INTERACTION BETWEEN COEFFICIENT CONDITIONS AND SOLUTION CONDITIONS OF DIFFERENTIAL EQUATIONS IN THE UNIT DISK

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The influence of the normality of the coefficient $A(z)$ of the differential equation $f^{(k)} + A(z)f = 0$ on a solution f and also the influence of the normality of a solution f on $A(z)$ are investigated in the unit disk. In particular, an estimate of P. Lappan is used to determine restrictions on the growth of a meromorphic function $A(z)$ when a solution f is α-normal.

1. Introduction and statement of results

The concept of a normal function in the unit disk D was introduced by Noshiro [11]. He defined a function f to be *normal* in D if it is a meromorphic function in D for which the set of functions $f \circ S$ is a normal family in D where S ranges over the conformal mappings of D onto itself. He also characterized normal functions as those meromorphic functions in D for which

$$\sup_{z\in D}\left(1-|z|^2\right)f^{\#}(z) < \infty, \quad |z| < 1, \tag{1.1}$$

where $f^{\#}(z) = |f'(z)|/(1+|f(z)|^2)$ is the *spherical derivative* of f. In Pommerenke [12] and more recent studies of Heittokangas [8], Benbourenane [3], and Chen and Shon [5], investigations have been made for linear differential equations in the unit disk D regarding the interplay between the behavior of the equation's coefficients and that of its solutions.

In this paper we consider in D the equation

$$f^{(k)} + A(z)f = 0, \tag{1.2}$$

where k is a positive integer and A is a meromorphic function in D. We note that examples show that if A is a normal function in D, a solution f to (1.2) need not be a normal function. Conversely, if f is a normal function in D which is a solution to (1.2), then the

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coefficient function A need not be a normal function. However, while resulting functions in these two settings need not be normal, we can establish a measure of their closeness to being normal by considering the wider class of the so-called α-normal functions.

If α is a positive real number, then a meromorphic function f in D is termed *α-normal* provided

$$\sup_{z\in D}\left(1-|z|^2\right)^{\alpha} f^{\#}(z) < \infty. \tag{1.3}$$

We denote the set of such α-normal functions by $\mathcal{N}^{\alpha}$ and observe that $\mathcal{N}^{1}$ is the set of normal functions. A result of Heittokangas [8, Theorem 5.2] can be used to see that if A is an analytic coefficient in (1.2) and $f \not\equiv 0$ is a normal solution of (1.2), then A is in a set of analytic functions shown by Zhu [17, Proposition 7] to be a subset of $\mathcal{N}^{\alpha}$ for some $\alpha \geq 1$.

Our first theorem gives estimates restricting the growth of the coefficient A in (1.2) when f is an α-normal solution of (1.2) in D.

Theorem 1.1. *Let f be an α-normal solution of (1.2), where the coefficient A is a meromorphic function in D. Then there exist constants $C(f)$ and $P_k(f)$ such that*

(i)

$$\frac{|f|}{1+|f|^{k+2}}\left(1-|z|^2\right)^{\alpha(k+1)}\frac{|A'(z)|}{1+|A(z)|^2} \leq C(f), \tag{1.4}$$

(ii)

$$\frac{|f|}{1+|f|^{k+1}}\left(1-|z|^2\right)^{\alpha k}|A(z)| \leq P_k(f) \tag{1.5}$$

for all $z \in D$.

The estimates in Theorem 1.1 enable us to determine a specific β for which A is β-normal when the behavior of f is restricted further. Theorem 1.2 provides such results.

Theorem 1.2. *For $\alpha \geq 1$, suppose f is an α-normal function which is a solution of (1.2) where A is a meromorphic function in D.*

(i) If there exist constants ϵ and L such that $0 < \epsilon \leq |f(z)| \leq L$ for some r_0 with $r_0 < |z| < 1$, then A is $\alpha(k+1)$-normal in D.

(ii) If for each compact set $K \subset D$, there exists a constant $C(K)$ such that

$$\frac{1+|f\circ T|^{k+2}}{|f\circ T|} \leq C(K) < \infty, \tag{1.6}$$

for all conformal mappings T of D onto itself and all $z \in K$, then A is an $(1+\alpha(k+1))$-normal function in D.

