CLASSICAL ORTHOGONAL POLYNOMIALS AND LEVERRIER-FADDEEV ALGORITHM FOR THE MATRIX PENCILS $sE - A$

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In this contribution we present an extension of the Leverrier-Faddeev algorithm for the simultaneous computation of the determinant and the adjoint matrix $B(s)$ of a pencil $sE - A$ where E is a singular matrix but $\det(sE - A) \not\equiv 0$. Using a previous result by the authors we express $B(s)$ and $\det(sE - A)$ in terms of classical orthogonal polynomials.

1. Introduction

Consider a linear, time-invariant, multivariable singular system described in the state space as follows:

$$\begin{aligned} E\dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned} \tag{1.1}$$

where $E \in \mathbb{C}^{n\times n}$ is a singular matrix, x is the n-dimensional state vector, u is the m-dimensional input vector, y is the r-dimensional output vector, and A, B, and C are matrices with complex entries and appropriate dimension.

We can take the Laplace transform of our system (1.1). If $\det(sE - A) \not\equiv 0$, then the following transfer function appears:

$$H(s) = C(sE - A)^{-1}B, \tag{1.2}$$

which, in general, is a strictly proper rational matrix (see [1, 5] and references therein).

The computation of $(sE - A)^{-1}$ can be carried out by using the Cramer rule, which requires the evaluation of n^2 determinants of $(n-1)\times(n-1)$ polynomial matrices. Clearly, this is not a practical procedure for large n. We will describe an extension of the classical Leverrier-Faddeev algorithm using families of classical orthogonal polynomials following our previous contribution [2] when instead of a singular matrix E we used I_n. Here we generalize a recent result [6] based on the Chebyshev polynomials, a very

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particular family of classical orthogonal polynomials. Notice that in [3, 5] an alternative approach using the canonical basis (x^n) in the linear space of polynomials with complex coefficients was given for linear pencils. Along the paper, we will assume that the pencil $sE-A$ is regular, that is, $\det(sE-A)\not\equiv 0$.

The structure of the manuscript is the following. In Section 2 we summarize our algorithm presented in [2] as well as we introduce the basic background about monic classical orthogonal polynomials. In Section 3 we describe the algorithm to find the adjoint matrix $B(s)$ as well as the determinant of a regular pencil $sE-A$, where E is a singular matrix. We also cover a gap in [6] concerning the connection between $\det(sE-A)$ and the adjoint matrix of $(sE-A)$. Finally, in Section 4, some numerical examples in order to test our algorithm will be shown.

2. Leverrier-Faddeev algorithm and classical orthogonal polynomials

For a matrix $A\in\mathbb{C}^{n\times n}$ an algorithm attributed to Leverrier, Faddeev, and others allows the simultaneous determination of the characteristic polynomial of A and the adjoint matrix of sI_n-A. As it is shown in [1], if

$$\begin{aligned} p_A(s) &= \det\left(sI_n-A\right) = s^n+\sum_{k=0}^{n-1}\hat{a}_{n-k}s^k,\\ \widetilde{A}(s) &= \operatorname{Adj}\left(sI_n-A\right) = s^{n-1}I_n+\sum_{k=0}^{n-2}s^k\widehat{B}_{n-k-1}, \end{aligned}\tag{2.1}$$

then the relation between the coefficients $(\hat{a}_k)$ and the matrices $(\widehat{B}_k)$ follows by identification of the coefficients of the monomials in the following two equations:

$$\begin{aligned} (sI_n-A)\widetilde{A}(s) &= p_A(s)I_n,\\ \frac{dp_A(s)}{ds} &= \operatorname{tr}\widetilde{A}(s). \end{aligned}\tag{2.2}$$

From a numerical point of view, the accuracy of this algorithm is not so good. This is the reason why in [2] we have presented an alternative approach using in (2.1) the representation of $p_A(s)$ and $\widetilde{A}(s)$ in terms of a family of monic classical orthogonal polynomials.

The main reason to do it is related to the following fact.

Proposition 2.1 (see [4]). *$(P_n)_{n=0}^{\infty}$ is a family of monic classical orthogonal polynomials (Hermite, Laguerre, Jacobi, and Bessel) if and only if there exist sequences of real numbers (r_n) and (s_n) such that*

$$P_n(s)=\frac{P'_{n+1}(s)}{n+1}+r_n\frac{P'_n(s)}{n}+s_n\frac{P'_{n-1}(s)}{n-1}\quad \text{for } n\geqslant 2.\tag{2.3}$$

The coefficients that appear in (2.3) are given in Table 2.1.

Notice that the Hermite case appears when $r_n=s_n=0$, $n\geqslant 2$. The Laguerre case appears when $s_n=0$, $n\geqslant 2$. Finally, the Jacobi and the Bessel cases are related to the case $s_n\neq 0$ for every $n\geqslant 2$.

