CONVERGENCE THEOREMS FOR I-NONEXPANSIVE MAPPING

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We establish the weak convergence of a sequence of Mann iterates of an I-nonexpansive map in a Banach space which satisfies Opial's condition.

1. Introduction and preliminaries

Let K be a closed convex bounded subset of uniformly convex Banach space $X = (X, \|\cdot\|)$ and T self-mappings of X. Then T is called nonexpansive on K if

$$\|Tx - Ty\| \leq \|x - y\| \tag{1.1}$$

for all $x, y \in K$. Let $F(T) = \{x \in K : Tx = x\}$ be denoted as the set of fixed points of a mapping T.

The first nonlinear ergodic theorem was proved by Baillon [1] for general nonexpansive mappings in Hilbert space $\mathcal{H}$: if K is a closed and convex subset of $\mathcal{H}$ and T has a fixed point, then for every $x \in K$, $\{T^n x\}$ is weakly almost convergent, as $n \to \infty$, to a fixed point of T. It was also shown by Pazy [7] that if $\mathcal{H}$ is a real Hilbert space and $(1/n)\sum_{i=0}^{n-1} T^i x$ converges weakly, as $n \to \infty$, to $y \in K$, then $y \in F(T)$.

The concept of a quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. Diaz and Metcalf [2] and Dotson [3] studied quasi-nonexpansive mappings in Banach spaces. Recently, this concept was given by Kirk [5] in metric spaces which we adapt to a normed space as follows: T is called a quasi-nonexpansive mapping provided

$$\|Tx - f\| \leq \|x - f\| \tag{1.2}$$

for all $x \in K$ and $f \in F(T)$.

Remark 1.1. From the above definitions it is easy to see that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and linear quasi-nonexpansive mappings are nonexpansive. But it is easily seen that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive.

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There are many results on fixed points on nonexpansive and quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, the strong and weak convergence of the sequence of certain iterates to a fixed point of quasi-nonexpansive maps was studied by Petryshyn and Williamson [8]. Their analysis was related to the convergence of Mann iterates studied by Dotson [3]. Subsequently, the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces was discussed by Ghosh and Debnath [4]. In [10], the weakly convergence theorem for I-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. In [11], convergence theorems of iterative schemes for nonexpansive mappings have been presented and generalized.

In this paper, we consider T and I self-mappings of K, where T is an I-nonexpansive mapping. We establish the weak convergence of the sequence of Mann iterates to a common fixed point of T and I.

Let X be a normed linear space, let K be a nonempty convex subset of X, and let $T : K \to K$ be a given mapping. The Mann iterative scheme $\{x_n\}$ is defined by $x_0 = x \in K$ and

$$x_{n+1} = (1-k_n)x_n + k_n T x_n \tag{1.3}$$

for every $n \in \mathbb{N}$, where k_n is a sequence in $(0,1)$.

Recall that a Banach space X is said to satisfy Opial's condition [6] if, for each sequence $\{x_n\}$ in X, the condition $x_n \rightharpoonup x$ implies that

$$\overline{\lim_{n\to\infty}} ||x_n - x|| < \overline{\lim_{n\to\infty}} ||x_n - y|| \tag{1.4}$$

for all $y \in X$ with $y \neq x$. It is well known from [6] that all l_p spaces for $1 < p < \infty$ have this property. However, the L_p spaces do not, unless $p = 2$.

The following definitions and statements will be needed for the proof of our theorem.

Let K be a subset of a normed space $X = (X, \|\cdot\|)$ and T and I self-mappings of K. Then T is called I-nonexpansive on K if

$$\|Tx - Ty\| \leq \|Ix - Iy\| \tag{1.5}$$

for all $x, y \in K$ [9].

T is called I-quasi-nonexpansive on K if

$$\|Tx - f\| \leq \|Ix - f\| \tag{1.6}$$

for all $x \in K$ and $f \in F(T) \cap F(I)$.

2. The main result

Theorem 2.1. *Let K be a closed convex bounded subset of uniformly convex Banach space X, which satisfies Opial's condition, and let T, I self-mappings of K with T be an I-nonexpansive mapping, I a nonexpansive on K. Then, for $x_0 \in K$, the sequence $\{x_n\}$ of Mann iterates converges weakly to common fixed point of $F(T) \cap F(I)$.*

Proof. If $F(T)\cap F(I)$ is nonempty and a singleton, then the proof is complete. We will assume that $F(T)\cap F(I)$ is nonempty and that $F(T)\cap F(I)$ is not a singleton.

$$\begin{aligned}
||x_{n+1}-f|| &= ||(1-k_n)x_n+k_nTx_n-(1-k_n+k_n)f|| \\
&= ||(1-k_n)(x_n-f)+k_n(Tx_n-f)|| \\
&\le (1-k_n)||x_n-f||+k_n||Tx_n-f|| \\
&\le (1-k_n)||x_n-f||+k_n||Ix_n-f|| \\
&\le (1-k_n)||x_n-f||+k_n||x_n-f|| \\
&= ||x_n-f||,
\end{aligned} \tag{2.1}$$

where $\{k_n\}$ is a sequence in $(0,1)$.

Thus, for $k_n\neq 0$, $\{\|x_n-f\|\}$ is a nonincreasing sequence. Then, $\lim_{n\to\infty}\|x_n-f\|$ exists.

Now we show that $\{x_n\}$ converges weakly to a common fixed point of T and I. The sequence $\{x_n\}$ contains a subsequence which converges weakly to a point in K. Let $\{x_{n_k}\}$ and $\{x_{m_k}\}$ be two subsequences of $\{x_n\}$ which converge weakly to f and q, respectively. We will show that $f=q$. Suppose that X satisfies Opial's condition and that $f\neq q$ is in weak limit set of the sequence $\{x_n\}$. Then $\{x_{n_k}\}\to f$ and $\{x_{m_k}\}\to q$, respectively. Since $\lim_{n\to\infty}\|x_n-f\|$ exists for any $f\in F(T)\cap F(I)$, by Opial's condition, we conclude that

$$\begin{aligned}
\lim_{n\to\infty}||x_n-f|| &= \lim_{k\to\infty}||x_{n_k}-f|| < \lim_{k\to\infty}||x_{n_k}-q|| \\
&= \lim_{n\to\infty}||x_n-q|| = \lim_{j\to\infty}||x_{m_j}-q|| \\
&< \lim_{j\to\infty}||x_{m_j}-f|| = \lim_{n\to\infty}||x_n-f||.
\end{aligned} \tag{2.2}$$

This is a contradiction. Thus $\{x_n\}$ converges weakly to an element of $F(T)\cap F(I)$. □

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