A NONEXISTENCE RESULT FOR A NONLINEAR PDE WITH ROBIN CONDITION

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Received 16 May 2005; Revised 24 April 2006; Accepted 4 May 2006

Under the assumption $\lambda > 0$ and f verifying $f(x,y,0) = 0$ in D, $2F(x,y,u) - uf(x,y,u) \geq 0$, $u \neq 0$, and if $\Omega = R \times D$, we show the convexity of function $E(t) = \iint_D |u(t,x,y)|^2 dx\,dy$, where $u : \Omega \to \mathbb{R}$ is a solution of problem $\lambda(\partial^2 u/\partial t^2) - (\partial/\partial x)(p(x,y)(\partial u/\partial x)) - (\partial/\partial y)(q(x,y)(\partial u/\partial y)) + f(x,y,u) = 0$ in Ω, $u + \varepsilon(\partial u/\partial n) = 0$ on $\partial\Omega$, considered in $H^2(\Omega) \cap L^\infty(\Omega)$, $p, q : \overline{D} \to \mathbb{R}$ are two nonnull functions on D, ε is a positive real number, and $D =]a_1, b_1[\times]a_2, b_2[$, $(F(x,y,s) = \int_0^s f(x,y,t)dt)$.

1. Introduction

In this paper we consider the question of absence of nontrivial solutions of the following boundary value problem:

$$\begin{aligned} \lambda \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,u) = 0 \quad \text{in } \Omega, \\ u + \varepsilon \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{P}$$

where

$$p, q : \overline{D} \longrightarrow \mathbb{R}, \tag{1.1}$$

are two continuous functions > 0 or < 0 in D, ε is a positive real number, and f is a locally Lipschitz continuous function

$$f : D \times \mathbb{R} \longrightarrow \mathbb{R} \tag{1.2}$$

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 62601, Pages 1–12
DOI 10.1155/IJMMS/2006/62601

such that

$$f(x,y,0)=0 \quad \text{in } D, \tag{1.3}$$

so that

$$u\equiv 0 \tag{1.4}$$

is a solution of problem (P).

This problem is considered in the Sobolev space

$$H^2(\Omega)\cap L^\infty(\Omega), \tag{1.5}$$

with

$$\Omega=\mathbb{R}\times D, \quad D=\,]a_1,b_1[\,\times\,]a_2,b_2[. \tag{1.6}$$

This question has interested much researchers and a significant number of works, were carried out. We quote by the way of examples the works of De Figueiredo and Jianfu [2], Esteban and Lions [3], Pucci and Serrin [6], Pohožaev [5], and Van der Vorst [7].

Esteban and Lions show that the Dirichlet problem

$$\begin{gathered} -\Delta u+f(u)=0, \quad u\in C^2(\overline{\Omega}),\\ u=0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.7}$$

satisfying

$$\begin{gathered} \nabla u\in L^2(\Omega), \qquad f(0)=0,\\ F(u)=\int_0^u f(s)ds\in L^1(\Omega), \end{gathered} \tag{1.8}$$

where Ω is a connected unbounded domain of $\mathbb{R}^N$ such that

$$\exists \Lambda\in\mathbb{R}^N, \quad \|\Lambda\|=1, \quad \langle n(x),\Lambda\rangle\geq 0 \quad \text{on } \partial\Omega, \quad \langle n(x),\Lambda\rangle\neq 0, \tag{1.9}$$

does not have nontrivial solution.

Berestycky et al. [1] established that the problem

$$-\Delta u-u^3+u=0, \quad u\in H^2(\mathbb{R}^2), \tag{1.10}$$

admits a radial solution. This solution satisfies

$$\begin{aligned} -\Delta u - u^3 + u &= 0 \quad \text{in } \Omega =]0,+\infty[\times \mathbb{R}, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \{0\} \times \mathbb{R}. \end{aligned} \tag{1.11}$$

This shows that the analog of the Dirichlet problem for the Neumann problem is not true.