TABLE 2.1. Coefficients in the relation of Proposition 2.1.

	r_n	s_n
Hermite	0	0
Laguerre	n	0
Jacobi	$\frac{2n(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$	$-\frac{4n(n-1)(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}$
Bessel	$\frac{4n}{(2n+\alpha)(2n+\alpha+2)}$	$\frac{4n(n-1)}{(2n+\alpha-1)(2n+\alpha)^2(2n+\alpha+1)}$

TABLE 2.2. Coefficients in the three-term recurrence relation (2.4).

	β_n	γ_n
Hermite	0	$\frac{n}{2}$
Laguerre	$2n+\alpha+1$	$n(n+\alpha)$
Jacobi	$\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}$	$\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}$
Bessel	$-\frac{2\alpha}{(2n+\alpha)(2n+\alpha+2)}$	$-\frac{4n(n+\alpha)}{(2n+\alpha-1)(2n+\alpha)^2(2n+\alpha+1)}$

The second ingredient for our algorithm is the fact that if $(P_n)_{n=0}^{\infty}$ is a family of monic classical orthogonal polynomials, then the following three-term recurrence relation holds:

$$\begin{gathered} sP_n(s) = P_{n+1}(s) + \beta_n P_n(s) + \gamma_n P_{n-1}(s), \quad n \geqslant 1 \text{ with } \gamma_n \neq 0, \\ P_0(s) = 1, \qquad P_1(s) = s - \beta_0. \end{gathered} \tag{2.4}$$

The coefficients that appear in (2.4) are given in Table 2.2.

If we expand the characteristic polynomial $p_A(s)$ of A as well as the adjoint matrix $\widetilde{A}(s)$ of $sI_n - A$ in terms of the above basis of monic classical orthogonal polynomials, that is,

$$p_A(s) = P_n(s) + \sum_{k=0}^{n-1} \hat{a}_{n-k} P_k(s), \qquad \widetilde{A}(s) = P_{n-1}(s) I_n + \sum_{k=0}^{n-2} P_k(s) \widehat{B}_{n-k-1}, \tag{2.5}$$

and take into account (2.2) together with (2.3) and (2.4), then we get the following.

PROPOSITION 2.2 (see [2]). (i) *For* $k = 1, \ldots, n$,

$$k\hat{a}_k = (\beta_{n-k} - r_{n-k}) \operatorname{tr} \widehat{B}_{k-1} + (\gamma_{n-k+1} - s_{n-k+1}) \operatorname{tr} \widehat{B}_{k-2} - \operatorname{tr}(A\widehat{B}_{k-1}); \tag{2.6}$$

Data: $\{\beta_k\}_{k=0}^{n-1}$, $\{\gamma_k\}_{k=1}^{n}$, $\{r_k\}_{k=0}^{n-1}$, $\{s_k\}_{k=1}^{n}$.
Initial Condition: $\widehat{B}_{-1} = 0$, $\widehat{B}_0 = I_n$.
For $k = 1,2,\dots,n-1$

$$\begin{aligned} \widehat{a}_k &= (1/k)\big[(\beta_{n-k} - r_{n-k})\operatorname{tr}\widehat{B}_{k-1} + (\gamma_{n-k+1} - s_{n-k+1})\operatorname{tr}\widehat{B}_{k-2} - \operatorname{tr}(A\widehat{B}_{k-1})\big], \\ \widehat{B}_k &= A\widehat{B}_{k-1} + \widehat{a}_k I_n - \gamma_{n-k+1}\widehat{B}_{k-2} - \beta_{n-k}\widehat{B}_{k-1}. \end{aligned} \tag{2.8}$$

End (For)

$$\widehat{a}_n = (1/n)\big[(\beta_0 - r_0)\operatorname{tr}\widehat{B}_{n-1} + (\gamma_1 - s_1)\operatorname{tr}\widehat{B}_{n-2} - \operatorname{tr}(A\widehat{B}_{n-1})\big]. \tag{2.9}$$

Algorithm 2.1

(ii) *for* $k = 1, 2, \dots, n-1$,

$$\widehat{B}_k = A\widehat{B}_{k-1} + \widehat{a}_k I_n - \gamma_{n-k+1}\widehat{B}_{k-2} - \beta_{n-k}\widehat{B}_{k-1}, \tag{2.7}$$

with the convention $\widehat{B}_{-1} = 0$, $r_0 = 0$, $s_1 = 0$.

Indeed the algorithm to find (a_k) and (B_k) is in Algorithm 2.1.

