STRUCTURE OF RINGS WITH CERTAIN CONDITIONS ON ZERO DIVISORS

HAZAR ABU-KHUZAM AND ADIL YAQUB

Received 4 May 2004; Revised 17 September 2004; Accepted 24 July 2006

Let R be a ring such that every zero divisor x is expressible as a sum of a nilpotent element and a potent element of $R: x = a + b$, where a is nilpotent, b is potent, and $ab = ba$. We call such a ring a D^*-ring. We give the structure of periodic D^*-ring, weakly periodic D^*-ring, Artinian D^*-ring, semiperfect D^*-ring, and other classes of D^*-ring.

1. Introduction

Throughout this paper, R is an associative ring; and N, C, $C(R)$, and J denote, respectively, the set of nilpotent elements, the center, the commutator ideal, and the Jacobson radical. An element x of R is called *potent* if $x^n = x$ for some positive integer $n = n(x) > 1$. A ring R is called *periodic* if for every x in R, $x^m = x^n$ for some distinct positive integers $m = m(x)$, $n = n(x)$. A ring R is called *weakly periodic* if every element of R is expressible as a sum of a nilpotent element and a potent element of $R: R = N + P$, where P is the set of potent elements of R. A ring R such that every zero divisor is nilpotent is called a *D-ring*. The structure of certain classes of D-rings was studied in [1]. Following [7], R is called *normal* if all of its idempotents are in C. A ring R is called a *D^*-ring*, if every zero divisor x in R can be written as $x = a + b$, where $a \in N$, $b \in P$, and $ab = ba$. Clearly every D-ring is a D^*-ring. In particular every nil ring is a D^*-ring, and every domain is a D^*-ring. A Boolean ring is a D^*-ring but not a D-ring. Our objective is to study the structure of certain classes of D^*-ring.

2. Main results

We start by stating the following known lemmas: Lemmas 2.1 and 2.2 were proved in [5], Lemmas 2.3 and 2.4 were proved in [4].

Lemma 2.1. *Let R be a weakly periodic ring. Then the Jacobson radical J of R is nil. If, furthermore, $xR \subseteq N$ for all $x \in N$, then $N = J$ and R is periodic.*

Lemma 2.2. *If R is a weakly periodic division ring, then R is a field.*

Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2006, Article ID 67692, Pages 1–6
DOI 10.1155/IJMMS/2006/67692

Lemma 2.3. *Let R be a periodic ring and x any element of R. Then*

(a) *some power of x is idempotent;*

(b) *there exists an integer $n > 1$ such that $x - x^n \in N$.*

Lemma 2.4. *Let R be a periodic ring and let $\sigma : R \to S$ be a homomorphism of R onto a ring S. Then the nilpotents of S coincide with $\sigma(N)$, where N is the set of nilpotents of R.*

Definition 2.5. A ring is said to be a *D-ring* if every zero divisor is nilpotent. A ring R is called a *D^*-ring* if every zero divisor x in R can be written as $x = a + b$, where $a \in N$, $b \in P$, and $ab = ba$.

Theorem 2.6. *A ring R is a D^*-ring if and only if every zero divisor of R is periodic.*

Proof. Assume R is a D^*-ring and let x be any zero divisor. Then

$$x = a + b, \quad a \in N,\ b \in P,\ ab = ba. \tag{2.1}$$

So, $(x - a) = b = b^n = (x - a)^n$. This implies, since x commutes with a, that $(x - a) = (x - a)^n = x^n +$ sum of pairwise commuting nilpotent elements.

Hence

$$x - x^n \in N \quad \text{for every zero divisor } x. \tag{2.2}$$

Since each such x is included in a subring of zero divisors, which is periodic by Chacron's theorem, x is periodic.