The problem

$$\begin{aligned} -\Delta u + u(u+1)(u+2) &= 0 \quad \text{in } \Omega = \mathbb{R}\times]0,\alpha[, \\ \frac{\partial u}{\partial n} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.12}$$

considered in the Sobolev space

$$H^2(\Omega) \cap L^\infty(\Omega), \tag{1.13}$$

is still open in

$$\Omega = \mathbb{R} \times]0,a[(a < \pi). \tag{1.14}$$

These considerations motivated us to explore more this question of the absence of nontrivial solutions which presents a double interest. It makes it possible to solve an open problem, and to ensure then that the single solution is the null solution.

The aim of this work is to extend the results of [4] to problem (P). We prove in Section 2 a Pohožaev-type identity.

In Section 3, we combine Theorem 2.1 with other results to obtain for the semilinear elliptic problems a corollary of nonexistence of solutions. For the semilinear hyperbolic problems, we obtain in the same section an interesting result which shows that the Klein-Gordon align does not have nontrivial solutions.

Finally in Section 4, we give some examples to illustrate Theorems 3.1 and 3.5.

Let us denote by

$$\Gamma = \partial\Omega = \mathbb{R} \times \partial D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \tag{1.15}$$

where

$$\begin{aligned} \Gamma_1 &= \{(t,a_1,y),\ t \in \mathbb{R}, a_2 < y < b_2\}, \\ \Gamma_2 &= \{(t,b_1,y),\ t \in \mathbb{R}, a_2 < y < b_2\}, \\ \Gamma_3 &= \{(t,x,a_2),\ t \in \mathbb{R}, a_1 < x < b_1\}, \\ \Gamma_4 &= \{(t,x,b_2),\ t \in \mathbb{R}, a_1 < x < b_1\}, \end{aligned} \tag{1.16}$$

the boundary of Ω, (t,x,y), the generic point of Ω,

$$n(t,\xi_1,\xi_2) = (n_1(t,\xi_1,\xi_2),\ n_2(t,\xi_1,\xi_2),\ n_3(t,\xi_1,\xi_2)), \tag{1.17}$$

the outward normal to $\partial\Omega$ at the point (t,ξ_1,ξ_2), $\partial^2 u(t,x,y)/\partial s^2$, the second derivative of u with respect to s at point (t,x,y), and

$$\begin{aligned}
\frac{\partial u}{\partial s}_{|l_1} &= \lim_{x\to a_1}\frac{\partial u}{\partial s}(t,x,b_2) = \lim_{y\to b_2}\frac{\partial u}{\partial s}(t,a_1,y),\\
\frac{\partial u}{\partial s}_{|l_2} &= \lim_{x\to a_1}\frac{\partial u}{\partial s}(t,x,a_2) = \lim_{y\to a_2}\frac{\partial u}{\partial s}(t,a_1,y),\\
\frac{\partial u}{\partial s}_{|l_3} &= \lim_{x\to b_1}\frac{\partial u}{\partial s}(t,x,b_2) = \lim_{y\to b_2}\frac{\partial u}{\partial s}(t,b_1,y),\\
\frac{\partial u}{\partial s}_{|l_4} &= \lim_{x\to b_1}\frac{\partial u}{\partial s}(t,x,a_2) = \lim_{y\to a_2}\frac{\partial u}{\partial s}(t,b_1,y),
\end{aligned} \tag{1.18}$$

where

$$\begin{aligned}
l_1 &= \Gamma_1\cap\Gamma_4 = \{(t,a_1,b_2),\ t\in\mathbb{R}\},\\
l_2 &= \Gamma_1\cap\Gamma_3 = \{(t,a_1,a_2),\ t\in\mathbb{R}\},\\
l_3 &= \Gamma_2\cap\Gamma_4 = \{(t,b_1,b_2),\ t\in\mathbb{R}\},\\
l_4 &= \Gamma_2\cap\Gamma_3 = \{(t,b_1,a_2),\ t\in\mathbb{R}\}.
\end{aligned} \tag{1.19}$$

2. Integral identity

We now give an integral identity in the form of theorem.