3. Regular pencils

Now, we are interested in the computation of $a(s) = \det(sE - A)$, assuming $sE - A$ is a regular pencil, and $B(s) = \operatorname{Adj}(sE - A)$, where $A, E \in \mathbb{C}^{n\times n}$ and E is a singular matrix. If in the expressions of the previous section we replace A by $A(s) = -sE + A$, then we get

$$\widetilde{a}(\lambda, s) := \det(\lambda I_n - A(s)) = P_n(\lambda) + \sum_{k=0}^{n-1} \widehat{a}_{n-k}(s) P_k(\lambda) \tag{3.1}$$

as well as

$$\widetilde{B}(\lambda, s) := \operatorname{Adj}(\lambda I_n - A(s)) = P_{n-1}(\lambda) I_n + \sum_{k=0}^{n-2} P_k(\lambda)\widehat{B}_{n-k-1}(s). \tag{3.2}$$

Thus, from (2.6) and (2.7) we get

$$\begin{aligned} k\widehat{a}_k(s) &= (\beta_{n-k} - r_{n-k})\operatorname{tr}\widehat{B}_{k-1}(s) - \operatorname{tr}(A(s)\widehat{B}_{k-1}(s)) \\ &\quad + (\gamma_{n-k+1} - s_{n-k+1})\operatorname{tr}\widehat{B}_{k-2}(s), \quad k = 1,\dots,n \end{aligned} \tag{3.3}$$

as well as

$$\widehat{B}_k(s) = \widehat{a}_k(s) I_n - \gamma_{n-k+1}\widehat{B}_{k-2}(s) - \beta_{n-k}\widehat{B}_{k-1}(s) + A(s)\widehat{B}_{k-1}(s) \tag{3.4}$$

for $k = 1,\dots,n-1$. Thus, if $\lambda = 0$ in (3.1) and (3.2), then we get

$$a(s) := \det(sE - A) = \widetilde{a}(0,s) = P_n(0) + \sum_{k=0}^{n-1} \widehat{a}_{n-k}(s)P_k(0), \tag{3.5}$$

$$B(s) := \operatorname{Adj}(sE - A) = \widetilde{B}(0,s) = P_{n-1}(0)I_n + \sum_{k=0}^{n-2} P_k(0)\widehat{B}_{n-k-1}(s). \tag{3.6}$$

Taking into account $\deg(P_k(s)) = k$ for all $k \geqslant 0$, (3.3), and (3.4), we can assure that the degrees of the polynomial $\widehat{a}_k(s)$, $k = 1,2,\dots,n$, and the polynomial matrix $\widehat{B}_k(s)$, $k = 1, 2,\dots, n-1$, are at most equal to k. Thus for $\widehat{a}_k(s)$ and $\widehat{B}_k(s)$ we get the expansions

$$\begin{aligned} \widehat{a}_k(s) &= \sum_{j=0}^{k} a_{k,j}P_j(s), \quad a_{k,j} \in \mathbb{C}, \\ \widehat{B}_k(s) &= \sum_{j=0}^{k} P_j(s)B_{k,j}, \quad B_{k,j} \in \mathbb{C}^{n\times n}. \end{aligned} \tag{3.7}$$

Substituting (3.7) in (3.3), we get

$$\begin{aligned} k\sum_{j=0}^{k} a_{k,j}P_j(s) = \operatorname{tr}\Bigg(&(\beta_{n-k} - r_{n-k})\sum_{j=0}^{k-1} P_j(s)B_{k-1,j} + (\gamma_{n-k+1} - s_{n-k+1})\sum_{j=0}^{k-2} P_j(s)B_{k-2,j} \\ &+ (sE - A)\sum_{j=0}^{k-1} P_j(s)B_{k-1,j}\Bigg). \end{aligned} \tag{3.8}$$

Applying in the right-hand side the three-term recurrence relation, we get

$$\begin{aligned} k\sum_{j=0}^{k} a_{k,j}P_j(s) = {}&\operatorname{tr}(EB_{k-1,k-1})P_k(s) \\ &+ \big[(\beta_{n-k} - r_{n-k})\operatorname{tr}B_{k-1,k-1} + \beta_{k-1}\operatorname{tr}(EB_{k-1,k-1}) \\ &\quad - \operatorname{tr}(AB_{k-1,k-1}) + \operatorname{tr}(EB_{k-1,k-2})\big]P_{k-1}(s) \\ &+ \sum_{j=1}^{k-2}\big[\gamma_{j+1}\operatorname{tr}(EB_{k-1,j+1}) + \beta_j\operatorname{tr}(EB_{k-1,j}) \\ &\qquad + (\beta_{n-k} - r_{n-k})\operatorname{tr}B_{k-1,j} + (\gamma_{n-k+1} - s_{n-k+1})\operatorname{tr}B_{k-2,j} \\ &\qquad - \operatorname{tr}(AB_{k-1,j}) + \operatorname{tr}(EB_{k-1,j-1})\big]P_j(s) \\ &+ \big[\gamma_1\operatorname{tr}(EB_{k-1,1}) + \beta_0\operatorname{tr}(EB_{k-1,0}) + (\beta_{n-k} - r_{n-k})\operatorname{tr}B_{k-1,0} \\ &\quad + (\gamma_{n-k+1} - s_{n-k+1})\operatorname{tr}B_{k-2,0} - \operatorname{tr}(AB_{k-1,0})\big]P_0(s). \end{aligned} \tag{3.9}$$

