and improves (2.4), which means that $\mathcal{L} > A$. In fact, by the above remarks, one can say that

$$E > A \Longleftrightarrow \mathcal{A} > \mathcal{L}. \tag{2.5}$$

(2) In [23], it was proven that $E$ is not comparable with $A_\lambda$ for $\lambda > 5/3$. Then in [17], we have shown, among others, that

$$A(a,b) < E(a,b) < A(a,b)\cdot e^{|b-a|/2}. \tag{2.6}$$

Now, if $|b-a|$ becomes small, clearly $e^{|b-a|/2}$ approaches to 1, that is, the conjecture $E > A_\lambda$ of [23] cannot be true for any $1 < \lambda \le 5/3$.

We get another double inequality from (1.5) and (1.6):

$$A < E < 2A. \tag{2.7}$$

These inequalities cannot be improved. Indeed, for $1 < \lambda < 2$, we have

$$\lim_{x\to\infty}\left[E(1,x) - \lambda A(1,x)\right] = \infty, \tag{2.8}$$

but

$$E(1,1)-\lambda A(1,1)=1-\lambda<0, \tag{2.9}$$

thus $E$ is not comparable with $\lambda A$.

On the other hand,

$$\overline{E}(a,b)=\frac{e^b(a+1)-e^a(b+1)}{e^b-e^a}=(a+1)(b+1)\cdot\frac{f(b)-f(a)}{e^b-e^a}, \tag{2.10}$$

where $f(x)=e^x/(x+1)$. By Cauchy's mean value theorem,

$$\frac{f(b)-f(a)}{e^b-e^a}=\frac{f'(c)}{e^c},\quad c\in(a,b). \tag{2.11}$$

Since

$$\frac{f'(c)}{e^c}=\frac{c}{(c+1)^2}\le\frac{1}{4}, \tag{2.12}$$

we get

$$0<2A-E\le\frac{(a+1)(b+1)}{4}. \tag{2.13}$$

(3) By using the series representation

$$\log\frac{I}{G}=\sum_{k=1}^{\infty}\frac{1}{2k+1}\left(\frac{b-a}{b+a}\right)^{2k}, \tag{2.14}$$

(see [9, 21]), we can deduce the following series representation:

$$(E-A)(a,b)=\sum_{k=1}^{\infty}\frac{1}{2k+1}\left(\frac{e^b-e^a}{e^b+e^a}\right)^{2k}. \tag{2.15}$$

By (2.1), $|e^b-e^a|/(e^b+e^a)<|b-a|/2$, thus we get the estimate

$$(E-A)(a,b)<\sum_{k=1}^{\infty}\frac{1}{2k+1}\left(\frac{b-a}{2}\right)^{2k}. \tag{2.16}$$

The series is convergent at least for $|b-a|<2$. Writing

$$\frac{A(e^a,e^b)}{L(e^a,e^b)}=e^{\mathscr{A}(a,b)-\mathscr{L}(a,b)}, \tag{2.17}$$

the identity (1.14) implies the relation

$$E-A=e^{\mathscr{A}-\mathscr{L}}-1. \tag{2.18}$$

This gives again the equivalence (2.5). But one can obtain also a stronger relation by writing $e^x > 1 + x + x^2/2$, for $x > 0$. Thus (2.18) gives

$$E - A > \mathcal{A} - \mathcal{L} + \frac{1}{2}(\mathcal{A} - \mathcal{L})^2. \tag{2.19}$$

(4) Consider the inequality proved in [10]:

$$\frac{2}{e}A < I < A. \tag{2.20}$$

By the exp-log method, we deduce

$$\log 2 - 1 + \mathcal{A} < E < \mathcal{A}. \tag{2.21}$$

From the inequality

$$I < \frac{2}{e}(A + G) = \frac{4}{e}\left(\frac{\sqrt{a} + \sqrt{b}}{2}\right)^2, \tag{2.22}$$

given in [5], we have, by the same method,

$$E(x, y) < 2\log 2 - 1 + 2\mathcal{A}\left(\frac{x}{2}, \frac{y}{2}\right). \tag{2.23}$$

Relation (2.23) may be compared with the left-hand side of (2.21). Take now the relation

$$L < L(A, G) = \frac{A - G}{\log(A/G)} \tag{2.24}$$

from [5]. Since $A - G = 1/2(\sqrt{a} - \sqrt{b})^2$, one obtains

$$\mathcal{A} - A < \frac{1}{2e^{\mathcal{L}}}\left(e^{x/2} - e^{y/2}\right)^2. \tag{2.25}$$

