THE CONVERGENCE OF AN IMPLICIT MEAN VALUE ITERATION

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We show the convergence of an implicit mean value iteration when applied to uniformly pseudocontractive maps. Remarks about other implicit mean value iterations are given.

1. Introduction

Let X be a real Banach space, $T : X \to X$ a map, and $x_0, u_0 \in X$. In [5], the following iteration is introduced:

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n T u_n, \tag{1.1}$$

where $\{\alpha_n\} \subset (0, 1)$.

In the rest of the paper, we will assume that $(I - tT)^{-1}$ exists for all $t \in (0, 1)$. Consider the following iteration, see [8]:

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_{n+1}, \tag{1.2}$$

where the sequence $\{\alpha_n\}$ is in $(0, 1)$.

The following example shows that iteration (1.2) is well defined. We recall the following well-known result.

LEMMA 1.1. *Let $\{\beta_n\}$ be a nonnegative sequence such that $\beta_n \in (0, 1]$, for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} \beta_n = \infty$, then $\prod_{n=1}^{\infty}(1 - \beta_n) = 0$.*

Inspired by [3, 8], we give an example which shows that Mann iterations (1.1) and (1.2) are independent.

Example 1.2. Let $X = \mathbb{R}^2$. Let $T : X \to X$ be the map given by

$$T(x, y) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \forall (x, y) \in \mathbb{R}^2. \tag{1.3}$$

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Iteration (1.1) is not convergent to the fixed point of T, while iteration (1.2), for $\{\alpha_n\} \subset (1/2, 1)$, converges to the fixed point of T.

Proof. Let $u = (x, y)$. For all $\lambda \in]0, 1[$,

$$
\begin{aligned}
\left| (1 - \lambda) u + \lambda T u \right|^2 &= \left| (1 - \lambda)(x, y) + \lambda T(x, y) \right|^2 \\
&= \left| (1 - \lambda)(x, y) + \lambda(x + y, -x + y) \right|^2 \\
&= \left| (1 - \lambda)x + \lambda x + \lambda y, (1 - \lambda)y - \lambda x + \lambda y \right|^2 \\
&= \left| (x + \lambda y, y - \lambda x) \right|^2 = (1 + \lambda^2)x^2 + (1 + \lambda^2)y^2 \\
&= (1 + \lambda^2)|u|^2 > |u|^2.
\end{aligned}
\tag{1.4}
$$

Hence the Mann iteration is not convergent to $(0,0)$, for all $\{\alpha_n\} \subset (0,1)$. Note that $(I_2 - tT)^{-1}$ exists for all $t \in (0,1)$. Moreover,

$$
(I_2 - tT) = \begin{pmatrix} 1 - t & -t \\ t & 1 - t \end{pmatrix},
$$

$$
(I_2 - tT)^{-1} = \frac{1}{(1 - t)^2 + t^2} \begin{pmatrix} 1 - t & t \\ -t & 1 - t \end{pmatrix}.
\tag{1.5}
$$

Thus, for $\{\alpha_n\} \in (0,1)$,

$$
x_{n+1} = \frac{1 - \alpha_n}{(1 - \alpha_n)^2 + \alpha_n^2} \begin{pmatrix} 1 - \alpha_n & \alpha_n \\ -\alpha_n & 1 - \alpha_n \end{pmatrix} x_n.
\tag{1.6}
$$

Denote $x_n := (a_n, b_n)'$ to obtain

$$
\begin{aligned}
\left| x_{n+1} \right|^2 &= a_{n+1}^2 + b_{n+1}^2 \\
&= \frac{(1 - \alpha_n)^2}{\left[(1 - \alpha_n)^2 + \alpha_n^2 \right]^2} \left[\left((1 - \alpha_n)a_n + \alpha_n b_n \right)^2 + \left(-\alpha_n a_n + (1 - \alpha_n)b_n \right)^2 \right] \\
&= \frac{(1 - \alpha_n)^2}{\left[(1 - \alpha_n)^2 + \alpha_n^2 \right]^2} \left[(1 - \alpha_n)^2 + \alpha_n^2 \right] (a_n^2 + b_n^2) = \frac{(1 - \alpha_n)^2}{\left[(1 - \alpha_n)^2 + \alpha_n^2 \right]} (a_n^2 + b_n^2) \\
&= \left(1 - \frac{\alpha_n^2}{(1 - \alpha_n)^2 + \alpha_n^2} \right) |x_n|^2 \leq (1 - \alpha_n) |x_n|^2.
\end{aligned}
\tag{1.7}
$$

The last inequality holds because $0 \geq 2\alpha_n^2 - 3\alpha_n + 1$ for $\alpha_n \in (1/2, 1)$. Lemma 1.1 assures that $\lim_{n \to \infty} |x_n| = 0$. $\qquad \square$

Take $T : [0,1) \to [0,1)$, $Tx = x^2$, to obtain a map for which Mann iteration converges to the fixed point, while implicit Mann iteration is not well defined and consequently does not converge at all. Using (1.2),