Thus, for $k = 1,2,\dots,n$,

$$\begin{aligned}
ka_{k,0} &= \gamma_1 \operatorname{tr}(EB_{k-1,1}) + \beta_0 \operatorname{tr}(EB_{k-1,0}) + (\beta_{n-k} - r_{n-k}) \operatorname{tr} B_{k-1,0} \\
&\quad - \operatorname{tr}(AB_{k-1,0}) + (\gamma_{n-k+1} - s_{n-k+1}) \operatorname{tr} B_{k-2,0}, \\
&\vdots \\
ka_{k,j} &= \gamma_{j+1} \operatorname{tr}(EB_{k-1,j+1}) + \beta_j \operatorname{tr}(EB_{k-1,j}) + \operatorname{tr}(EB_{k-1,j-1}) \\
&\quad + (\gamma_{n-k+1} - s_{n-k+1}) \operatorname{tr} B_{k-2,j} + (\beta_{n-k} - r_{n-k}) \operatorname{tr} B_{k-1,j} \\
&\quad - \operatorname{tr}(AB_{k-1,j}), \quad j = 1,\dots,k-2, \\
&\vdots \\
ka_{k,k-1} &= (\beta_{n-k} - r_{n-k}) \operatorname{tr} B_{k-1,k-1} + \operatorname{tr}(EB_{k-1,k-2}) \\
&\quad + \beta_{k-1} \operatorname{tr}(EB_{k-1,k-1}) - \operatorname{tr}(AB_{k-1,k-1}), \\
ka_{k,k} &= \operatorname{tr}(EB_{k-1,k-1}).
\end{aligned} \tag{3.10}$$

In an analogous way, substituting (3.7) in (3.4),

$$\begin{aligned}
\sum_{j=0}^{k} P_j(s) B_{k,j} &= \sum_{j=0}^{k} a_{k,j} P_j(s) I_n - \gamma_{n-k+1} \sum_{j=0}^{k-2} P_j(s) B_{k-2,j} \\
&\quad - \beta_{n-k} \sum_{j=0}^{k-1} P_j(s) B_{k-1,j} + (-sE + A) \sum_{j=0}^{k-1} P_j(s) B_{k-1,j}.
\end{aligned} \tag{3.11}$$

Using again the three-term recurrence relation, we get

$$\begin{aligned}
\sum_{j=0}^{k} P_j(s) B_{k,j} &= P_k(s) \left[a_{k,k} I_n - EB_{k-1,k-1} \right] \\
&\quad + P_{k-1}(s) \left[a_{k,k-1} I_n - EB_{k-1,k-2} + (A - \beta_{k-1} E - \beta_{n-k} I_n) B_{k-1,k-1} \right] \\
&\quad + \sum_{j=1}^{k-2} P_j(s) \big[a_{k,j} I_n - EB_{k-1,j-1} + (A - \beta_j E - \beta_{n-k} I_n) B_{k-1,j} \\
&\qquad - \gamma_{j+1} EB_{k-1,j+1} - \gamma_{n-k+1} B_{k-2,j} \big] \\
&\quad + P_0(s) \big[a_{k,0} I_n + (A - \beta_0 E - \beta_{n-k} I_n) B_{k-1,0} \\
&\qquad - \gamma_1 EB_{k-1,1} - \gamma_{n-k+1} B_{k-2,0} \big].
\end{aligned} \tag{3.12}$$

Data: $\{\beta_k\}_{k=0}^{n-1}$, $\{\gamma_k\}_{k=1}^{n}$, $\{r_k\}_{k=0}^{n-1}$, $\{s_k\}_{k=1}^{n}$.

Initial Condition: $B_{i,j} = 0$, if $i < j$ or $j < 0$, $a_{0,0} = 1$, $B_{0,0} = I_n$.

For $k = 1, \ldots, n-1$

$\alpha_{n-k} = \beta_{n-k} - r_{n-k}$.

$\delta_{n-k+1} = \gamma_{n-k+1} - s_{n-k+1}$.

$A_k = A - \beta_{n-k} I_n$.