(iii) If there exists a number $R > 0$ and a constant $M(R)$ such that

$$\frac{1+|f(z)|^{k+2}}{|f(z)|} \leq M(R) \tag{1.7}$$

for $\{z : |A(z)| < R\}$, then A is an $\alpha(k+1)$-normal function in D.

Remarks 1.3. (1) If A in part (i) of Theorem 1.2 is known to be an analytic function, then the second part of Theorem 1.1 combined with Zhu [17, Proposition 7] implies A is $(\alpha k+1)$-normal in D.

(2) Parts (ii) and (iii) follow from some characterizations of α-normal functions by Wulan [15].

The estimates involved in the proof of Theorem 1.1 lead to the following more generally applicable result.

THEOREM 1.4. *Let f be an α-normal meromorphic function which is a solution in D of (1.2) where A is a meromorphic function in D. Then*

(i) *for $0<r<1$ there exist constants C_1 and C_2 depending on f such that*

$$\int_0^r\int_0^{2\pi}\log^+\left((1-|z|^2)^{(k+1)\alpha}\frac{|A'(z)|}{1+|A(z)|^2}\right)d\theta\,dr\le C_1+C_2T(r,f),\tag{1.8}$$

where $T(r,f)$ is the Nevanlinna characteristic function of f at r,

(ii) *for $0<r<1$ there exist constants K_1 and K_2 depending on f such that*

$$\int_0^r\int_0^{2\pi}\log^+\left((1-|z|^2)^{k\alpha-1}|A(z)|\right)d\theta\,dr\le K_1+K_2T(r,f),\tag{1.9}$$

where $T(r,f)$ is the Nevanlinna characteristic function of f at r.

Lehto and Virtanen [10] showed that if f is a normal meromorphic function in D, then there is a constant K so

$$T(r,f)\le K\log\frac{1}{1-r},\quad 0<r<1.\tag{1.10}$$

In [14] Shea and Sons studied the class F of functions defined as meromorphic in D for which

$$\limsup_{r\to1^-}\frac{T(r,f)}{\log(1/(1-r))}=\alpha(f)<\infty.\tag{1.11}$$

It is shown that the derivative of a function in F is in F. Thus, if f is a normal meromorphic function which satisfies (1.2), further considerations from Nevanlinna theory show A is in F since

$$T(r,A)\le T(r,f^{(k)})+T(r,f)\le\widetilde{K}\log\frac{1}{1-r}\tag{1.12}$$

for $0<r<1$ and a constant $\widetilde{K}$.

If f is an α-normal meromorphic function in D with $\alpha>1$, a simple calculation using the Ahlfors-Shimizu characteristic shows $T(r,f)\le\widetilde{C}(1-r)^{-(2\alpha-2)}$, $(0<r<1)$, where $\widetilde{C}$ is a positive constant.

It follows that if f is normal or α-normal for $\alpha>1$, then the double integrals in Theorem 1.4 are $O(\log(1/(1-r)))$, $(r\to1)$, or $O((1-r)^{-(2\alpha-2)})$, $(r\to1)$, respectively.

If f is a normal function in D which satisfies (1.2) and is bounded, then the integrals in Theorem 1.4 when $\alpha = 1$ are bounded. Also, if f is a normal function in D which satisfies (1.2) and is of bounded characteristic, then the integrals in Theorem 1.4 when $\alpha = 1$ are bounded. These latter considerations may be compared with [8, Theorem 4.5] which states the following.

Theorem 1.5. *Let A be the analytic coefficient of (1.2) in D for which*

$$\iint_D (1-|z|)^{k-1}\,|A(re^{i\theta})|\,r\,dr\,d\theta < \infty. \tag{1.13}$$

Then any solution f of (1.2) is a function of bounded characteristic in D.

Using Nevanlinna's theory one can also see that for $k = 2$ and $A = (-2(5-6z+3z^2))/(1-z)^6$, the function f defined by $f(z) = \exp(1/(1-z)^2)$ satisfies (1.2) and is not in class F, so f is certainly not a normal function in D. This example provides the expectation that in contrast with the integrand of the double integral in Theorem 1.5, the integrand in Theorem 1.4 involves a logarithm. Additional considerations along the lines of this example may be found in Benbourenane [3], Benbourenane and Sons [4], and Heittokangas [8]. In [8] Theorems 3.1.4 and 4.3 give restrictions on the growth of a solution f of (1.2) when A is an analytic function for which $|A(z)| \le \alpha/(1-|z|)^\beta$ for z in D where $\alpha > 0$ and $\beta \ge 0$.