The relation

$$L^3 > \left(\frac{A + G}{2}\right)^2 G, \tag{2.26}$$

from [13], gives similarly

$$3\mathcal{L}(x, y) > A(x, y) + 4\mathcal{A}\left(\frac{x}{2}, \frac{y}{2}\right), \tag{2.27}$$

while the inequality

$$\log\frac{I}{L} > 1 - \frac{G}{L}, \tag{2.28}$$

from [7], offers the relation

$$E - \mathcal{L} > 1 - e^{A - \mathcal{L}}. \tag{2.29}$$

(5) The exp-log method applied to the inequality

$$L > \sqrt{GI}, \tag{2.30}$$

given in [2, 11], implies that

$$\mathcal{L} > \frac{A+E}{2} > \frac{2A+\mathcal{A}}{3}. \tag{2.31}$$

On the other side, the inequality

$$I > \sqrt{AL}, \tag{2.32}$$

proven in [11], gives on the same way the inequality

$$E > \frac{\mathcal{A}+\mathcal{L}}{2}. \tag{2.33}$$

After all, we have the double inequality

$$\frac{\mathcal{A}+\mathcal{L}}{2} < E < 2\mathcal{L} - A. \tag{2.34}$$

(6) Consider now the inequality

$$3I^2 < 2A^2 + G^2, \tag{2.35}$$

from [20]. It gives

$$\log 3 + 2E < \log\left(e^{2A} + 2e^{2\mathcal{A}}\right). \tag{2.36}$$

Similarly

$$I > \frac{2A+G}{3}, \tag{2.37}$$

given in [8], implies that

$$\log 3 + E > \log\left(2e^{A} + e^{\mathcal{A}}\right). \tag{2.38}$$

In fact, the relation

$$I > \frac{A+L}{2}, \tag{2.39}$$

from [7], gives

$$\log 2 + E > \log\left(e^{\mathcal{L}} + e^{\mathcal{A}}\right), \tag{2.40}$$

but this is weaker than (2.38), as follows from [8]. The inequalities (2.33) and (2.40) can be combined as

$$E > \log\left(\frac{e^{\mathcal{L}} + e^{\mathcal{A}}}{2}\right) > \frac{\mathcal{L}+\mathcal{A}}{2}, \tag{2.41}$$

where the second inequality is a consequence of the concavity of the logarithmic function. We notice also that by

$$L + I < A + G, \tag{2.42}$$

given in [1], one can write

$$e^{\mathscr{L}} + e^{E} < e^{\mathscr{A}} + e^{A}. \tag{2.43}$$

(7) In [9] Sándor proved the inequality

$$I(a^2, b^2) < \frac{A^4(a,b)}{I^2(a,b)}. \tag{2.44}$$

By the exp-log method, we get

$$E(2x, 2y) < 4\mathscr{A}(x,y) - 2E(x,y). \tag{2.45}$$

It is interesting to note that by the equality

$$\log\frac{I^2(\sqrt{a},\sqrt{b})}{I(a,b)} = \frac{G(a,b)}{L(a,b)} - 1, \tag{2.46}$$

given in [7], we have the identity

$$2E\left(\frac{x}{2},\frac{y}{2}\right) - E(x,y) = e^{A(x,y)-\mathscr{L}(x,y)} - 1. \tag{2.47}$$

Putting $x \to x/2$, $y \to y/2$ in (2.45), and taking into account (2.47), we can write

$$2E(x,y) + e^{A(x,y)-\mathscr{L}(x,y)} - 1 < 4\mathscr{A}\left(\frac{x}{2},\frac{y}{2}\right). \tag{2.48}$$

This may be compared to (2.23).

(8) We consider now applications of the special Gini mean

$$S = S(a,b) = (a^a b^b)^{1/(a+b)} \tag{2.49}$$

(see [15]). Its attached mean (by the exp-log method)

$$\mathscr{S}(x,y) = \frac{xe^x + ye^y}{e^x + e^y} = \log S(e^x, e^y) \tag{2.50}$$

is a special case of

$$M_f(x,y) = \frac{xf(x) + yf(y)}{f(x) + f(y)} \tag{2.51}$$

which was defined in [18]. Using the inequality

$$\left(\frac{S}{A}\right)^2 < \left(\frac{I}{G}\right)^3 \tag{2.52}$$

from [15], we get

$$2\mathcal{S} - 2\mathcal{A} < 3E - 3A. \tag{2.53}$$

The inequalities

$$\frac{A^2}{I} < S < \frac{A^4}{I^3} < \frac{A^2}{G} \tag{2.54}$$

given in [15] imply that

$$2\mathcal{A} - E < \mathcal{S} < 4\mathcal{A} - 3E < 2\mathcal{A} - A. \tag{2.55}$$

These offer connections between the exponential means $E$ and $\mathcal{S}$.

Let now the mean

$$U = U(a,b) = \frac{1}{3}\sqrt{(2a+b)(a+2b)}. \tag{2.56}$$

In [12], it is proved that

$$G < \sqrt[4]{U^3G} < I < \frac{U^2}{A} < U < A. \tag{2.57}$$

By the exp-log method, we get

$$A < \frac{1}{4}(3\mathcal{U} + A) < E < 2\mathcal{U} - \mathcal{A} < \mathcal{U} < \mathcal{A}. \tag{2.58}$$

These relations offer a connection between the means $E$ and $\mathcal{U}$.

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