For $j = 0, 1, \ldots, k$

$$a_{k,j} := (1/k)\left[\gamma_{j+1}\operatorname{tr}(EB_{k-1,j+1}) + \beta_j \operatorname{tr}(EB_{k-1,j}) + \alpha_{n-k}\operatorname{tr} B_{k-1,j} + \operatorname{tr}(EB_{k-1,j-1}) + \delta_{n-k+1}\operatorname{tr} B_{k-2,j} - \operatorname{tr}(AB_{k-1,j})\right].$$

$$B_{k,j} := a_{k,j} I_n - EB_{k-1,j-1} + (A_k - \beta_j E)B_{k-1,j} - \gamma_{j+1} EB_{k-1,j+1} - \gamma_{n-k+1} B_{k-2,j}.$$

End (For j).

End (For k).

For $j = 0, 1, \ldots, n$

$$a_{n,j} := (1/n)\left[\gamma_{j+1}\operatorname{tr}(EB_{n-1,j+1}) + \beta_j \operatorname{tr}(EB_{n-1,j}) + \beta_0 \operatorname{tr} B_{n-1,j} + \operatorname{tr}(EB_{n-1,j-1}) + \gamma_1 \operatorname{tr} B_{n-2,j} - \operatorname{tr}(AB_{n-1,j})\right].$$

End.

Algorithm 3.1

Thus, for $k = 1, 2, \ldots, n-1$,

$$\begin{aligned}
B_{k,0} &= a_{k,0} I_n + (A - \beta_0 E - \beta_{n-k} I_n) B_{k-1,0} - \gamma_1 EB_{k-1,1} - \gamma_{n-k+1} B_{k-2,0}, \\
&\vdots \\
B_{k,j} &= a_{k,j} I_n - EB_{k-1,j-1} + (A - \beta_j E - \beta_{n-k} I_n) B_{k-1,j} \\
&\quad - \gamma_{j+1} EB_{k-1,j+1} - \gamma_{n-k+1} B_{k-2,j}, \quad j = 1, \ldots, k-2, \\
&\vdots \\
B_{k,k-1} &= a_{k,k-1} I_n - EB_{k-1,k-2} + (A - \beta_{k-1} E - \beta_{n-k} I_n) B_{k-1,k-1}, \\
B_{k,k} &= a_{k,k} I_n - EB_{k-1,k-1}.
\end{aligned} \tag{3.13}$$

As a conclusion, the algorithm for the computation of the coefficients $a_{i,j}$ in (3.5) and $B_{i,j}$ in (3.6) is as in Algorithm 3.1.

Notice that formula (3.10) in [6] is not right as a simple computation shows. Indeed for a regular pencil it is enough to consider the expression of $a(s)$ and $B(s)$ in the example provided in [6, Section 4].

Next we will give the right result.

THEOREM 3.1. *Let* $A,E \in \mathbb{C}^{n\times n}$, $a(s) = \det(sE - A)$, *and* $B(s) = \operatorname{Adj}(sE - A)$. *Then*

$$\frac{d}{ds}a(s) = \operatorname{tr}\big(EB(s)\big). \tag{3.14}$$

Proof. First, assume that E is a nonsingular matrix. Then $sE - A = (sI_n - AE^{-1})E$ and

$$\begin{aligned}
\frac{d}{ds}a(s) &= \det(E)\frac{d}{ds}\big(\det\big(sI_n - AE^{-1}\big)\big)\\
&= \det(E)\operatorname{tr}\big(\operatorname{Adj}\big(sI_n - AE^{-1}\big)\big)\\
&= \det(E)\det(E)^{-1}\det(sE - A)\operatorname{tr}\big(E(sE - A)^{-1}\big)\\
&= \det(sE - A)\operatorname{tr}\big(E(sE - A)^{-1}\big)\\
&= \operatorname{tr}\big(EB(s)\big).
\end{aligned} \tag{3.15}$$

Next, if E is a singular matrix, then consider $\varepsilon > 0$, such that $\varepsilon < \min\{|\lambda_i| : \lambda_i$ is an eigenvalue of $E, \lambda_i \neq 0\}$.

Then $E_\varepsilon := E + \varepsilon I_n$ is a nonsingular matrix. Using the first part of the proof,

$$\frac{d}{ds}a_\varepsilon(s) = \operatorname{tr}\big(E_\varepsilon B_\varepsilon(s)\big), \tag{3.16}$$

where $a_\varepsilon(s) = \det(sE_\varepsilon - A)$ and $B_\varepsilon(s) := \operatorname{Adj}(sE_\varepsilon - A)$.