Theorem 2.1. *Let u be an element of $H^2(\Omega)\cap L^\infty(\Omega)$, a solution of problem (P), then for each $t\in\mathbb{R}$ and $\varepsilon>0$,*

$$\begin{aligned}
&\iint_D\left[\frac{\lambda}{2}\left|\frac{\partial u}{\partial t}\right|^2+\frac{p(x,y)}{2}\left|\frac{\partial u}{\partial x}\right|^2+\frac{q(x,y)}{2}\left|\frac{\partial u}{\partial y}\right|^2+F(x,y,u)\right]dx\,dy\\
&\quad+\frac{1}{2\varepsilon}\left(\int_{a_1}^{b_1}\left[p(x,a_2)\,|\,u(t,x,a_2)\,|^2+q(x,b_2)\,|\,u(t,x,b_2)\,|^2\right]dx\right.\\
&\qquad\left.+\int_{a_2}^{b_2}\left[p(a_1,y)\,|\,u(t,a_1,y)\,|^2+q(b_1,y)\,|\,u(t,b_1,y)\,|^2\right]dy\right)=0.
\end{aligned} \tag{2.1}$$

Proof. For $t\in\mathbb{R}$, we consider a function K defined by

$$K(t)=\iint_D\left[\frac{\lambda}{2}\left|\frac{\partial u}{\partial t}\right|^2+\frac{p(x,y)}{2}\left|\frac{\partial u}{\partial x}\right|^2+\frac{q(x,y)}{2}\left|\frac{\partial u}{\partial y}\right|^2+F(x,y,u)\right]dx\,dy. \tag{2.2}$$

The hypotheses on u, p, q, and f imply that K is absolutely continuous and thus differentiable almost everywhere on $\mathbb{R}$; we have

$$K'(t)=\iint_D\left[\lambda\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2}+p(x,y)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}+q(x,y)\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}+\frac{\partial u}{\partial t}f(x,y,u)\right]dx\,dy. \tag{2.3}$$

Fubini's theorem and an integration by part make it possible to write

$$\begin{aligned}
&\iint_D p(x,y)\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}\right)(t,x,y)dx\,dy\\
&\quad=\int_{a_1}^{b_1}\int_{a_2}^{b_2} p(x,y)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}(t,x,y)dx\,dy\\
&\quad=\int_{a_2}^{b_2}\left[\int_{a_1}^{b_1} p(x,y)\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x\partial t}\right)(t,x,y)dx\right]dy\\
&\quad=\int_{a_1}^{b_1}\int_{a_2}^{b_2}-\frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t}(t,x,y)dx\,dy\\
&\qquad+\int_{a_2}^{b_2}\left[p(b_1,y)\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right)(t,b_1,y)-p(a_1,y)\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right)(t,a_1,y)\right]dy.
\end{aligned}\tag{2.4}$$

In the same way, we find

$$\begin{aligned}
&\iint_D q(x,y)\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y\partial t}(t,x,y)dx\,dy\\
&\quad=\int_{a_1}^{b_1}\int_{a_2}^{b_2}-\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right)\frac{\partial u}{\partial t}(t,x,y)dx\,dy\\
&\qquad+\int_{a_1}^{b_1}\left[q(b_1,y)\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right)(t,b_1,y)-q(a_1,y)\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right)(t,a_1,y)\right]dx.
\end{aligned}\tag{2.5}$$

Replacing in (2.3), we find

$$\begin{aligned}
K'(t)=&\iint_D\left[\lambda\frac{\partial^2 u}{\partial t^2}-\frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right)-\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right)+f(x,y,u)\right]\frac{\partial u}{\partial t}dx\,dy\\
&+\int_{a_1}^{b_1}\left[q(x,b_2)\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right)(t,x,b_2)-q(x,a_2)\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right)(t,x,a_2)\right]dx\\
&+\int_{a_2}^{b_2}\left[p(b_1,y)\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right)(t,b_1,y)-p(a_1,y)\left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\right)(t,a_1,y)\right]dy.
\end{aligned}\tag{2.6}$$