Suppose, conversely, that each zero divisor is periodic. Then by the proof of [4, Lemma 1], R is a D^*-ring. □

Theorem 2.7. *If R is any normal D^*-ring, then either R is periodic or R is a D-ring. Moreover, $aR \subseteq N$ for each $a \in N$.*

Proof. If R is a normal D^*-ring which is not a D-ring, then R has a central idempotent zero divisor e. Then $R = eR \oplus A(e)$, where eR and $A(e)$ both consist of zero divisors of R, hence (in view of Theorem 2.6) are periodic. Therefore R is periodic.

Now consider $a \in N$ and $x \in R$. Since ax is a zero divisor, hence a periodic element, $(ax)^j = e$ is a central idempotent for some j. Thus $(ax)^{j+1} = (ax)^j ax = a^2 y$ for some $y \in R$. Repeating this argument, one can show that for each positive integer k, there exists m such that $(ax)^m = a^{2^k} w$ for some $w \in R$. Therefore $aR \subseteq N$. □

Corollary 2.8. *Let R be a D^*-ring which is not a D-ring. If $N \subseteq C$, then R is commutative.*

Proof. Since $N \subseteq C$, R is normal. Therefore commutativity follows from Theorem 2.7 and a theorem of Herstein. □

Now, we prove the following result for D^*-rings.

Theorem 2.9. *Let R be a normal D^*-ring.*

(i) *If R is weakly periodic, then N is an ideal of R, R is periodic, and R is a subdirect sum of nil rings and/or local rings R_i. Furthermore, if N_i is the set of nilpotents of the local ring R_i, then R_i/N_i is a periodic field.*

(ii) *If R is Artinian, then N is an ideal and R/N is a finite direct product of division rings.*

Proof. (i) Using Theorem 2.7, we have

$$aR \subseteq N \quad \text{for every } a \in N. \tag{2.3}$$

This implies, using Lemma 2.1, that $N = J$ is an ideal of R, and R is periodic.
As is well-known, we have

$$R \cong \text{ a subdirect sum of subdirectly irreducible rings } R_i. \tag{2.4}$$

Let $\sigma : R \to R_i$ be the natural homomorphism of R onto R_i. Since R is periodic, R_i is periodic and by Lemma 2.4,

$$N_i = \text{ the set of nilpotents of } R_i = \sigma(N) \text{ is an ideal of } R_i. \tag{2.5}$$

We now distinguish two cases.

Case 1 $1 \notin R_i$. Let $x_i \in R_i$, and let $\sigma : x \to x_i$. Then by Lemma 2.3, x^k is a central idempotent of R, and hence x_i^k is a central idempotent in the subdirectly irreducible ring R_i, for some positive integer k. Hence $x_i^k = 0$ $(1 \notin R_i)$. Thus $R_i = N_i$ is a nil ring.

Case 2 $1 \in R_i$. The above argument in Case 1 shows that x_i^k is a central idempotent in the subdirectly irreducible ring R_i. Hence $x_i^k = 0$ or $x_i^k = 1$ for all $x_i \in R_i$. So, R_i is a local ring and for every $x_i + N_i \in R_i/N_i$,

$$x_i + N_i = N_i \quad \text{or} \quad (x_i + N_i)^k = 1 + N_i. \tag{2.6}$$

So R_i/N_i is a periodic division ring, and hence by Lemma 2.2, R_i/N_i is a periodic field.

(ii) Suppose R is Artinian. Using (2.3), aR is a nil right ideal for every $a \in N$. So, $N \subseteq J$. But $J \subseteq N$ since R is Artinian. Therefore $N = J$ is an ideal of R and $R/N = R/J$ is semisimple Artinian. This implies that R/N is isomorphic to a finite direct product $R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a complete $t_i \times t_i$ matrix ring over a division ring D_i. Since R is Artinian, the idempotents of R/J lift to idempotents in R [2], and hence the idempotents of R/J are central. If $t_j > 1$, then $E_{11} \in R_j$, and $(0,\dots,0,E_{11},0,\dots,0)$ is an idempotent element of R/J which is not central in R/J. This is a contradiction. So $t_i = 1$ for every i. Therefore each R_i is a division ring and R/N is isomorphic to a finite direct product of division rings. □

The next result deals with a special kind of D^*-rings.

