VARIATIONAL INEQUALITY PROBLEMS IN H-SPACES

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The concept of η-invex set is explored and the concept of T-η-invex function is introduced. These concepts are applied to the generalized vector variational inequality problems in ordered topological vector spaces. The study of variational inequality problems is extended to H-spaces and differentiable n-manifolds. The solution of complementarity problem is also studied in the presence of fixed points or Lefschetz number.

1. Introduction

Variational inequality theory has become a rich source of inspiration in pure and applied mathematics. In recent years, classical variational inequality problem has been extended to study a wide class of problems arising in mechanics, physics, optimization and control, nonlinear programming, economics, finance, regional, structural, transportation, elasticity and applied sciences, and so forth. They have been extended and generalized in different directions by using novel and innovative techniques and ideas. In this paper, we extend the study of variational inequality problems to H-spaces and n-manifolds. First we extend the concept of η-invex sets and introduce the concept of T-η-invex function and study their applications to the generalized vector variational inequality problems in ordered topological vector spaces.

2. η-invex set and T-η-invex function in ordered topological vector spaces

The notion of invexity was introduced by Hanson [8] as a generalization of the concept of convexity. Now this concept is broadly used in the theory of optimization. Many authors have studied different types of convex and invex functions in different vector spaces. Suneja et al. [13] have studied K-convex functions in finite-dimensional vector spaces. Mititelu [11] has studied the concept of η-invex functions in the differentiable manifolds. We recall the concept of η-invex set.

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Definition 2.1 [8]. Let X be a topological vector space and $K \subset X$ a nonempty subset of X. K is said to be an *η-invex set* if there exists a vector function $\eta : K \times K \to X$such that $y + t\eta(x,y) \in K$ for all $x, y \in K$ and for all $t \in (0,1)$. For an example of η-invex set, see Example 2.7.

We make the following definitions for the vector function η for our need.

Definition 2.2 (Condition C_0). Let X be a topological vector space and K a nonempty subset of X. A vector function $\eta : K \times K \to X$ is said to *satisfy condition C_0* if the following hold:

$$\begin{gathered} \eta(x' + \eta(x,x'),x') + \eta(x',x' + \eta(x,x')) = 0, \\ \eta(x' + t\eta(x,x'),x') + t\eta(x,x') = 0 \end{gathered} \tag{2.1}$$

for all $x, x' \in K$, for all $t \in (0,1)$.

Example 2.3 (Condition C_0). Let $X = \mathbb{R}$. Let $K = [0,\infty)$ be any nonempty convex subset of X. Define the vector function $\eta : K \times K \to X$ by the rule $\eta(x,x') = x' - x$ for all $x, x' \in K$. $\eta(x' + \eta(x,x'),x') + \eta(x',x' + \eta(x,x')) = x' - x' - \eta(x,x') + x'_{+}\eta(x,x') - x' = 0$ and for all $t \in (0,1)$, $\eta(x' + t\eta(x,x'),x') = x' - x' - t\eta(x,x') = -t\eta(x,x')$. Thus $\eta(x' + t\eta(x,x'),x') + t\eta(x,x') = 0$ for all x, $x' \in K$.

Theorem 2.4. *Let $K \subset X$ be a nonempty pointed subset of the topological vector space X. Let $p : X \to X$ be a linear projective map ($p^2 = p$). Let $\eta : K \times K \to X$ be a vector-valued mapping defined by the rule $\eta(x,x') = p(x') - p(x)$. Then η satisfies condition C_0.*

Proof. For all $x, x' \in K$, $\eta(x' + \eta(x,x'),x') + \eta(x',x' + \eta(x,x')) = p(x') - p(x' + \eta(x,x')) + p(x' + \eta(x,x')) - p(x') = 0$. If p is projective, then $\eta(x' + t\eta(x,x'),x') = p(x') - p(x' + t\eta(x,x')) = p(x') - p(x') - p(t\eta(x,x')) = -tp(\eta(x,x')) = -tp(p(x') - p(x)) = -t(p^2(x') - p^2(x)) = -t(p(x') - p(x)) = -t\eta(x,x')$. □

We use the following result of Ky Fan in our work.

Theorem 2.5 [6, Theorem 4.3.1, page 116]. *Let K be an arbitrary nonempty set in a Housdorff topological vector space X. Let the set-valued mapping $F : K \to 2^X$ be a KKM map such that*

(a) *$F(x)$ is closed for all $x \in K$,*

(b) *$F(x)$ is compact for at least one $x \in K$.*

Then $\cap_{x \in K} F(x) \neq \emptyset$.

Theorem 2.6. *Let X be a topological vector space and let K be any nonempty η-invex subset of X. Let (Y,P) be an ordered topological vector space equipped with the closed convex pointed cone P with $\operatorname{int}P \neq \emptyset$. Let $L(X,Y)$ be the set of linear continuous functionals from X to Y. Let $T : K \to L(X,Y)$ and $\eta : K \times K \to X$ be continuous mappings. Assume that*

(a) *$\langle T(x),\eta(x,x)\rangle \notin -\operatorname{int}P$ for all $x \in K$,*

(b) *for each $u \in K$, the set $B(u) = \{x \in K : \langle T(u),\eta(x,u)\rangle \in -\operatorname{int}P\}$ is an η-invex set,*

(c) *the map $u \mapsto \langle T(u),\eta(x,u)\rangle$ is continuous on the finite-dimensional subspaces (or at least hemicontinuos),*

(d) *for at least one $x \in K$, the set $\{u \in K : \langle T(u),\eta(x,u)\rangle \notin -\operatorname{int}P\}$ is compact.*

Then there exists an $x_0 \in K$ such that $\langle T(x_0),\eta(x,x_0)\rangle \notin -\operatorname{int}P$ for all $x \in K$.

Proof. For each $u \in K$, consider the set-valued mapping $F : K \to 2^X$ defined by the rule $F(x) = \{u \in K : \langle T(u), \eta(x,u)\rangle \notin -\operatorname{int} P\}$ for all $x \in K$.