Taking into account $E_\varepsilon \to E$, $a_\varepsilon(s) \to a(s)$, and $B_\varepsilon(s) \to B(s)$, when $\varepsilon \to 0$, we deduce our statement. □

4. Examples

Let $A,E \in \mathbb{C}^{3\times 3}$ given by

$$A = \begin{bmatrix} 1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}. \tag{4.1}$$

Notice that $\operatorname{rank} E = 2$. It is straightforward to prove that

$$\begin{aligned}
a(s) &= \det(sE - A) = -s^2,\\
B(s) &= \operatorname{Adj}(sE - A) = \begin{bmatrix} -s & 0 & s\\ 0 & -s & s\\ s & s & s^2 - 2s\end{bmatrix}.
\end{aligned} \tag{4.2}$$

Applying the algorithm of the previous section for Hermite polynomials $\{H_k(s)\}_{k=0}^{n}$, we get

$$
\begin{aligned}
a_{1,0} &= -\operatorname{tr} A = -3, \qquad a_{1,1} = \operatorname{tr} E = 2;\\
B_{1,0} &= a_{1,0}I_3 + A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \qquad B_{1,1} = a_{1,1}I_3 - E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix};\\
a_{2,0} &= \frac{1}{2}\left[\frac{1}{2}\operatorname{tr}(EB_{1,1}) - \operatorname{tr}(AB_{1,0}) + 3\right] = 2,\\
a_{2,1} &= \frac{1}{2}\left[\operatorname{tr}(EB_{1,0}) - \operatorname{tr}(AB_{1,1})\right] = -4,\\
a_{2,2} &= \frac{1}{2}\operatorname{tr}(EB_{1,1}) = 1;\\
B_{2,0} &= a_{2,0}I_3 + AB_{1,0} - \frac{1}{2}EB_{1,1} - I_3 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix},\\
B_{2,1} &= a_{2,1}I_3 + AB_{1,1} - EB_{1,0} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix},\\
B_{2,2} &= a_{2,2}I_3 - EB_{1,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix};\\
a_{3,0} &= \frac{1}{3}\left[\frac{1}{2}\operatorname{tr}(EB_{2,1}) - \operatorname{tr}(AB_{2,0}) + \frac{1}{2}\operatorname{tr} B_{1,0}\right] = -2,\\
a_{3,1} &= \frac{1}{3}\left[\operatorname{tr}(EB_{2,2}) - \operatorname{tr}(AB_{2,1}) + \operatorname{tr}(EB_{2,0}) + \frac{1}{2}\operatorname{tr} B_{1,1}\right] = 1,\\
a_{3,2} &= \frac{1}{3}\left[\operatorname{tr}(EB_{2,1}) - \operatorname{tr}(AB_{2,2})\right] = -1,\\
a_{3,3} &= \frac{1}{3}\operatorname{tr}(EB_{2,2}) = 0.
\end{aligned} \tag{4.3}
$$

Thus

$$
\begin{aligned}
\hat{a}_1(s) &= a_{1,0}H_0(s) + a_{1,1}H_1(s) = -3H_0(s) + 2H_1(s),\\
\hat{a}_2(s) &= a_{2,0}H_0(s) + a_{2,1}H_1(s) + a_{2,2}H_2(s) = 2H_0(s) - 4H_1(s) + H_2(s),\\
\hat{a}_3(s) &= a_{3,0}H_0(s) + a_{3,1}H_1(s) + a_{3,2}H_2(s) + a_{3,3}H_3(s) = -2H_0(s) + H_1(s) - H_2(s);\\
\widehat{B}_1(s) &= H_0(s)B_{1,0} + H_1(s)B_{1,1},\\
\widehat{B}_2(s) &= H_0(s)B_{2,0} + H_1(s)B_{2,1} + H_2(s)B_{2,2}.
\end{aligned} \tag{4.4}
$$

Now, the determinant $a(s)$ and the adjoint $B(s)$ of $sE - A$ are given by

$$\begin{aligned} a(s) &= H_3(0) + \hat{a}_1(s)H_2(0) + \hat{a}_2(s)H_1(0) + \hat{a}_3(s)H_0(0) \\ &= -\frac{1}{2}\hat{a}_1(s) + \hat{a}_3(s) = -H_2(s) - \frac{1}{2}H_0(s), \\ B(s) &= H_2(0)\widehat{B}_0(s) + H_1(0)\widehat{B}_1(s) + H_0(0)\widehat{B}_2(s) = -\frac{1}{2}I_3 + \widehat{B}_2(s) \\ &= H_0(s)\left[-\frac{1}{2}I_3 + B_{2,0}\right] + H_1(s)B_{2,1} + H_2(s)B_{2,2}. \end{aligned} \tag{4.5}$$