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n x_{n+1}^2. \tag{1.8}$$

Solving for x_{n+1} yields

$$x_{n+1} = \frac{1 - \left(1 - 4\alpha_n(1 - \alpha_n)x_n\right)^{1/2}}{2\alpha_n} \tag{1.9}$$

or

$$x_{n+1} = \frac{1 + \left(1 - 4\alpha_n(1 - \alpha_n)x_n\right)^{1/2}}{2\alpha_n}. \tag{1.10}$$

In the later case, x_{n+1} is no longer inside the interval $[0,1)$. Suppose one always takes the first case. With the choice that each $\alpha_n = 1/2$, we have

$$x_{n+1} = 1 - (1 - x_n)^{1/2}, \tag{1.11}$$

that is, $1 - x_{n+1} = (1 - x_n)^{1/2}$. Set $a_n = 1 - x_n > 0$ to obtain $\lim_{n \to \infty} a_n = 0$. Thus, $\{x_n\}$ converges to 1, which is not in the interval $[0,1)$.

The map $J : X \to 2^{X^*}$, given by $J(x) := \{f \in X^* : \langle x, f \rangle = \|x\|^2, \|f\| = \|x\|\}$, for all $x \in X$, is called *the normalized duality mapping*. It is easy to see that

$$\langle y, j(x) \rangle \le \|x\| \|y\|, \quad \forall x, y \in X, \ \forall j(x) \in J(x). \tag{1.12}$$

Define

$$\Psi := \{\psi \mid \psi : [0, +\infty) \longrightarrow [0, +\infty) \text{ is a strictly increasing map such that } \psi(0) = 0\}. \tag{1.13}$$

The following definition can be found, for example, in [4].

Definition 1.3. A map $T : X \to X$ is called uniformly pseudocontractive if there exist a map $\psi \in \Psi$ and a $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \le \|x - y\|^2 - \psi(\|x - y\|), \quad \forall x, y \in X. \tag{1.14}$$

Taking $\psi(a) := \psi(a) \cdot a$, for all $a \in [0, +\infty)$, $(\psi \in \Psi)$, gives the usual definitions of ψ-strongly pseudocontractivity. The choice $\psi(a) := \gamma \cdot a^2$, $\gamma \in (0,1)$, for all $a \in [0, +\infty)$, $(\psi \in \Psi)$, yields the usual definition of strong pseudocontractivity.

The convergence of (1.2) dealing with strongly pseudocontractive maps was proved in [8]. We will prove the convergence of iteration (1.2) when applied to uniformly pseudocontractive maps. For this purpose, we need the following result.

LEMMA 1.4 [9]. *Let $\{a_n\}$ be a nonnegative bounded sequence which satisfies the following inequality:*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n a_{n+1} - \alpha_n \frac{\psi(a_{n+1})}{a_{n+1}} + \alpha_n \varepsilon_n, \quad \forall n \ge n_0, \tag{1.15}$$

where $\psi(\cdot) \in \Psi$, $\alpha_n \in (0,1)$, $\varepsilon_n \ge 0$, for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, and $\lim_{n \to \infty} \varepsilon_n = 0$. Then $\lim_{n \to \infty} a_n = 0$.

2. Main results

THEOREM 2.1. *Let X be a real Banach space and $T : X \to X$ a uniformly pseudocontractive map with a fixed point such that*

$$\exists (I - tT)^{-1} \quad \forall t \in (0,1). \tag{2.1}$$

If $\{\alpha_n\} \subset (0,1)$ satisfies

$$\lim_{n \to \infty} \alpha_n = 0, \qquad \sum_{n=0}^{\infty} \alpha_n = \infty, \tag{2.2}$$

and $\{x_n\}$ is bounded, then for $x_0 \in X$ the iteration (1.2) converges to the fixed point of T.

Proof. The uniqueness of the fixed point comes from (1.14). Let x^* be the fixed point of T.

If there exists a nonnegative integer n for which $x_n = x^*$, then from (1.2),

$$x_{n+1} = (1 - \alpha_n)x^* + \alpha_n T x_{n+1} \tag{2.3}$$

or

$$(I - \alpha_n T)x_{n+1} = (I - \alpha_n T)x^*, \tag{2.4}$$

which, using (2.1), implies that $x_{n+1} = x^*$. By induction, $x_m = x^*$ for all $m \ge n$.