We assert that $F(x)$ is closed for each $x \in K$. Let $\{u_n\}$ be a sequence in $F(x)$ such that $u_n \to u$. Since $u_n \in F(x)$, we have $\langle T(u_n), \eta(x,u_n)\rangle \notin -\operatorname{int} P$ for all $x \in K$, that is, $\langle T(u_n), \eta(x,u_n)\rangle \in (Y - \{-\operatorname{int} P\})$ for all $x \in K$. Since T and η are continuous, we have $\langle T(u_n), \eta(x,u_n)\rangle \to \langle T(u), \eta(x,u)\rangle$. Since $(Y - \{-\operatorname{int} P\})$ is a closed set, $\langle T(u), \eta(x,u)\rangle \in (Y - \{-\operatorname{int} P\})$ for all $x \in K$, that is, $\langle T(u), \eta(x,u)\rangle \notin -\operatorname{int} P$ for all $x \in K$. Thus $u \in F(x)$ and $F(x)$ is closed.

We claim that F is a *KKM* mapping. If not there exists a finite subset $\{x_1, x_2, \dots, x_n\}$ of K such that $C_h(\{x_1, x_2, \dots, x_n\}) \not\subset \cup\{F(x) : x \in \{x_1, x_2, \dots, x_n\}\}$, that is, $C_h(\{x_1, x_2, \dots, x_n\}) \not\subset F(x)$ for any $x \in \{x_1, x_2, \dots, x_n\}$, where $C_h(\{x_1, x_2, \dots, x_n\})$ denotes the convex hull of $\{x_1, x_2, \dots, x_n\}$. Let $w \in C_h(\{x_1, x_2, \dots, x_n\})$ such that $w \notin \cup\{F(x) : x \in \{x_1, x_2, \dots, x_n\}\}$, that is, $w \notin F(x)$ for any $x \in \{x_1, x_2, \dots, x_n\}$. Note that $w \in K$. Thus $\langle T(w), \eta(x,w)\rangle \in -\operatorname{int} P$ for all $x \in \{x_1, x_2, \dots, x_n\}$; this shows that $x \in \{x_1, x_2, \dots, x_n\} \subset C_h(\{x_1, x_2, \dots, x_n\}) \subset B(w)$ (since every convex set is a subset of invex set). Hence $w \in B(w)$, that is, $\langle T(w), \eta(w,w)\rangle \in -\operatorname{int} P$, which contradicts (a). Thus F is a *KKM* mapping. Hence by Theorem 2.6, $\cap\{F(x) : x \in K\} \neq \emptyset$, that is, there exists $x_0 \in K$ such that $\langle T(x_0), \eta(x,x_0)\rangle \notin -\operatorname{int} P$ for all $x \in K$. $\square$

We illustrate Theorem 2.6 by an example.

Example 2.7. Let $X = \{si : s \in (-\infty, \infty)\}$, $K = \{si : s \in [0,\infty)\}$, $Y = \mathbb{R}$, $P = [0,\infty)$. Let $\eta : K \times K \to X$ be defined by $\eta(u,v) = u - v$. Let $T : K \to L(X,Y)$ be defined by $T(x) = -x$ for all $x \in K$ and $\langle T(u), x\rangle = T(u) \cdot x$ for all $u \in K$ and $x \in X$.

(a) For all $x \in K$, $\langle T(x), \eta(x,x)\rangle = x(x-x) =_P 0$.

(b) Let $a, b \in B(u)$. We show $b + t\eta(a,b) \in B(u)$. First we show that $t\langle T(u), \eta(a,u)\rangle + (1-t)\langle T(u), \eta(b,u)\rangle - \langle T(u), \eta(b + t\eta(a,b), u)\rangle = 0$ for all $a, b \in B(u)$ and for all $t \in (0,1) : t\langle T(u), \eta(a,u)\rangle + (1-t)\langle T(u), \eta(b,u)\rangle - \langle T(u), \eta(b + t\eta(a,b), u)\rangle = t(-u)(a-u) + (1-t)(-u)(b-u) - (-u)\eta(b + t(a-b), u) = (-u)(ta - tu + b - u - tb + tu) - (-u)(b + ta - tb - u) = (-u)(ta - tu + b - u - tb + tu - b - ta + tb + u) = (-u)0 = 0$. Hence for each $u \in K$, $tT(u)$, $\eta(a,u) + (1-t)T(u)$, $\eta(b,u) - T(u)$, $\eta(b + t\eta(a,b), u) = 0 \notin -\operatorname{int} P$ for all $a, b \in B(u)$.

As $a, b \in B(u)$, $\langle T(u), \eta(a,u)\rangle \in -\operatorname{int} P$, $\langle T(u), \eta(b,u)\rangle \in -\operatorname{int} P$. Now for any $t \in (0,1)$, $\langle T(u), \eta(b + t\eta(a,b), u)\rangle = t\langle T(u), \eta(a,u)\rangle + (1-t)\langle T(u), \eta(b,u)\rangle \in -\operatorname{int} P$ for all $a, b \in B(u)$. Hence $b + t\eta(a,b) \in B(u)$ for all $a, b \in B(u)$ and for all $t \in (0,1)$.

(c) It is obvious that the map $u \mapsto \langle T(u), \eta(x,u)\rangle$ is continuous.

(d) At $x = u$, $\langle T(u), \eta(x,u)\rangle = T(u)(x-u) = 0$ for all $u \in K$ showing for at least one $x = 0 \in K$, the set $\{u \in K : \langle T(u), \eta(x,u)\rangle \notin -\operatorname{int} P\}$ is compact.

All the conditions of Theorem 2.6 are satisfied. Hence there exists $0x_0 = 0 \in K$ solving the problem $\langle T(x_0), \eta(x,x_0)\rangle \notin -\operatorname{int} P$ for all $x \in K$.

We introduce the concept of T-η-invexity.

Let X be a topological vector space and let (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $\operatorname{int} P \neq \emptyset$. Let $L(X,Y)$ be the set of linear functional from X to Y and let $\eta : X \times X \to X$ be a vector-valued function, where K is any subset of X. Let $T : K \to L(X,Y)$ be an operator.

Definition 2.8. f is said to be T-η-*invex* in K if $f(x)-f(x')-\langle T(x'),\eta(x,x')\rangle \geq_P 0$ (i.e., $\notin -\operatorname{int}P$).