Let us write on $\partial\Omega$ the expression $u+\varepsilon\partial u/\partial n=0$ in an equivalent way:

$$\begin{aligned}
u+\varepsilon\frac{\partial u}{\partial n}=0\Longleftrightarrow\ & u-\varepsilon\frac{\partial u}{\partial x}=0\quad\text{on }\{(t,a_1,y),\ t\in\mathbb{R},\ a_2<y<b_2\},\\
& u+\varepsilon\frac{\partial u}{\partial x}=0\quad\text{on }\{(t,b_1,y),\ t\in\mathbb{R},\ a_2<y<b_2\},\\
& u-\varepsilon\frac{\partial u}{\partial y}=0\quad\text{on }\{(t,x,a_2),\ t\in\mathbb{R},\ a_1<x<b_1\},\\
& u+\varepsilon\frac{\partial u}{\partial y}=0\quad\text{on }(t,x,b_2),\ t\in\mathbb{R},\ a_1<x<b_1.
\end{aligned}\tag{2.7}$$

For $0 < \varepsilon < +\infty$, we can write

$$\begin{aligned}
&\int_{a_1}^{b_1} \left[q(x,b_2) \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \right)(t,x,b_2) - q(x,a_2) \frac{\partial u}{\partial x} \frac{\partial u}{\partial t}(t,x,a_2) \right] dx \\
&\quad = \frac{1}{\varepsilon} \left[\int_{a_1}^{b_1} \left[q(x,b_2) \left(u \frac{\partial u}{\partial t} \right)(t,x,b_2) - q(x,a_2) \left(u \frac{\partial u}{\partial t} \right)(t,x,a_2) \right] dx \right] \\
&\quad = -\frac{1}{2\varepsilon} \frac{d}{dt} \left(\int_{a_1}^{b_1} \left[q(x,b_2) \,|\, u(t,x,b_2) \,|^2 + q(x,a_2) \,|\, u(t,x,a_2) \,|^2 \right] dx \right)
\end{aligned} \tag{2.8}$$

and in the same manner

$$\begin{aligned}
&\int_{a_2}^{b_2} \left[p(b_1,y) \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \right)(t,b_1,y) - p(a_1,y) \left(\frac{\partial u}{\partial x} \frac{\partial u}{\partial t} \right)(t,a_1,y) \right] dy \\
&\quad = -\frac{1}{2\varepsilon} \frac{d}{dt} \left(\int_{a_2}^{b_2} \left[p(b_1,y) \,|\, u(t,b_1,y) \,|^2 + p(a_1,y) \,|\, u(t,a_1,y) \,|^2 \right] dy \right);
\end{aligned} \tag{2.9}$$

K', derivative of K, verifies therefore

$$\begin{aligned}
&\frac{d}{dt} \left(K(t) + \frac{1}{2\varepsilon} \left(\int_{a_1}^{b_1} [q(x,b_2) u^2(t,x,b_2) + q(x,a_2) u^2(t,x,a_2)] dx \right. \right. \\
&\qquad \left. \left. + \int_{a_2}^{b_2} \left[p(b_1,y) \,|\, u(t,b_1,y) \,|^2 + p(a_1,y) \,|\, u(t,a_1,y) \,|^2 \right] dy \right) \right) = 0.
\end{aligned} \tag{2.10}$$