The remaining sections of this paper proceed as follows. Section 2 provides some examples which further illuminate the theorems. Section 3 contains the proof of Theorem 1.1 which relies on a generalization of a result of Lappan [9]. The proof of Theorem 1.4 is in Section 4. Finally, Section 5 gives a proof for Theorem 1.2, additional results, and some concluding discussion.

Earlier versions of Theorems 1.1 and 1.2 appeared in Fowler [6].

2. Examples

Some well-known examples of normal functions in D are (i) bounded analytic functions in D; (ii) analytic univalent functions in D; (iii) analytic functions in D which omit two values; (iv) meromorphic functions in D which omit three values; and (v) Bloch functions which are analytic functions f in D for which

$$\sup_{z\in D}(1-|z|^2)\,|f'(z)| < \infty. \tag{2.1}$$

(See Schiff [13] for discussions of these classes of functions.)

We relate some examples to (1.2).

Example 2.1. The analytic function f defined in D by

$$f(z) = 2(1-z)\exp\left(\frac{2+z}{1-z}\right) \tag{2.2}$$

satisfies the equation

$$f'(z) - \frac{2+z}{(1-z)^2} f(z) = 0. \tag{2.3}$$

The function $A(z) = -(2+z)/(1-z)^2$ satisfies $|A(z)| \geq 1/4$ in D and is thus a normal function, while f is not a normal function (cf. Hayman and Storvick [7]). So, even for $k = 1$ a normal coefficient in (1.2) need not lead to a normal function.

Example 2.2. Bagemihl and Seidel noted in [2] that the function f defined in D by

$$f(z) = \prod_{n=1}^{\infty} \left(\frac{z_n - z}{1 - z_n z} \right), \tag{2.4}$$

where $z_n = 1 - 1/n^2$ for $n = 1, 2, 3, \ldots$, is a normal function, but the function $A(z) = -f'(z)/f(z)$ is not a normal function in D (cf. Fowler [6, page 9]). Hence for $k = 1$, a normal solution to (1.2) need not imply the equation's coefficient is normal.

Examples for other positive integers k in (1.2) similar in character to those in Examples 2.1 and 2.2 can be found in Fowler [6].

Example 2.3. Let f be defined in D by

$$f(z) = \int_0^z \exp\left(\frac{t+1}{t-1} \right) dt + 2. \tag{2.5}$$

It is easy to see that $1 \leq |f(z)| \leq 3$ for $z \in D$, and thus f is a normal function in D. Thus by part (i) of Theorem 1.2, $A(z) = -f^{(k)}(z)/f(z)$ is $(k+1)$-normal in D. It also follows from part (iii) of Theorem 1.2 that A is $(k+1)$-normal in D, whereas part (ii) of Theorem 1.2 gives A to be $(k+2)$-normal in D.

No information regarding Example 2.2 above is a consequence of parts (i) and (ii) of Theorem 1.2, but part (iii) applies when $\alpha = 1$ to give $A(z)$ is 2-normal.

Details related to all of the above examples appear in Fowler [6].

3. Proof of Theorem 1.1

Our proof of Theorem 1.1 will use the following theorem which is a generalization of Lappan [9, Theorem 4].

Lemma 3.1. *Let $0 < \alpha < \infty$. If f is an α-normal meromorphic function in D, then for each positive integer n, there exists a positive constant $P_n(f)$ such that*

$$\frac{|f^{(n)}(z)|\,(1-|z|^2)^{\alpha n}}{1 + |f(z)|^{n+1}} \leq P_n(f) \tag{3.1}$$

for each $z \in D$.

Proof of the lemma. We proceed by induction on n.

For $n = 1$, the result is trivially true by the definition of an α-normal function.

So we suppose the lemma is true for $k < n$. Then by Xu [16, Lemma 2] there exists a constant $E_n(f,\alpha,1)$ such that

$$\left(1-|z|^2\right)^{\alpha n}\left|f^{(n)}(z)\right| \le E_n(f,\alpha,1) \tag{3.2}$$

for each $z \in D$ such that $|f(z)| \le 1$.