Remark 2.9. If $X=\mathbb{R}$, $Y=\mathbb{R}$, $K=(0,1)$, $T=\nabla f$, and $P=[0,\infty)$ and if f is differentiable, then Definition 2.8 coincides with the definition of a differentiable convex function.

Definition 2.10. T is said to be η-*monotone* if there exists a vector function $\eta: K\times K\to X$ such that $\langle T(x'),\eta(x,x')\rangle+\langle T(x),\eta(x',x)\rangle \notin \operatorname{int}P$ for all $x,x'\in K$.

Example 2.11. Let $X=\mathbb{R}$, $K=\mathbb{R}_+$, $Y=\mathbb{R}^2$, $P=\mathbb{R}^2_+$. Let $f:K\to Y$ be defined by $f(u)=\left[\begin{smallmatrix}u^2\\0\end{smallmatrix}\right]$ for all $u\in K$ and let $T:K\to L(X,Y)$ be defined by $T(u)=\left[\begin{smallmatrix}-2u\\0\end{smallmatrix}\right]$ for all $u\in K$, where $\langle T(u),x\rangle=T(u)\cdot x$, $u\in K$, $x\in X$. Define a vector function $\eta:K\times K\to X$ by $\eta(u,v)=u+v$ for all $u,v\in K$. Now for all $u,v\in K$, we have

$$\begin{aligned}
f(u)-f(v)-\langle T(v),\eta(u,v)\rangle &= \begin{bmatrix}u^2\\0\end{bmatrix}-\begin{bmatrix}v^2\\0\end{bmatrix}-\begin{bmatrix}-2v\\0\end{bmatrix}[u+v]\\
&=\begin{bmatrix}u^2-v^2+2v(u+v)\\0\end{bmatrix}\\
&=\begin{bmatrix}(u+v)^2\\0\end{bmatrix}\notin -\operatorname{int}P,\\
f(v)-f(u)-\langle T(u),\eta(v,u)\rangle &= \begin{bmatrix}v^2\\0\end{bmatrix}-\begin{bmatrix}u^2\\0\end{bmatrix}-\begin{bmatrix}-2u\\0\end{bmatrix}[v+u]\\
&=\begin{bmatrix}v^2-u^2+2u(v+u)\\0\end{bmatrix}\\
&=\begin{bmatrix}(v+u)^2\\0\end{bmatrix}\notin -\operatorname{int}P
\end{aligned}\tag{2.2}$$

showing f is T-η-invex in K. Next we have

$$\begin{aligned}
\langle T(v),\eta(u,v)\rangle+\langle T(u),\eta(v,u)\rangle &= \begin{bmatrix}-2v\\0\end{bmatrix}[u+v]+\begin{bmatrix}-2u\\0\end{bmatrix}[v+u]\\
&=\begin{bmatrix}-2(u+v)\\0\end{bmatrix}[u+v]\\
&=-\begin{bmatrix}2(u+v)\\0\end{bmatrix}[u+v]\notin \operatorname{int}P
\end{aligned}\tag{2.3}$$

for all $u,v\in K$, showing that T is η-monotone in K.

Example 2.12. Let $X = \mathbb{R}$, $K = \mathbb{R}_+$, $Y = \mathbb{R}^2$, $P = \mathbb{R}^2_+$. Let $T : K \to L(X,Y)$ be defined by $T(u) = \left[\begin{smallmatrix} -2u \\ -u \end{smallmatrix}\right]$ for all $u \in K$, where $\langle T(u), x\rangle = T(u)\cdot x$, $u \in K$, $x \in X$. Define a vector function $\eta : K \times K \to X$ by $\eta(u,v) = u+v$ for all $u,v \in K$. Now for all $u,v \in K$, we have

$$\begin{aligned}\langle T(v), \eta(u,v)\rangle + \langle T(u), \eta(v,u)\rangle &= \begin{bmatrix} -2v \\ -v \end{bmatrix}(u+v) + \begin{bmatrix} -2u \\ -u \end{bmatrix}(v+u) \\ &= \begin{bmatrix} -2 \\ -1 \end{bmatrix}(u+v)^2 \notin \operatorname{int} P \quad \forall u,v \in K,\end{aligned} \tag{2.4}$$

showing that T is η- monotone in K.

Proposition 2.13. *Let X be a topological vector space and let (Y,P) be an ordered topological vector space equipped with a closed pointed cone P such that $\operatorname{int} P \neq \emptyset$. Let $L(X,Y)$ be the set of linear functional from X to Y and let $\eta : X \times X \to X$ be a vector-valued function, where K is any subset of X. Let $T : K \to L(X,Y)$ be an operator. Let the function $f : K \to Y$ be T-η-invex in K. Then T is η-monotone.*

Proof. Let f be T-η-invex in K, then for all $x, x' \in K$, we have $f(x) - f(x') - \langle T(x'), \eta(x, x')\rangle \notin -\operatorname{int} P$. Interchanging x and x', we have $f(x') - f(x) - \langle T(x), \eta(x', x)\rangle \notin -\operatorname{int} P$. Adding the above we get $-\langle T(x'), \eta(x,x')\rangle - \langle T(x), \eta(x', x)\rangle \notin -\operatorname{int} P$, that is, $\langle T(x'), \eta(x, x')\rangle + \langle T(x), \eta(x', x)\rangle \notin \operatorname{int} P$ for all $x, x' \in K$, showing T is η-monotone. □

The converse of Proposition 2.13 is not true, as the following example shows.

Example 2.14. Let $X = Y = \mathbb{R}$ and $K = [-\pi/2, \pi/2]$, $P = [0,\infty]$, $T = \nabla f$ (derivative of f). Let $f : K \to \mathbb{R}$ be defined by $f(x) = \sin x$ for all $x \in K$ and $\eta : K \times K \to X$ defined by $\eta(x,x') = \cos x - \cos x'$ for all $(x,x') \in K \times K$. Then $\langle T(x'), \eta(x,x')\rangle + \langle T(x), \eta(x',x)\rangle = \nabla f(x')\eta(x,x') + \nabla f(x)\eta(x',x) = \cos x'(\cos x - \cos x') + \cos x(\cos x' - \cos x) = -(\cos x - \cos x')^2 \leq 0$ for all $x, x' \in K$, showing that T is η-monotone. But at $x = -\pi/3$ and $x' = \pi/6$, we have $f(x) - f(x') - \langle T(x'), \eta(x,x')\rangle < 0$, showing f is not T-η-invex.