Integrating with respect to t, we obtain

$$\begin{aligned}
&K(t) + \frac{1}{2\varepsilon} \left(\int_{a_1}^{b_1} [q(x,b_2) u^2(t,x,b_2) + q(x,a_2) u^2(t,x,a_2)] dx \right. \\
&\qquad \left. + \int_{a_2}^{b_2} \left[p(b_1,y) \,|\, u(t,b_1,y) \,|^2 + p(a_1,y) \,|\, u(t,a_1,y) \,|^2 \right] dy \right) = \text{const}
\end{aligned} \tag{2.11}$$

and by noticing that

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \left[K(t) + \frac{1}{2\varepsilon} \left(\int_{a_1}^{b_1} [q(x,b_2) u^2(t,x,b_2) + q(x,a_2) u^2(t,x,a_2)] dx \right. \right. \\
&\qquad \left. \left. + \int_{a_2}^{b_2} \left[p(b_1,y) \,|\, u(t,b_1,y) \,|^2 + p(a_1,y) \,|\, u(t,a_1,y) \,|^2 \right] dy \right) \right] dt < \infty.
\end{aligned} \tag{2.12}$$

The constant is null; this shows the theorem. □

3. Main results

The parameter λ plays in fact an important part as it allows problem (P) to be dealt with in two manners according to whether its value is positive or negative.

3.1. Semilinear hyperbolic problems. By employing identity (2.1) we obtain the following nonexistence result.

Theorem 3.1. *Let,* $\lambda > 0$,

$$p(x,y), q(x,y) > 0 \quad \text{in } D, \tag{3.1}$$

and f *satisfying*

$$F(x,y,u) \geq 0, \quad u \neq 0. \tag{A}$$

Then problem (P) considered in $H^2(\Omega) \cap L^\infty(\Omega)$ *does not have nontrivial solutions.*

Proof. Applying formula (2.1) we immediately obtain

$$\frac{\partial u}{\partial t}(t,x,y) = \frac{\partial u}{\partial x}(t,x,y) = \frac{\partial u}{\partial y}(t,x,y) = 0 \quad \text{in } \Omega. \tag{3.2}$$

□

The following theorem gives a nonexistence result if f satisfies another type of nonlinearity.

Theorem 3.2. *Let,* $\lambda > 0$,

$$u \in H^2(\Omega) \cap L^\infty(\Omega) \tag{3.3}$$

be a solution of problem (P),

$$p(x,y), q(x,y) > 0 \quad \text{or} \quad < 0 \quad \text{in } D, \tag{3.4}$$

and f *a function verifying*

$$2F(x,y,u) - uf(x,y,u) \geq 0, \quad u \neq 0. \tag{B}$$

Then the function

$$E(t) = \iint_D |u(t,x,y)|^2 dx\,dy \tag{3.5}$$

is convex on $\mathbb{R}$.

Remark 3.3. The convexity of the function $E(t)$ on $\mathbb{R}$ implies the triviality of the solution $u(t,x,y)$ of the problem (P).

Proof. It is easy to see that almost everywhere in Ω, we have

$$\left(u\frac{\partial^2 u}{\partial t^2}\right)(t,x,y) = \left(\frac{1}{2}\frac{\partial^2(u^2)}{\partial t^2} - \left|\frac{\partial u}{\partial t}\right|^2\right)(t,x,y). \tag{3.6}$$

Let us multiply the first equation of (P) by $u/2$ and integrate the new align over D, we obtain

$$\begin{aligned}
&\iint_D \left[\lambda\frac{\partial^2 u}{\partial t^2}\frac{u}{2} - \frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right)\frac{u}{2} - \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right)\frac{u}{2} + \frac{1}{2}uf(x,y,u)\right](t,x,y)\,dx\,dy \\
&\quad = \iint_D \left[\frac{\lambda}{4}\frac{\partial^2(u^2)}{\partial t^2} - \frac{\lambda}{2}\left|\frac{\partial u}{\partial t}\right|^2\right](t,x,y)\,dx\,dy \\
&\qquad + \iint_D \left[-\frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right)\frac{u}{2} - \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right)\frac{u}{2} + \frac{1}{2}uf(x,y,u)\right](t,x,y)\,dx\,dy.
\end{aligned} \tag{3.7}$$