Theorem 2.10. *Let R be a ring such that every zero divisor x can be written uniquely as $x = a + e$, where $a \in N$ and e is idempotent.*

(i) *If R is weakly periodic, then N is an ideal of R, and R/N is isomorphic to a subdirect sum of fields.*

(ii) *If R is Artinian, then N is an ideal and R/N is a finite direct product of division rings.*

Proof. Let $e^2 = e \in R$, $x \in R$, and let $f = e + ex - exe$. Then $f^2 = f$ and hence $(ef - e)f = 0$. So if f is not a zero divisor, then $ef - e = 0$. So $ef = e$, and thus $f = e$, which implies that $ex = exe$. The net result is $ex - exe = 0$ if f is not a zero divisor. Next, suppose f is a zero divisor. Then since

$$\begin{gathered} f = (ex - exe) + e; \quad ex - exe \in N,\ e \text{ idempotent;} \\ f = 0 + f, \end{gathered} \tag{2.7}$$

it follows from uniqueness that $ex - exe = 0$, and hence $ex = exe$ in all cases. Similarly $xe = exe$, and thus

$$\text{all idempotents of } R \text{ are central, and hence } R \text{ is a normal } D^*\text{-ring.} \tag{2.8}$$

(i) Using (2.8), R satisfies all the hypotheses of Theorem 2.9(i), and hence N is an ideal, and R is periodic. Using Lemma 2.2, for each $x \in R$, there exists an integer $k > 1$, such that $x - x^k \in N$, and hence

$$(x+N)^k = (x+N), \quad k = k(x) > 1. \tag{2.9}$$

By a well-known theorem of Jacobson [6], (2.9) implies that R/N is a subdirect sum of fields.

(ii) If R is Artinian, then using (2.8), R satisfies the hypotheses of Theorem 2.9(ii). Therefore N is an ideal and R/N is a finite direct product of division rings. □

Theorem 2.11. *Let R be a semiprime D^*-ring with N commutative. Then R is either a domain or a J-ring.*

Proof. As in the proof of [3, Theorem 1] we can show that if $a^k = 0$, then $(ar)^k = 0$ for all $r \in R$. Therefore, by Levitzki's theorem, $N = \{0\}$. Assume R is not a domain, and let a be any nonzero divisor of zero. Then a is potent and aR consists of zero divisors, hence is a J-ring containing a. Therefore $[ax, a] = 0$ for all $x \in R$, hence $(ax)^n = a^n x^n$ for all $x \in R$, and all $n \geq 2$. For x not a zero divisor, choose $n > 1$ such that $a^n = a$ and $(ax)^n = ax$. Then $a^n x^n = ax$, so $a(x^n - x) = 0$ and $x^n - x$ is a zero divisor, hence is periodic. It follows by Chacron's theorem that R is a periodic ring; and since $N = \{0\}$, R is a J-ring. □

Example 2.12. Let

$$R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}, \quad 0, 1 \in GF(2). \tag{2.10}$$

Then R is a normal weakly periodic D^*-ring with commuting nilpotents. R is not semiprime since the set of nilpotent elements N is a nonzero nilpotent ideal. This example shows that we cannot drop the hypothesis "R is semiprime" in Theorem 2.11.

In Theorem 2.14 below, we study the structure of a special kind of D^*-rings, the class of rings in which every zero divisor is potent. Recall that a ring is semiperfect [2] if and only if R/J is semisimple (Artinian) and idempotents lift modulo J. We need the following lemma.

Lemma 2.13. *Let R be a ring in which every zero divisor is potent. Then $N = \{0\}$ and R is normal. Moreover, If R is not a domain, then $J = \{0\}$.*

Proof. If $a \in N$, then a is a zero divisor and hence potent by hypothesis. So $a^n = a$ for some positive integer n, and since $a \in N$, there exists a positive integer k such that $0 = a^{n^k} = a$. So $N = \{0\}$. Let e be any idempotent element of R and x is any element of R. Then $ex - exe \in N$, and hence $ex - exe = 0$. Similarly, $xe = exe$. So $ex = xe$ and R is normal.