3. A complementarity problem

In this section we study a vector complementarity problem in topological vector spaces. We use the notations of the following result.

Lemma 3.1 [5, Lemma 2.1]. *Let (V,P) be an ordered topological vector space with a closed, pointed, and convex cone P with $\operatorname{int} P \neq \emptyset$. Then, for all $y, z \in V$,*
(i) *$y - z \in \operatorname{int} P$ and $y \notin \operatorname{int} P$ imply $z \notin \operatorname{int} P$;*
(ii) *$y - z \in P$ and $y \notin \operatorname{int} P$ imply $z \notin \operatorname{int} P$;*
(iii) *$y - z \in -\operatorname{int} P$ and $y \notin -\operatorname{int} P$ imply $z \notin -\operatorname{int} P$;*
(iv) *$y - z \in -P$ and $y \notin -\operatorname{int} P$ imply $z \notin -\operatorname{int} P$.*

Remark 3.2. For simplicity, we use the following terminologies:
(a) $y \notin -\operatorname{int} P$ if and only if $y \geq_P 0$;
(b) $y \in \operatorname{int} P$ if and only if $y >_P 0$;
(c) $y \notin \operatorname{int} P$ if and only if $y \leq_P 0$;
(d) $y \in -\operatorname{int} P$ if and only if $y <_P 0$;

(e) $y - z \notin -\operatorname{int} P$ if and only if $y - z \geq_P 0$ (i.e., $y \geq_P z$);
(f) $y - z \notin \operatorname{int} P$ if and only if $y - z \leq_P 0$ (i.e., $y \leq_P z$);
(g) $y - z \notin (\operatorname{int} P \cup (-\operatorname{int} P))$ if and only if $y - z =_P 0$, (i.e., $y =_P z$).

We also use the following terminologies as and when required:
(A) $y - z \notin -P$ and $z \notin -\operatorname{int} P$ imply $y \notin -\operatorname{int} P$;
(B) $y - z \notin -\operatorname{int} P$ and $z \notin -\operatorname{int} P$ imply $y \notin -\operatorname{int} P$;
(C) $y - z \notin -P$ and $y \in -\operatorname{int} P$ imply $z \in -\operatorname{int} P$;
(D) $y - z \notin -\operatorname{int} P$ and $y \in -\operatorname{int} P$ imply $z \in -\operatorname{int} P$;
(E) $y - z \in -\operatorname{int} P$ and $z \in -\operatorname{int} P$ imply $y \in -\operatorname{int} P$;
(F) $y \notin -\operatorname{int} P$ if and only if $-y \notin \operatorname{int} P$;
(G) $y \notin -\operatorname{int} P$ and $z \notin -\operatorname{int} P$ imply $y + z \notin -\operatorname{int} P$.

Definition 3.3. Let X be a topological vector space and let (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $\operatorname{int} P \neq \emptyset$. Let K be any subset of X, let $L(X,Y)$ be the set of all linear functionals from X to Y, and let $\eta : K \times K \to X$ be a vector-valued function. Let $T : K \to L(X,Y)$ be an arbitrary map.

The *generalized vector variational inequality problem* (GVVI) and the *generalized vector complementarity problem* (GVCP) are defined as follows.

GVVI: find $x_0 \in K$ such that $\langle T(x_0), \eta(x,x_0)\rangle \notin -\operatorname{int} P$ for all $x \in K$.

GVCP: find $x_0 \in K$ such that $\langle T(x_0), \eta(x,x_0)\rangle \in (Y - (\operatorname{int} P \cup (-\operatorname{int} P)))$ for all $x \in K$.

We prove the following result concerning GVCP.

Theorem 3.4. *Let K be a nonempty compact cone in a topological vector space X and let (Y,P) be an ordered topological vector space equipped with a convex pointed cone P such that $\operatorname{int} P \neq \emptyset$. Let $L(X,Y)$ be the set of all linear functionals from X to Y and let $\eta : K \times K \to X$ be a continuous vector-valued function. Let $T : K \to L(X,Y)$ be an arbitrary continuous map and let K be η-invex.*

Let the following conditions hold:
(a) $\langle T(x), \eta(x,x)\rangle =_P 0$ *for all* $x \in K$,
(b) *for each* $u \in K$, *the set* $B(u) = \{x \in K : \langle T(u)\eta(x,u)\rangle \in -\operatorname{int} P\}$ *is an η-invex set,*
(c) η *satisfies condition* C_0.

Then there exists $x_0 \in K$ *such that* $\langle T(x_0), \eta(x,x_0)\rangle \in (Y - (\operatorname{int} P \cup (-\operatorname{int} P)))$ *for all* $x \in K$.

Proof. By Theorem 2.6, there exists $x_0 \in K$ such that $\langle T(x_0), \eta(x,x_0)\rangle \notin -\operatorname{int} P$ for all $x \in K$, that is, $\langle T(x_0), \eta(x,x_0)\rangle \geq_P 0$ for all $x \in K$. Since K is η-invex cone, $x_0 + t\eta(x,x_0) \in K$ for all $x \in K$. Replacing x by $x_0 + t\eta(x,x_0)$, we get $\langle T(x_0), \eta(x_0 + t\eta(x,x_0), x_0)\rangle \geq_P 0$ for all $x \in K$. Thus $0 \leq_P \langle T(x_0), \eta(x_0 + t\eta(x,x_0), x_0)\rangle = \langle T(x_0), -t\eta(x,x_0)\rangle = -t\langle T(x_0), \eta(x,x_0)\rangle$ (by (c)) and hence $-t\langle T(x_0), \eta(x,x_0)\rangle \notin -\operatorname{int} P$ for all $x \in K$. Since $t > 0$, we have $\langle T(x_0), \eta(x,x_0)\rangle \notin \operatorname{int} P$. Hence $\langle T(x_0), \eta(x,x_0)\rangle \notin \{-\operatorname{int} P\} \cup \{\operatorname{int} P\}$ for all $x \in K$ showing $\langle T(x_0), \eta(x,x_0)\rangle \in (Y - (\operatorname{int} P \cup (-\operatorname{int} P)))$ for all $x \in K$. □

4. Variational inequality problems in H-spaces

In the recent past H-spaces have become an interesting area of research domain for studying variational-type inequality [1, 14] because most of the pivotal concepts such as convex sets, weakly convex sets, and *KKM* maps in Banach spaces are, respectively, replaced by

H-convex sets, H-weakly convex sets, and H-KKM maps in H-spaces. In this section we establish an inequality in H-space and obtain the traditional variational and variational-type inequalities as particular cases of the newly obtained inequality. We also discuss the uniqueness of the solutions of the inequality with examples.