Let us transform

$$\begin{aligned}
&\iint_D \left[\frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right)\frac{u}{2}\right](t,x,y)\,dx\,dy \\
&\quad = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right)\frac{u}{2}(t,x,y)\,dx\,dy \\
&\quad = \int_{a_2}^{b_2}\left[\int_{a_1}^{b_1} \frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right)\frac{u}{2}(t,x,y)\,dx\right]dy \\
&\quad = -\iint_D \frac{p(x,y)}{2}\left|\frac{\partial u(t,x,y)}{\partial x}\right|^2 dx\,dy \\
&\qquad + \int_{a_2}^{b_2}\left[p(b_1,y)\left(\frac{u}{2}\frac{\partial u}{\partial x}\right)(t,b_1,y) - p(a_1,y)\left(\frac{u}{2}\frac{\partial u}{\partial x}\right)(t,a_1,y)\right]dy.
\end{aligned} \tag{3.8}$$

Similary,

$$\begin{aligned}
&\iint_D \left[\frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right)\frac{u}{2}\right](t,x,y)\,dx\,dy \\
&\quad = -\iint_D \frac{q(x,y)}{2}\left|\frac{\partial u(t,x,y)}{\partial y}\right|^2 dx\,dy \\
&\qquad + \int_{a_1}^{b_1}\left[q(x,b_2)\left(\frac{u}{2}\frac{\partial u}{\partial y}\right)(t,x,b_2) - q(x,a_2)\left(\frac{u}{2}\frac{\partial u}{\partial y}\right)(t,x,a_2)\right]dx,
\end{aligned} \tag{3.9}$$

the substitution of these formulas in (3.7) gives

$$\begin{aligned}\iint_D \left[\frac{\lambda}{4}\frac{\partial^2(u^2)}{\partial t^2} - \frac{\lambda}{2}\left|\frac{\partial u}{\partial t}\right|^2 - \frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right)\frac{u}{2} - \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right)\frac{u}{2} + \frac{1}{2}uf(x,y,u)\right]dx\,dy \\ = -\frac{1}{2\varepsilon}\left(\int_{a_1}^{b_1}[p(x,a_2)\,|\,u(t,x,a_2)\,|^2 + p(x,b_2)\,|\,u(t,x,b_2)\,|^2]dx \right. \\ \left. + \int_{a_2}^{b_2}[q(a_1,y)\,|\,u(t,a_1,y)\,|^2 + q(b_1,y)\,|\,u(t,b_1,y)\,|^2]dy\right),\end{aligned} \tag{3.10}$$

which combined with (2.1) yields

$$\frac{\lambda}{4}\frac{d^2}{dt^2}\left(\iint_D |u(t,x,y)|^2 dx\,dy\right) = \iint_D \left[\lambda\left|\frac{\partial u}{\partial t}\right|^2 + F(x,y,u) - \frac{1}{2}uf(x,y,u)\right)(t,x,y]dx\,dy, \tag{3.11}$$

the assumption (B) and $\lambda > 0$ enable us to affirm that

$$\frac{\lambda}{4}\frac{d^2}{dt^2}\left(\iint_D |\,u(t,x,y)\,|^2 dx\,dy\right) \geq \lambda\iint_D \left|\frac{\partial u(t,x,y)}{\partial t}\right|^2 dx\,dy \geq 0. \tag{3.12}$$

This proves the theorem. □

Corollary 3.4. *The result of Theorems 2.1 and 3.1 are still true if the condition of Robin is replaced either by Dirichlet or Neumann condition.*

3.2. Semilinear elliptic problems. For the elliptic case, there is a nonexistence result which is stated as follows.

Theorem 3.5. *Let*

$$u \in H^2(\Omega) \cap L^\infty(\Omega) \tag{3.13}$$

be a solution of (P), $\lambda < 0$, and f satisfying

$$2F(x,y,u) - uf(x,y,u) \leq 0, \quad u \neq 0. \tag{C}$$

Then the function $E(t)$ defined in Theorem 3.2 is convex on $\mathbb{R}$.