Now let $g(z) = 1/f(z)$. Then $|g(z)| < 1$ where $|f(z)| > 1$. Since $g^{\#}(z) = f^{\#}(z)$, g is also an α-normal function in D. We differentiate the equation $f(z)g(z) \equiv 1$ n times to get

$$f^{(n)}(z)g(z) = -\sum_{k=0}^{n-1}\binom{n}{k} f^{(k)}(z)g^{(n-k)}(z), \tag{3.3}$$

which when $|f(z)| > 1$ gives

$$\left|f^{(n)}(z)\right| \le \sum_{k=0}^{n-1}\binom{n}{k}\left|f(z)f^{(k)}(z)g^{(n-k)}(z)\right|, \tag{3.4}$$

and thus

$$\begin{aligned}
&\frac{\left|f^{(n)}(z)\right|\left(1-|z|^2\right)^{\alpha n}}{1+\left|f(z)\right|^{n+1}}\\
&\quad\le \sum_{k=0}^{n-1}\binom{n}{k}\frac{\left|f^{(k)}(z)\right|\left(1-|z|^2\right)^{\alpha k}}{1+\left|f(z)\right|^{k+1}}\left|g^{(n-k)}(z)\right|\left(1-|z|^2\right)^{\alpha(n-k)}\\
&\qquad\cdot\frac{\left|f(z)\right|+\left|f(z)\right|^{k+2}}{1+\left|f(z)\right|^{n+1}}\\
&\quad\le \sum_{k=0}^{n-1}\binom{n}{k}P_k(f)E_{n-k}(g,\alpha,1)\left(\frac{\left|f(z)\right|+\left|f(z)\right|^{k+2}}{1+\left|f(z)\right|^{n+1}}\right),
\end{aligned} \tag{3.5}$$

where $P_0(f) = 1$ and we further use Xu [16, Lemma 2]. It is easy to see that $(|f(z)| + |f(z)|^{k+2})/(1+|f(z)|^{n+1}) \le 2$ when $k < n$ and $|f(z)| > 1$. Hence, setting

$$P_n(f) = \max\left\{E_n(f,\alpha,1), \sum_{k=0}^{n-1}2\binom{n}{k}P_k(f)E_{n-k}(g,\alpha,1)\right\} \tag{3.6}$$

completes the proof of the lemma. □

Proof of Theorem 1.1. Equation (1.2) gives $f^{(k+1)} + Af' + A'f = 0$ for all $z \in D$, and thus

$$|A'f| = |A'||f| \le \left|f^{(k+1)}\right| + |A||f'|. \tag{3.7}$$

We then see for all $z \in D$,

$$\begin{aligned}
&\frac{|f|}{1+|f|^{k+2}}(1-|z|^2)^{\alpha(k+1)}\frac{|A'|}{1+|A|^2} \\
&\quad\leq \frac{(1-|z|^2)^{\alpha(k+1)}|f^{(k+1)}|}{1+|f|^{k+2}}\cdot\frac{1}{1+|A|^2} \\
&\qquad+\frac{(1-|z|^2)^{\alpha(k+1)}|f'|}{1+|f|^{k+2}}\cdot\frac{|A|}{1+|A|^2}.
\end{aligned}\tag{3.8}$$

Hence, using the lemma above, we get for all $z \in D$,

$$\begin{aligned}
&\frac{|f|}{1+|f|^{k+2}}(1-|z|^2)^{\alpha(k+1)}\frac{|A'|}{1+|A|^2} \\
&\quad\leq P_{k+1}(f)\frac{1}{1+|A|^2} \\
&\qquad+\frac{(1-|z|^2)^{\alpha}|f'|}{1+|f|^2}\cdot\frac{(1-|z|^2)^{\alpha k}(1+|f|^2)}{1+|f|^{k+2}}\cdot\frac{|A|}{1+|A|^2},
\end{aligned}\tag{3.9}$$

and further,

$$\frac{|f|}{1+|f|^{k+2}}(1-|z|^2)^{\alpha(k+1)}\frac{|A'|}{1+|A|^2}\leq P_{k+1}(f)+P_1(f)\cdot 2=C(f).\tag{3.10}$$