Several generalizations of the celebrated Ky Fan minimax inequality [7] have already appeared. This study requires the use of *KKM* theorem. In [1] Bardaro and Ceppitelli have explained the necessity of generalizing the reformulation of the *KKM* theorem for generalizing minimax inequality for functions taking values in ordered vector spaces.

In this section we prove certain results in H-spaces.

Definition 4.1 [1]. Let X be a topological space and let $\{\Gamma_A\}$ be a given family of nonempty contractible subsets of X, indexed by finite subsets of X. A pair $(X,\{\Gamma_A\})$ is said to be an H-*space* if $A\subset B$ implies $\Gamma_A\subset\Gamma_B$.

Let $(X,\{\Gamma_A\})$ be an H-space. A subset $D\subset X$ is said to be H-*convex* if $\Gamma_A\subset D$ for every finite subset $A\subset D$.

A subset $D\subset X$ is said to be *weakly* H-*convex* if $\Gamma_A\cap D$ is nonempty and contractible for every finite subset $A\subset D$. This is equivalent to saying that the pair $(D,\{\Gamma_A\cap D\})$ is an H-space.

A subset $K\subset X$ is said to be H-*compact* if there exists a compact and weakly H-convex set $D\subset X$ such that $K\cup A\subset D$ for every finite subset $A\subset X$.

In a given H-space a multifunction $F:X\to 2^X$ is said to be H-*KKM* if $\Gamma_A\subset\cup\{F(x):x\in A\}$ for every finite subset $A\subset X$.

In this section we present an application of [1, Theroem 1, page 486]. In fact we establish an inequality associated with the variational inequality or variational-type inequality.

Let X be a topological vector space, let (Y,P) be an ordered topological vector space equipped with closed convex pointed cone with $\operatorname{int}P\neq\varnothing$, and let $L(X,Y)$ be the set of continuous linear functionals from X to Y. Let the value of $f\in L(X,Y)$ at $x\in X$ be denoted by $\langle f,x\rangle$. Let K be a convex set in X with $0\in K$ and let $T:K\to L(X,Y)$ be any map. The *variational inequality problem* is to find $x_0\in K$ such that $\langle T(x_0),v-x_0\rangle\notin -\operatorname{int}P$ for all $v\in K$.

Theorem 4.2. *Let $(X,\{\Gamma_A\})$ be an H-space. Assume that X is Hausdorff. Let N be a subset of $X\times X$ having the following properties.*

(a) *For each $x\in X$, $(x,x)\in N$.*

(b) *For each fixed $y\in X$, the set $N(y)=\{x\in X:(x,y)\in N\}$ is closed in X.*

(c) *For each $x\in X$, the set $M(x)=\{y\in X:(x,y)\notin N\}$ is H-convex.*

(d) *There exists a compact set $L\subset X$ and an H-compact set $W\subset X$ such that for each weakly H-convex set D with $W\subset D\subset X$, $\cap_{y\in D}(\{x\in X:(x,y)\in N\}\cap D)\subset L$.*

Then there exists $x_0\in X$ such that $\{x_0\}\times X\subset N$.

Proof. Define a set-valued map $F:X\to 2^X$ by the rule $F(y)=\{x\in X:(x,y)\in N\}$. By (a), $F(y)\neq\varnothing$ for each $y\in X$. Since X is Hausdorff, by (b), $F(y)$ is compactly closed for each $y\in X$. We assert that F is an H-KKM map. Suppose to the contrary that F is not an H-KKM map .Then there exists a finite set $A\subset X$ such that $\Gamma_A\not\subset\cup_{y\in A}F(y)$. Thus there exists some $u\in\Gamma_A$ such that $(u,y)\notin N$ for all $y\in A$. Let $M(u)=\{y\in X:(u,y)\notin N\}$. By (c), $M(u)$ is H-convex . We observe that $A\subset M(u)$. By the H-convexity of $M(u)$, $\Gamma_A\subset M(u)$.

Thus $u \in M(u)$, that is, $(u,u) \notin N$, which is a contradiction to (a). Hence F is an H-KKM map.

By (d) there exists a compact set $L \subset X$ and an H-compact set $W \subset X$ such that for each weakly H-convex set D with $W \subset D \subset X$, we have $\cap_{y\in D}(\{x \in X : (x,y) \in N\} \cap D) \subset L$, that is, $\cap_{y\in D}(F(y) \cap D) \subset L$. Thus by [1, Theorem 1, page 486], $\cap_{y\in X} F(y) \neq \emptyset$. Hence there exists $x_0 \in X$ such that $\{x_0\} \times X \subset N$. □

The following result is a slightly different version of Theorem 4.2.

Theorem 4.3. *Let $(X,\{\Gamma_A\})$ be an H-space and let (Y,P) be an ordered topological vector space equipped with closed convex pointed cone with $\operatorname{int} P \neq \emptyset$. Let K be a convex set in X, with $0 \in K$. Assume that X is Hausdorff. Let $f : K \times K \to Y$ be a continuous map having the following properties.*

(a) *For each $x \in X$, $f(x,x) \notin -\operatorname{int} P$.*
(b) *For each fixed $v \in K$, the set $\{x \in K : f(x,v) \notin -\operatorname{int} P\}$ is closed in X.*
(c) *For each $x \in K$, the set $\{v \in K : f(x,v) \in -\operatorname{int} P\}$ is H-convex.*
(d) *There exists a compact set $L \subset X$ and an H-compact set $W \subset X$, such that for each weakly H-convex set D with $W \subset D \subset X$, $\cap_{v\in D}(\{x \in K : f(x,v) \notin -\operatorname{int} P\} \cap D) \subset L$.*

Then there exists $x_0 \in K$ such that $f(x_0,v) \notin -\operatorname{int} P$ for all $v \in K$.