Next, applying the algorithm for the family $\{L_k^\alpha(s)\}_{k=0}^n$ (Laguerre polynomials with parameter α), we get

$$a_{1,0} = (1+\alpha)\operatorname{tr}E + 3(3+\alpha) - \operatorname{tr}A = 8 + 5\alpha, \qquad a_{1,1} = \operatorname{tr}E = 2;$$

$$B_{1,0} = (a_{1,0} - 5 - \alpha)I_3 + A - (1+\alpha)E = \begin{bmatrix} 3+3\alpha & 1 & 1 \\ 1 & 3+3\alpha & 1 \\ 1 & 1 & 4+4\alpha \end{bmatrix},$$

$$B_{1,1} = a_{1,1}I_3 - E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix};$$

$$\begin{aligned} a_{2,0} &= \frac{1}{2}\big[(1+\alpha)\big(\operatorname{tr}(EB_{1,1}) + \operatorname{tr}(EB_{1,0})\big) + (2+\alpha)\big(\operatorname{tr}B_{1,0} + 6\big) - \operatorname{tr}(AB_{1,0})\big] \\ &= 4(1+\alpha)(3+2\alpha), \\ a_{2,1} &= \frac{1}{2}\big((2+\alpha)\operatorname{tr}B_{1,1} + \operatorname{tr}(EB_{1,0}) + (3+\alpha)\operatorname{tr}(EB_{1,1}) - \operatorname{tr}(AB_{1,1})\big) = 8 + 6\alpha, \\ a_{2,2} &= \frac{1}{2}\operatorname{tr}(EB_{1,1}) = 1; \\ B_{2,0} &= \big(a_{2,0} - 4 - 2\alpha\big)I_3 + \big(A - (1+\alpha)E - (3+\alpha)I_3\big)B_{1,0} - (1+\alpha)EB_{1,1} \\ &= (1+\alpha)\begin{bmatrix} 2\alpha & 1 & 2 \\ 1 & 2\alpha & 2 \\ 2 & 2 & 2+4\alpha \end{bmatrix}, \\ B_{2,1} &= a_{2,1}I_3 + EB_{1,0} + \big(A - (3+\alpha)(E+I_3)\big)B_{1,1} = \begin{bmatrix} \alpha & 0 & 1 \\ 0 & \alpha & 1 \\ 1 & 1 & 4+4\alpha \end{bmatrix}, \end{aligned}$$

$$
\begin{aligned}
B_{2,2} &= a_{2,2}I_3 - EB_{1,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \\
a_{3,0} &= \frac{1}{3}\big[(1+\alpha)\big(\operatorname{tr}(EB_{2,1}) + \operatorname{tr}(EB_{2,0}) + \operatorname{tr}B_{2,0} + \operatorname{tr}B_{1,0}\big) - \operatorname{tr}(AB_{2,0})\big] \\
&= 2\alpha(1+\alpha)(3+2\alpha), \\
a_{3,1} &= \frac{1}{3}\big(2(2+\alpha)\operatorname{tr}(EB_{2,2}) + (3+\alpha)\operatorname{tr}(EB_{2,1}) + \operatorname{tr}(EB_{2,0}) + +(1+\alpha)\operatorname{tr}B_{2,1} \\
&\quad + (1+\alpha)\operatorname{tr}B_{1,1} - \operatorname{tr}(AB_{2,1})\big) = 2\alpha(3+2\alpha), \\
a_{3,2} &= \frac{1}{3}\big((1+\alpha)\operatorname{tr}B_{2,2} + \operatorname{tr}(EB_{2,1}) + (5+\alpha)\operatorname{tr}(EB_{2,2}) - \operatorname{tr}(AB_{2,2})\big) = \alpha, \\
a_{3,3} &= \frac{1}{3}\operatorname{tr}(EB_{2,2}) = 0.
\end{aligned}
\tag{4.6}
$$

Thus

$$
\begin{aligned}
\hat{a}_1(s) &= a_{1,0}L_0^\alpha(s) + a_{1,1}L_1^\alpha(s) = (8+5\alpha)L_0^\alpha(s) + 2L_1^\alpha(s), \\
\hat{a}_2(s) &= a_{2,0}L_0^\alpha(s) + a_{2,1}L_1^\alpha(s) + a_{2,2}L_2^\alpha(s) \\
&= 4(1+\alpha)(3+2\alpha)L_0^\alpha(s) + (8+6\alpha)L_1^\alpha(s) + L_2^\alpha(s), \\
\hat{a}_3(s) &= a_{3,0}L_0^\alpha(s) + a_{3,1}L_1^\alpha(s) + a_{3,2}L_2^\alpha(s) + a_{3,3}L_3^\alpha(s) \\
&= 2\alpha(1+\alpha)(3+2\alpha)L_0^\alpha(s) + 2\alpha(3+2\alpha)L_1^\alpha(s) + \alpha L_2^\alpha(s); \\
\hat{B}_1(s) &= L_0^\alpha(s)B_{1,0} + L_1^\alpha(s)B_{1,1}, \\
\hat{B}_2(s) &= L_0^\alpha(s)B_{2,0} + L_1^\alpha(s)B_{2,1} + L_2^\alpha(s)B_{2,2}.
\end{aligned}
\tag{4.7}
$$