To see part (ii) we note that for all $z \in D$ (1.2) gives

$$\frac{|f^{(k)}(z)|(1-|z|^2)^{\alpha k}}{1+|f(z)|^{k+1}}=\frac{|A(z)||f(z)|(1-|z|^2)^{\alpha k}}{1+|f(z)|^{k+1}},\tag{3.11}$$

so the lemma shows

$$\frac{|A(z)||f(z)|(1-|z|^2)^{\alpha k}}{1+|f(z)|^{k+1}}\leq P_k(f),\tag{3.12}$$

for all $z \in D$. □

4. Proof of Theorem 1.4

Since f is an α-normal meromorphic function satisfying (1.2), we have $f^{(k+1)}+Af'+A'f=0$ for all $z \in D$ and thus

$$\begin{aligned}
&(1-|z|^2)^{(k+1)\alpha}\frac{|A'(z)|}{1+|A(z)|^2}|f(z)| \\
&\quad\leq\frac{(1-|z|^2)^{(k+1)\alpha}|f^{(k+1)}(z)|}{1+|A(z)|^2}+(1-|z|^2)^{(k+1)\alpha}\frac{|A(z)|}{1+|A(z)|^2}|f'(z)|,
\end{aligned}\tag{4.1}$$

for all $z \in D$. The lemma in Section 3 implies for $z \in D$,

$$\begin{aligned}(1-|z|^2)^{(k+1)\alpha}&\frac{|A'(z)|}{1+|A(z)|^2}|f(z)| \\ &\le P_{k+1}(f)(1+|f(z)|^{k+2})+(1-|z|^2)^{k\alpha}P_1(f)(1+|f(z)|^2).\end{aligned} \tag{4.2}$$

Using properties of $\log^+$, we then see for $z = re^{i\theta}$ in D

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}&\log^+\left((1-|z|^2)^{(k+1)\alpha}\frac{|A'(z)|}{1+|A(z)|^2}|f(z)|\right)d\theta \\ &\le K_1+K_2\left(\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(z)|\,d\theta\right),\end{aligned} \tag{4.3}$$

where K_1 and K_2 are positive constants.

If f has no zeros or poles on $|z| = r$, we observe that

$$\begin{aligned}\frac{1}{2\pi}&\int_0^{2\pi}\log^+\left((1-|z|^2)^{(k+1)\alpha}\frac{|A'|}{1+|A|^2}\right)d\theta \\ &\le \frac{1}{2\pi}\int_0^{2\pi}\log^+\left((1-|z|^2)^{(k+1)\alpha}\frac{|A'|}{1+|A|^2}|f|\right)d\theta+\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|f|}d\theta.\end{aligned} \tag{4.4}$$

The first fundamental theorem of Nevanlinna's theory gives

$$\frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|f(re^{i\theta})|}d\theta \le T\left(r,\frac{1}{f}\right) \le T(r,f)+K, \tag{4.5}$$

where K is a constant. Combining this fact with (4.3) and (4.4) gives part (i) of the theorem upon integration.

For part (ii) we observe that for $z \in D$,

$$(1-|z|^2)^{k\alpha-1}|A(z)f(z)| = (1-|z|^2)^{k\alpha-1}|f^{(k)}(z)|. \tag{4.6}$$

Then using the lemma in Section 3 we have for $z \in D$,

$$(1-|z|^2)^{k\alpha-1}|A(z)f(z)| \le (1-|z|^2)^{-1}P_k(f)(1+|f(z)|^{k+1}). \tag{4.7}$$

Properties of $\log^+$ give for $z \in D$

$$\log^+((1-|z|^2)^{k\alpha-1}|A(z)||f(z)|) \le C_1+C_2\log^+|f(z)|+C_3\log^+\frac{1}{1-|z|^2} \tag{4.8}$$

for some positive constants C_1, C_2, and C_3. Hence

$$\int_0^r\left(\frac{1}{2\pi}\int_0^{2\pi}\log^+((1-|z|^2)^{k\alpha-1}|A(z)||f(z)|)\,d\theta\right)dr \le C_4+C_5T(r,f). \tag{4.9}$$