Proof. Let $N = \{(x,v) : f(x,v) \notin -\operatorname{int} P\} \subset K \times K$. By (a), N is nonempty since $(x,x) \in N$ for each $x \in K$. For each $v \in K$ consider the set $N(v) = \{x \in X : (x,v) \in N\} = \{x \in X : f(x,v) \notin -\operatorname{int} P\}$. By (b), $N(v)$ is closed for each $v \in K$. By (c), for each $x \in K$, the set $M(x) = \{v \in K : (x,v) \notin N\} = \{v \in K : f(x,v) \in -\operatorname{int} P\}$ is H-convex . By (d), there exists a compact set $L \subset X$ and an H-compact set $W \subset X$ such that for each weakly H-convex set D with $W \subset D \subset X$, we have $\cap_{y\in D}(\{x \in K : f(x,v) \in N\} \cap D) \subset L$. Thus all the conditions of Theorem 4.2 are satisfied and hence there exists $x_0 \in K$ such that $\{x_0\} \times K \subset N$, that is, $(x_0,v) \in N$ for all $v \in K$. This means there exists $x_0 \in K$ such that $f(x_0,v) \notin -\operatorname{int} P$ for all $v \in K$. □

Remarks 4.4. In Theorem 4.3 we take X to be a Hausdorff topological real vector space with dual X^*, $Y = \mathbb{R}$ and $P = [0,\infty)$. Clearly X is an H-space. Let K be a nonempty convex subset of X. Let $T : K \to X^*$, $\eta : K \times K \to X$, $\theta : K \times K \to \mathbb{R}$, $g : K \to X$ be continuous functions satisfying some appropriate conditions as and when required. We consider the following cases.

Case 1. In Theorem 4.3 if we define $f : K \times K \to \mathbb{R}$ by the rule $f(x,y) = \langle Tx, y - x\rangle$, then there exists $x_0 \in K$ such that $\langle Tx_0, y - x_0\rangle \geq 0$ for all $y \in K$, which is the variational inequality as given in [4, Theorem 1, page 780]; also see [9, Theorem 4.32, page 116], [12, Theorem 1, page 90].

Case 2. In Theorem 4.3 if we define $f : K \times K \to \mathbb{R}$ by the rule $f(x,y) = \langle Tx, \eta(y,x)\rangle$, then there exists $x_0 \in K$ such that $\langle Tx_0, \eta(y,x_0)\rangle \geq 0$ for all $y \in K$, which is the variational-type inequality as given in [2, Theorem 2.1, Theorem 2.2, page 184].

Case 3. In Theorem 4.3 if we define $f : K \times K \to \mathbb{R}$ by the rule $f(x,y) = \langle Tx, y - g(x)\rangle$, then there exists $x_0 \in K$ such that $\langle Tx_0, y - g(x_0)\rangle \geq 0$ for all $y \in K$, which is the variational inequality as given in [9, Proposition 6. 2.2, page 170].

Case 4. In Theorem 4.3 if we define $f: K \times K \to \mathbb{R}$ by the rule $f(x,y) = \langle Tx, \eta(y,x)\rangle + \theta(x,y)$, then there exists $x_0 \in K$ such that $\langle Tx_0, \eta(y,x_0)\rangle + \theta(x_0,y) \geq 0$ for all $y \in K$, which is the variational-type inequality as given in [3, Theorem 2.1, page 346; Theorem 2.2, page 347].

The following result characterizes the uniqueness of the solution of the inequality $f(x_0,v) \notin -\operatorname{int} P$ obtained in Theorem 4.3.

Theorem 4.5. *Let $(X,\{\Gamma_A\})$ be an H-space and let (Y,P) be an ordered topological vector space equipped with closed convex pointed cone with* $\operatorname{int} P \neq \emptyset$. *Let K be a convex set in X, with $0 \in K$. Assume that X is Hausdorff. Let $f: K \times K \to Y$ be a continuous map such that*

(a) $f(x,v) + f(v,x) \notin \operatorname{int} P$ *for all* $x, v \in K$,

(b) $f(x,v) + f(v,x) =_P 0$ *implies* $x = v$.

Then if the problem, find x_0 such that $f(x_0,v)$ for all $v \in K$, is solvable, then it has a unique solution.

Proof. Let $x_1, x_2 \in K$ be such that $f(x_1,v) \notin -\operatorname{int} P$ and $f(x_2,v) \notin -\operatorname{int} P$ for all $v \in K$; putting $v = x_2$ in the former inequality and $v = x_1$ in the later inequality we see that $f(x_1,x_2) \notin -\operatorname{int} P$ and $f(x_2,x_1) \notin -\operatorname{int} P$ and on adding we get $f(x_1,x_2) + f(x_2,x_1) \notin -\operatorname{int} P$. This combined with inequality (a) gives $f(x_1,x_2) + f(x_2,x_1) =_P 0$. Hence by (b), we have $x_1 = x_2$. □

The following examples illustrate Theorem 4.5. Example 4.6, given below, shows that fulfillment of conditions (a) and (b) does not guarantee the existence of the solution of the problem, stated in Theorem 4.5.

Example 4.6. Let $X = \mathbb{R}$ and define $f: X \times X \to \mathbb{R}$ by $f(x,v) = -e^{-x}|x - v|$. Clearly $f(x,v) + f(v,x) = -(e^{-x} + e^{-v})|x - v| \leq 0$. Furthermore $f(x,v) + f(v,x) = 0$ implies that $x = v$. It is clear that there is no $x_0 \in K$ satisfying $f(x_0,v) = -e^{-x_0}|x_0 - v| \geq 0$ for all $v \in X$.

In Example 4.7 the function $f: X \times X \to \mathbb{R}$ satisfies conditions (a) and (b) of Theorem 4.5 and at the same time the problem stated in Theorem 4.5 has a unique solution.