We may therefore assume that each $x_n \ne x^*$. Using (1.2)–(1.14),

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \langle x_{n+1} - x^*, j(x_{n+1} - x^*) \rangle \\
&= \langle (1 - \alpha_n)(x_n - x^*) + \alpha_n (T x_{n+1} - T x^*), j(x_{n+1} - x^*) \rangle \\
&= (1 - \alpha_n)\langle (x_n - x^*), j(x_{n+1} - x^*) \rangle + \alpha_n \langle T x_{n+1} - T x^*, j(x_{n+1} - x^*) \rangle \\
&\le (1 - \alpha_n)\|x_n - x^*\|\,\|x_{n+1} - x^*\| + \alpha_n \|x_{n+1} - x^*\|^2 - \alpha_n \psi(\|x_{n+1} - x^*\|) \\
&\le \|x_{n+1} - x^*\| \left((1 - \alpha_n)\|x_n - x^*\| + \alpha_n \|x_{n+1} - x^*\| - \alpha_n \frac{\psi(\|x_{n+1} - x^*\|)}{\|x_{n+1} - x^*\|} \right).
\end{aligned}
\tag{2.5}
$$

Dividing by $\|x_{n+1} - x^*\|$ and defining $a_n = \|x_n - x^*\|$ yield (1.15) with each $\varepsilon_n = 0$. From Lemma 1.4, $\lim_{n \to \infty} a_n = 0$. $\qquad \square$

The following remark indicates some ways in which Theorem 2.1 can be applied to certain accretive maps.

Remark 2.2. (1) The operator T is a (uniformly, ψ-strongly, strongly) pseudocontractive map if and only if $(I - T)$ is a (uniformly, ψ-strongly, strongly) accretive map.

(2) Let $T, S : X \to X$, and $f \in X$ be given. A fixed point for the map $Tx = f + (I - S)x$, for all $x \in X$, is a solution for $Sx = f$, and conversely.

(3) Consider iteration (1.2) with $Tx = f + (I - S)x$ to obtain a convergence result to the solution of $Sx = f$.

(4) Let $f \in X$ be given. If the operator S is accretive, then $f - S$ is a strongly pseudocontractive map.

(5) Let $T, S : X \to X$. A fixed point for the map $Tx = f - Sx$, for all $x \in X$, is a solution for $x + Sx = f$, and conversely.

(6) Consider iteration (1.2) with $Tx = f - Sx$ to obtain a convergence result to the solution of $x + Sx = f$.

Remark 2.3. If (1.14) is also true for all $x \in X$, and $y := x^*$, the fixed point, then such a map is called uniformly hemicontractive. Obviously, our result holds for the uniformly hemicontractive case.

3. Remarks about implicit mean value iterations

Let X be a real Banach space and B a nonempty convex subset, $u_0, x_0 \in B$. Consider for $\{\alpha_n\} \subset (0,1)$, a finite family of functions $\{T_i\}_{i=1}^N : B \to B$ and the following two iterations:

$$x_1 = \alpha_1 x_0 + (1 - \alpha_1) T_1 x_1,$$
$$x_2 = \alpha_2 x_1 + (1 - \alpha_2) T_2 x_2,$$
$$\vdots$$
$$x_N = \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N, \tag{3.1}$$
$$x_{N+1} = \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1 x_{N+1},$$
$$\vdots$$

$$u_1 = \alpha_1 u_0 + (1 - \alpha_1) T_1 u_1,$$
$$u_2 = \alpha_2 u_1 + (1 - \alpha_2) T_2 u_2,$$
$$\vdots$$
$$x_N = \alpha_N x_{N-1} + (1 - \alpha_N) T_N x_N,$$
$$x_{N+1} = \alpha_{N+1} x_N + (1 - \alpha_{N+1}) T_1^2 x_{N+1}, \tag{3.2}$$
$$\vdots$$
$$x_{2N} = \alpha_{2N} x_{2N-1} + (1 - \alpha_{2N}) T_N^2 x_{2N},$$
$$\vdots$$

Iteration (3.1) has been considered in [2, 6, 7, 11, 12]. Iteration (3.1) has been discussed in [1, 10]. Note that iteration (1.2) is a particular case of (3.1). However, as far as we know, no such paper is dedicated to the convergence of the implicit iteration dealing with uniformly pseudocontractive maps.

Condition (2.1) forces iteration (1.2) to be well defined. The papers listed above do not impose such a condition, and consequently, the resulting implicit mean value iterations need not be well defined, as the following example illustrates.

Example 3.1. Take $T_i : [0,1] \to [0,1]$, $i = 1,2$, $T_1(x) = x^2$, and $T_2(x) = (1/2)x$ with the common fixed point $x^* = 0$. Then for $x_0 = 1$, $x_1 = \alpha_1 x_0 + (1 - \alpha_1)x_1^2$, one obtains $x_1 = 1$ and $x_1 = \alpha_1/(1 - \alpha_1)$. Take now $u_0 = 1$, $u_1 = \alpha_1 u_0 + (1 - \alpha_1)u_1^2$, and $u_2 = \alpha_2 u_1 + (1 - \alpha_2)(1/4)u_2$. Observe that there are two possible values for u_2.

Remark 3.2. The existence of $(I - tT_i)^{-1}$, for all $t \in \,]0,1[$, $i = 1, N$, should be added to the hypotheses of the results of [2, 6, 7, 11, 12] in order to have well-defined iterations. The existence of $(I - tT_i^p)^{-1}$, for all $t \in (0,1)$, $i = 1, N$, for all $p \geq 1$, should be added to the hypotheses of the results of [1, 10].

Acknowledgment

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