The determinant $a(s)$ and the adjoint $B(s)$ of $sE - A$ are given by

$$
\begin{aligned}
a(s) &= L_3^\alpha(0) + \hat{a}_1(s)L_2^\alpha(0) + \hat{a}_2(s)L_1^\alpha(0) + \hat{a}_3(s)L_0^\alpha(0) \\
&= -(1+\alpha)(2+\alpha)L_0^\alpha(s) - 2(2+\alpha)L_1^\alpha(s) - L_2^\alpha(s), \\
B(s) &= L_2^\alpha(0)\hat{B}_0(s) + L_1^\alpha(0)\hat{B}_1(s) + L_0^\alpha(0)\hat{B}_2(s) \\
&= L_0^\alpha(s)\big[(1+\alpha)(2+\alpha)I_3 - (1+\alpha)B_{1,0} + B_{2,0}\big] \\
&\quad + L_1^\alpha(s)\big[-(1+\alpha)B_{1,1} + B_{2,1}\big] + L_2^\alpha(s)B_{2,2}.
\end{aligned}
\tag{4.8}
$$

Finally, if we consider the family $\{T_k(s)\}_{k=0}^{n}$ of the Chebyshev polynomials of first kind, applying the algorithm we get

$$\begin{gathered}
a_{1,0} = -\operatorname{tr} A = -3, \qquad a_{1,1} = \operatorname{tr} E = 2; \\
B_{1,0} = a_{1,0} I_3 + A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \qquad B_{1,1} = a_{1,1} I_3 - E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \\
a_{2,0} = \frac{1}{2}\left(\frac{1}{4}\operatorname{tr}(EB_{1,1}) - \operatorname{tr}(AB_{1,0}) + \frac{3}{2}\right) = \frac{5}{4}, \\
a_{2,1} = \frac{1}{2}\left(\operatorname{tr}(EB_{1,0}) - \operatorname{tr}(AB_{1,1})\right) = -4, \qquad a_{2,2} = \frac{1}{2}\left(\operatorname{tr}(EB_{1,1})\right) = 1; \\
B_{2,0} = a_{2,0} I_3 + AB_{1,0} - \frac{1}{4} EB_{1,1} - \frac{1}{4} I_3 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
B_{2,1} = a_{2,1} I_3 - EB_{1,0} + AB_{1,1} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 1 & -2 \end{bmatrix}, \\
B_{2,2} = a_{2,2} I_3 - EB_{1,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \\
a_{3,0} = \frac{1}{3}\left(\frac{1}{4}\operatorname{tr}(EB_{2,1}) + \frac{1}{2}\operatorname{tr} B_{1,0} - \operatorname{tr}(AB_{2,0})\right) = -2, \\
a_{3,1} = \frac{1}{3}\left(\frac{1}{4}\operatorname{tr}(EB_{2,2}) + \operatorname{tr}(EB_{2,0}) + \frac{1}{2}\operatorname{tr} B_{1,1} - \operatorname{tr}(AB_{2,1})\right) = 1, \\
a_{3,2} = \frac{1}{3}\left(\operatorname{tr}(EB_{2,1}) - \operatorname{tr}(AB_{2,2})\right) = -1, \\
a_{3,3} = \frac{1}{3}\operatorname{tr}(EB_{2,2}) = 0.
\end{gathered} \tag{4.9}$$

Thus

$$\begin{gathered}
\hat{a}_1(s) = a_{1,0} T_0(s) + a_{1,1} T_1(s) = -3T_0(s) + 2T_1(s), \\
\hat{a}_2(s) = a_{2,0} T_0(s) + a_{2,1} T_1(s) + a_{2,2} T_2(s) = \frac{5}{4} T_0(s) - 4T_1(s) + T_2(s), \\
\hat{a}_3(s) = a_{3,0} T_0(s) + a_{3,1} T_1(s) + a_{3,2} T_2(s) + a_{3,3} T_3(s) = -2T_0(s) + T_1(s) - T_2(s); \\
\widehat{B}_1(s) = T_0(s) B_{1,0} + T_1(s) B_{1,1}, \widehat{B}_2(s) = T_0(s) B_{2,0} + T_1(s) B_{2,1} + T_2(s) B_{2,2}.
\end{gathered} \tag{4.10}$$

The determinant $a(s)$ and the adjoint $B(s)$ of $sE - A$ are given by

$$\begin{aligned} a(s) &= T_3(0) + \hat{a}_1(s)T_2(0) + \hat{a}_2(s)T_1(0) + \hat{a}_3(s)T_0(0) \\ &= -\frac{1}{2}T_0(s) - T_2(s), \\ B(s) &= T_2(0)\hat{B}_0(s) + T_1(0)\hat{B}_1(s) + T_0(0)\hat{B}_2(s) \\ &= T_0(s)\big(B_{2,0} - \frac{1}{2}I_3\big) + T_1(s)B_{2,1} + T_2(s)B_{2,2}. \end{aligned} \tag{4.11}$$

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