Example 4.7. Let $X = [0,\infty)$ and define $f: X \times X \to \boldsymbol{R}$ by $f(x,v) = -x^2|x - v|$. Clearly $f(x,v) + f(v,x) = -(x^2 + v^2)|x - v| \leq 0$. Furthermore $f(x,v) + f(v,x) = 0$ implies that either $x^2 + v^2 = 0$ or $|x - v| = 0$; since x^2 and v^2 are nonnegative, when $x^2 + v^2 = 0$, we have $x = 0$ and $v = 0$ and when $|x - v| = 0$ we have certainly $x = v$. Thus conditions (a) and (b) of Theorem 4.5 hold. In this example we have a unique solution $x_0 = 0$ to the problem of Theorem 4.5, for $f(x_0,v) \geq 0$ for all $v \in X$ implies $-x_0^2|x_0 - v| \geq 0$ for all $v \in X$; since $|x_0 - v| \neq 0$, the only solution is $x_0 = 0$.

We explore some characteristics of generalized vector variational inequality problems in H-spaces.

Theorem 4.8. *Let $(X,\{\Gamma_A\})$ be an H-space and let (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that* $\operatorname{int} P \neq \emptyset$ *and $K \subset X$. Let $L(X,Y)$ be the set of all linear functionals from X to Y and let $T: K \to L(X,Y)$ be a mapping.*

Let $\eta : K \times K \to X$ be a vector valued function and let $f : K \to Y$ be continuous. Assume that the following conditions hold:

(a) *$f(x) \notin -\operatorname{int} P$ for all $x \in K$;*

(b) *$f : K \to Y$ is T-η-invex in K;*

(c) *for each $u \in K$ the set $B_u = \{x \in X : f(u) - f(x) \in -\operatorname{int} P\}$ is either H-convex or empty.*

Then $f(x') - \langle T(x'), \eta(x,x')\rangle \notin -\operatorname{int} P$ for all $x, x' \in K$.

Proof. Define a set-valued mapping $F : K \to 2^X$ by the rule $F(x) = \{x' \in X : f(x') - \langle T(x'), \eta(x,x')\rangle \notin -\operatorname{int} P\}$ for each $x \in K$. Clearly $F(x)$ is nonempty for each $x \in K$.

It is enough to prove that F is an H-*KKM* mapping. If not, then there exists a finite set $A \subset K$ such that $\Gamma_A \notin \cup\{F(x) : x \in A\}$. Let $z \in \Gamma_A$ such that $z \notin \cup\{F(x) : x \in A\}$. Thus $z \notin F(x)$ for all $x \in A$, that is, $f(z) - \langle T(z), \eta(x,z)\rangle \in -\operatorname{int} P$ for all $x \in A$. Since f is T-η-invex in X, at z, we have $f(x) - f(z) - \langle T(z), \eta(x,z)\rangle \notin -\operatorname{int} P$ for all $x \in A$. By (a), $f(z) \notin -\operatorname{int} P$, so by Lemma 3.1(i) we have $f(x) - \langle T(z), \eta(x,z)\rangle \notin -\operatorname{int} P$ for all $x \in A$. Thus $f(x) - \langle T(z), \eta(x,z)\rangle \in -\operatorname{int} P$ for all $x \in A$, that is, $\langle T(z), \eta(x,z)\rangle - f(x) \in -\operatorname{int} P$ for all $x \in A$ and we obtain $f(z) - f(x) \in -\operatorname{int} P$ for all $x \in A$. Hence $x \in B_z$ for all $x \in A$. Thus $A \subset B_z$ and by H-convexity of B_z, $\Gamma_A \subset B_z$. Since $z \in \Gamma_A$, we have $z \in B_z$, that is, $f(z) - f(z) \in -\operatorname{int} P$ giving $0 \in -\operatorname{int} P$, which is a contradiction because by the pointedness condition of P, $0 \in P \cap (-P)$ implies that $0 \notin (\operatorname{int} P) \cup (-\operatorname{int} P)$. Hence F is an H-*KKM* mapping. □

Theorem 4.9. *Let $(X, \{\Gamma_A\})$ be an H-space and let (Y, P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $\operatorname{int} P \neq \emptyset$. Let $L(X,Y)$ be the set of all linear functionals from X to Y and let $T : K \to L(X,Y)$ be a mapping. Let $\eta : K \times K \to X$ be a vector-valued function and let $f : K \to Y$ be continuous. Assume that the following conditions hold:*

(a) *$f(x) \notin -\operatorname{int} P$ for all $x \in K$;*

(b) *$f : K \to Y$ is T-η-invex in K;*

(c) *for each $u \in K$ the set $B_u = \{x \in X : f(u) - f(x) \in -\operatorname{int} P\}$ is either H-convex or empty;*

(d) *the mapping $v \mapsto f(v) - \langle T(v), \eta(x,v)\rangle$ of K into Y is continuous;*

(e) *there exists a compact set $L \subset X$ and an H-compact $C \subset X$, such that, for each weakly H-convex set D with $C \subset D \subset X$, $\cap\{x \in D : f(x') - \langle T(x'), \eta(x,x')\rangle \notin -\operatorname{int} P\} \subset L$.*

Then there exists $x_0 \in K$ such that $f(x_0) - \langle T(x_0), \eta(x,x_0)\rangle \notin -\operatorname{int} P$ for all $x \in K$.