Proof. The proof is similar to that of Theorem 3.2. □

4. Examples

In this paragraph we will discuss some examples to demonstrate the use of Theorems 3.1 and 3.2.

Example 4.1. Let

$$\theta : D \longrightarrow R \tag{4.1}$$

be a continuous function

$$\begin{gathered} p(x,y), q(x,y) > 0 \quad \text{or} \quad < 0 \quad \text{in } D, \\ f(x,y,u) = \theta(x,y)|u|^{\gamma-1}u. \end{gathered} \tag{4.2}$$

Then problem

$$\begin{gathered} \lambda \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,u) = 0 \quad \text{in } \Omega, \\ u + \varepsilon \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{P)$_1$}$$

does not have nontrivial solutions. In the following situations:

$$\begin{gathered} \lambda > 0, \\ p, q, \theta > 0 \quad \text{in } D, \end{gathered} \tag{4.3}$$

or

$$\begin{gathered} \lambda > 0, \\ p(x,y), q(x,y) > 0 \quad \text{or} \quad < 0, \qquad \theta \le 0 \quad \text{in } D, \end{gathered} \tag{4.4}$$

we can see that in the first case

$$F(x,y,u) = \frac{\theta(x,y)}{\gamma+1}|u|^{\gamma+1}, \tag{4.5}$$

and in the second case

$$F(x,y,u) - \frac{1}{2}uf(x,y,u) = \theta(x,y)\left(\frac{1}{\gamma+1} - \frac{1}{2}\right)|u|^{\gamma+1}. \tag{4.6}$$

Example 4.2. Let

$$p, q, \rho : D \longrightarrow \mathbb{R} \tag{4.7}$$

be continuous functions and

$$f(x,y,u) \equiv u. \tag{4.8}$$

The problem

$$\begin{gathered} \lambda \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + \rho(x,y)u = 0 \quad \text{in } \Omega, \\ u + \varepsilon \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{P)$_2$}$$

considered in $H^2(\Omega) \cap L^\infty(\Omega)$ does not have nontrivial solutions.

Example 4.3. Let $\lambda > 0$,

$$f(x,y,u) = mu - \theta_1(x,y)|u|^{r-1}u - \theta_2(x,y)|u|^{s-1}u, \quad u \in H^2(\Omega) \cap L^\infty(\Omega), \tag{4.9}$$

be a solution of

$$\lambda \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,u) = 0 \quad \text{in } \Omega, \qquad u + \varepsilon \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{P)$_3$}$$

where $m > 0$ is the mass of a particle, let

$$\theta_1, \theta_2 : D \longrightarrow \mathbb{R} \tag{4.10}$$

be two continuous nonnegative functions and r and s two numbers larger than one. The problem $(P)_3$ does not have nontrivial solutions. It suffices to remark that

$$F(x,y,u) - \frac{1}{2}uf(x,y,u) = \theta_1(x,y)\left(\frac{1}{2} - \frac{1}{r+1}\right)|u|^{r+1} + \theta_2(x,y)\left(\frac{1}{2} - \frac{1}{s+1}\right)|u|^{s+1}. \tag{4.11}$$

Theorem 3.2 gives the desired result.

Example 4.4. Let

$$\theta : D \longrightarrow \mathbb{R} \tag{4.12}$$

be a nonnegative continuous function,

$$p, q : D \longrightarrow \mathbb{R} \tag{4.13}$$

two continuous functions > 0 or < 0 in D,

$$f(x,y,u) = \theta(x,y)(\omega u + |u|^{\gamma-1}u), \tag{4.14}$$

where ω is a real parameter and $\gamma \geq 1$; the problem

$$-\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(p(x,y)\frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(q(x,y)\frac{\partial u}{\partial y}\right) + f(x,y,u) = 0 \quad \text{in } \Omega, \qquad u + \varepsilon \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{P)$_4$}$$

does not have nontrivial solutions in $H^2(\Omega) \cap L^\infty(\Omega)$.

Acknowledgment

The author would like to thank the referee for several helpful comments.

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