Let x be a nonzero divisor of zero. Then xJ consists of zero divisors, which are potent. Therefore $xJ = \{0\}$. But then J consists of zero divisors, hence potent elements, and therefore $J = \{0\}$. □

Theorem 2.14. *Let R be a ring such that every zero divisor is potent.*

(i) *If R is weakly periodic, then every element of R is potent and R is a subdirect sum of fields.*

(ii) *If R is prime, then R is a domain.*

(iii) *If R is Artinian, then R is a finite direct product of division rings.*

(iv) *If R is semiperfect, then R/J is a finite direct product of division rings.*

Proof. (i) Since R is weakly periodic, every element $x \in R$ can be written as

$$x = a + b, \quad \text{where } a \in N,\ b \text{ is potent.} \tag{2.11}$$

But $N = \{0\}$ (Lemma 2.13), so every $x \in R$ is potent and hence R is isomorphic to a subdirect sum of fields by a well-known theorem of Jacobson.

(ii) Suppose R is a prime, then R is a prime ring with $N = \{0\}$, and hence R is a domain.

(iii) Let R be an Artinian ring such that every zero divisor is potent. Since $N = \{0\}$ (Lemma 2.13) and R is Artinian, $J = N = \{0\}$. So R is semisimple Artinian and hence it is isomorphic to a finite direct product $R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a complete $t_i \times t_i$ matrix ring over a division ring D_i. If $t_j > 1$, then $E_{11} \in R_j$, and $(0,\ldots,0,E_{11},0,\ldots,0)$ is an idempotent element of R which is not central in R contradicting Lemma 2.13. So $t_i = 1$ for every i. Therefore each R_i is a division ring and R is isomorphic to a finite direct product of division rings.

(iv) Let R be a semiperfect ring such that every zero divisor is potent. Then R/J is semisimple Artinian and hence it is isomorphic to a finite direct product $R_1 \times R_2 \times \cdots \times R_n$, where each R_i is a complete $t_i \times t_i$ matrix ring over a division ring D_i. Since R is semiperfect, the idempotents of R/J lift to idempotents in R, and hence the argument of part (iii) above implies that each R_i is a division ring and R/J is isomorphic to a finite direct product of division rings. □

Acknowledgment

We wish to express our indebtedness and gratitude to the referee for the helpful suggestions and valuable comments.

References

[1] H. Abu-Khuzam, H. E. Bell, and A. Yaqub, *Structure of rings with a condition on zero divisors*, Scientiae Mathematicae Japonicae **54** (2001), no. 2, 219–224.
[2] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics, vol. 13, Springer, New York, 1974.
[3] H. E. Bell, *Some commutativity results for periodic rings*, Acta Mathematica Academiae Scientiarum Hungaricae **28** (1976), no. 3-4, 279–283.
[4] ______, *A commutativity study for periodic rings*, Pacific Journal of Mathematics **70** (1977), no. 1, 29–36.
[5] J. Grosen, H. Tominaga, and A. Yaqub, *On weakly periodic rings, periodic rings and commutativity theorems*, Mathematical Journal of Okayama University **32** (1990), 77–81.
[6] N. Jacobson, *Structure theory for algebraic algebras of bounded degree*, Annals of Mathematics **46** (1945), 695–707.
[7] H. Tominaga and A. Yaqub, *Some commutativity conditions for left s-unital rings satisfying certain polynomial identities*, Results in Mathematics **6** (1983), no. 2, 217–219.

Hazar Abu-Khuzam: Department of Mathematics, American University of Beirut, Beirut 1107 2020, Lebanon
E-mail address: hazar@aub.edu.lb

Adil Yaqub: Department of Mathematics, University of California, Santa Barbara, CA 93106-3080, USA
E-mail address: yaqub@math.ucsb.edu