We note as in part (i) that if f has no zeros or poles on $|z| = r$, then

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \log^+\left((1-|z|^2)^{k\alpha-1}|A(z)|\right)d\theta \\ &\quad\le \frac{1}{2\pi}\int_0^{2\pi} \log^+\left((1-|z|^2)^{k\alpha-1}|A(z)||f(z)|\right)d\theta + \frac{1}{2\pi}\int_0^{2\pi}\log^+\frac{1}{|f(z)|}d\theta. \end{aligned} \tag{4.10}$$

Using the first fundamental theorem of Nevanlinna's theory and combining (4.9) and (4.10) upon integration, we get part (ii) of Theorem 1.4.

5. Proof of Theorem 1.2 and discussion

The proof of part (ii) of Theorem 1.2 is based on the following theorem of H. Wulan.

Theorem 5.1 [15, Theorem 4.2.1]. *Let $0 < \alpha < \infty$, and let f be a meromorphic function in D. Then $f \in \mathcal{N}^\alpha$ if and only if for each compact subset K of D there exists a finite constant $C(K)$ such that*

$$(1-|T(z)|^2)^{\alpha-1}(f\circ T)^\#(z) \le C(K) \tag{5.1}$$

for all conformal mappings T of D onto itself and all $z \in K$.

Proof of part (ii) of Theorem 1.2. First note that any conformal mapping T of D onto itself can be written

$$T(z) = e^{i\theta}\frac{a+z}{1+\overline{a}z}, \tag{5.2}$$

where $a \in D$ and θ is a real number.

Suppose $z \in K$, for some compact subset $K \subset D$. If $a \in D$, $\theta \in \mathbb{R}$, and $0 < \alpha < \infty$, we have by Theorem 1.1 that

$$\begin{aligned} &(1-|T(z)|^2)^{\alpha(k+1)}(A\circ T)^\#(z) \\ &\quad= \left(1-\left|e^{i\theta}\frac{a+z}{1+\overline{a}z}\right|^2\right)^{\alpha(k+1)} A^\#\left(e^{i\theta}\frac{a+z}{1+\overline{a}z}\right)\frac{1-|a|^2}{|1+\overline{a}z|^2} \\ &\quad\le C(f)\frac{1+|f(e^{i\theta}((a+z)/(1+\overline{a}z)))|^{k+2}}{|f(e^{i\theta}((a+z)/(1+\overline{a}z)))|}\cdot\frac{1-|a|^2}{|1+\overline{a}z|^2} \\ &\quad\le C(f)C(K). \end{aligned} \tag{5.3}$$

Thus by Theorem 5.1 above, A is $(1+\alpha(k+1))$-normal in D. □

The proof of part (iii) of Theorem 1.2 uses the theorem of H. Wulan noted below.

Theorem 5.2 [15, Theorem 4.5.1]. *Let $1 \le \alpha < \infty$. Then a meromorphic function $f \in \mathcal{N}^\alpha$ if and only if there exists a number $R > 0$ and a constant $M(R)$ such that*

$$\sup\{|f'(z)|(1-|z|^2)^\alpha : |f(z)| < R\} < M(R). \tag{5.4}$$

Proof of part (iii) of Theorem 1.2. By Theorem 1.1 we have

$$|A'(z)|(1-|z|^2)^{\alpha(k+1)} \le C(f)\frac{1+|f(z)|^{k+2}}{|f(z)|}(1+|A(z)|^2), \tag{5.5}$$

which for $\{z : |A(z)| < R\}$ gives

$$|A'(z)|(1-|z|^2)^{\alpha(k+1)} < C(f)M(R)(1+R^2). \tag{5.6}$$

Thus by Theorem 5.2, A is $\alpha(k+1)$-normal in D. □

Other theorems of Wulan in [15] when combined with Theorem 1.1 provide information on A when f is an α-normal meromorphic solution of (1.2). In particular we observe that Corollary 4.3.1 and Lemma 4.2 each provide information on A in this setting. Also in [1] Aulaskari and Lappan prove an integral criterion for normal functions which combined with Theorem 1.1 provides information on the coefficient A when f is a normal solution of (1.2). For further discussion of these connections see Fowler [6, pages 37–39].

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