Proof. As in the proof of Theorem 4.8, for each $x \in K$ the set-valued mapping $F(x) = \{x' \in X : f(x') - \langle T(x')\eta(x,x')\rangle \notin -\operatorname{int} P\}$ is an H-*KKM* mapping. We show that $\cap\{F(x) : x \in X\} \neq \emptyset$. By [1, Theorem 1, page 486] we need to show that F is closed. Let $\{z_n\} \subset F(x)$ be a sequence in $F(x)$ where $z_n \to z$, then we show that $z \in F(x)$. Since the map $z \mapsto f(z) - \langle T(z), \eta(x,z)\rangle$ is continuous, we have $f(z) - \langle T(z_n), \eta(x,z_n)\rangle \to f(z) - \langle T(z)\eta(x,z)\rangle$. But we have $f(z_n) - \langle T(z_n), \eta(x,z_n)\rangle \notin -\operatorname{int} P$. Thus $f(z_n) - \langle T(z_n), \eta(x,z_n)\rangle \in (Y - \{-\operatorname{int} P\})$. But $(Y - \{-\operatorname{int} P\})$ is closed. Therefore $f(z) - \langle T(z), \eta(x,z)\rangle \in (Y - \{-\operatorname{int} P\})$ giving $f(z) - \langle T(z), \eta(x,z)\rangle \notin -\operatorname{int} P$. Hence $z \in F(x)$. Thus there exists $x_0 \in \cap\{F(x) : x \in X\}$, such that $f(x_0) - \langle T(x_0), \eta(x,x_0)\rangle \notin -\operatorname{int} P$ for all $x \in K$. This proves the theorem. □

The following result is a direct consequence of Theorem 4.9.

Theorem 4.10. *Let $(X,\{\Gamma_A\})$ be an H-space and let (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $\operatorname{int}P \neq \emptyset$. Let $L(X,Y)$ be the set of all linear functionals from X to Y and let $T: K \to L(X,Y)$ be a mapping. Let $\eta: K\times K \to X$ be a vector-valued function and let $f: K \to Y$ be continuous. Assume that the following conditions hold:*

(a) *$f(x) \notin -\operatorname{int}P$ for all $x \in K$;*
(b) *$f: K \to Y$ is T-η-invex in K;*
(c) *for each $u \in K$ the set $B_u = \{x \in X : f(u) - f(x) \in -\operatorname{int}P\}$ is either H-convex or empty;*
(d) *the mapping $v \mapsto f(v) - \langle T(v), \eta(x,v)\rangle$ of K into Y is continuous;*
(e) *there exists a compact set $L \subset X$ and an H-compact $C \subset X$, such that, for each weakly H-convex set D with $C \subset D \subset X$, $\cap\{x \in D : f(x') - \langle T(x'), \eta(x,x')\rangle \notin -\operatorname{int}P\} \subset L$;*
(f) *$\langle T(x'), \eta(x,x')\rangle \geq_P 0$ for all $x,x' \in K$, $x \neq x'$.*

Then there exists $x_0 \in K$ such that

(i) *$f(x) - T(x)$, $\eta(x_0,x) \notin -\operatorname{int}P$ for all $x \in K$;*
(ii) *$\langle T(x_0), \eta(x,x_0)\rangle - \langle T(x), \eta(x_0,x)\rangle \notin -\operatorname{int}P$ for all $x \in K$;*
(iii) *$f(x) - f(x_0) \notin -\operatorname{int}P$ for all $x \in K$;*
(iv) *$\{f(x) - \langle T(x), \eta(x_0,x)\rangle\} - \{f(x_0) - \langle T(x_0), \eta(x,x_0)\rangle\} \notin -\operatorname{int}P$ for all $x \in K$.*

Proof. (i) Theorem 4.9 gives the existence of $x_0 \in K$. Since T is η-monotone (Proposition 2.13) $\langle T(x_0), \eta(x,x_0)\rangle + \langle T(x), \eta(x_0,x)\rangle \leq_P 0$ for all $x \in K$. By (f), at $x' = x_0$, we have $-\langle T(x), \eta(x_0,x)\rangle \geq_P \langle T(x_0), \eta(x,x_0)\rangle \geq_P 0$ (by (f)) for all $x \in K$ and hence by (a) $f(x) - \langle T(x), \eta(x_0,x)\rangle \geq_P 0$ for all $x \in K$, that is, $f(x) - \langle T(x), \eta(x_0,x)\rangle \notin -\operatorname{int}P$ for all $x \in K$.

(ii) From (f), at $x' = x_0$, we have $\langle T(x_0), \eta(x,x_0)\rangle \geq_P 0$ for all $x \in K$. As in the proof of (i) $-\langle T(x), \eta(x_0,x)\rangle \geq_P 0$. Thus $\langle T(x_0), \eta(x,x_0)\rangle - \langle T(x), \eta(x_0,x)\rangle \geq_P 0$ for all $x \in K$, that is, $\langle T(x_0), \eta(x,x_0)\rangle - \langle T(x), \eta(x_0,x)\rangle \notin -\operatorname{int}P$ for all $x \in K$.

(iii) By T-η-invexity of f, $f(x) - f(x_0) - \langle T(x_0), \eta(x,x_0)\rangle \geq_P 0$ for all $x \in K$, that is, $f(x) - f(x_0) \geq_P \langle T(x_0), \eta(x,x_0)\rangle \geq_P 0$ (by (f)) for all $x \in K$, that is, $f(x) - f(x_0) \notin -\operatorname{int}P$ for all $x \in K$.

(iv) Addition of (ii) and (iii) gives $\{f(x) - \langle T(x), \eta(x_0,x)\rangle\} - \{f(x_0) - \langle T(x_0), \eta(x, x_0)\rangle\} \geq_P 0$ for all $x \in K$, that is, $\{f(x) - \langle T(x), \eta(x_0,x)\rangle\} - \{f(x_0) - \langle T(x_0), \eta(x,x_0)\rangle\} \notin -\operatorname{int}P$ for all $x \in K$. □

Theorem 4.11. *Let $(X,\{\Gamma_A\})$ be an H-space and let (Y,P) be an ordered topological vector space equipped with a closed convex pointed cone P such that $\operatorname{int}P \neq \emptyset$. Let $L(X,Y)$ be the set of all linear functionals from X to Y and let $T: K \to L(X,Y)$ be a mapping. Let $\eta: K\times K \to X$ be a vector-valued function and let $f: K \to Y$ be continuous. Assume that the following conditions hold:*

(a) *$f(x) \notin -\operatorname{int}P$ for all $x \in K$;*
(b) *$f: K \to Y$ is T-η-invex in K;*
(c) *for each $u \in K$ the set $B_u = \{x \in X : f(u) - f(x) \in -\operatorname{int}P\}$ is either H-convex or empty;*
(d) *the mapping $v \mapsto f(v) - \langle T(v), \eta(x,v)\rangle$ of K into Y is continuous;*
(e) *there exists a compact set $L \subset X$ and an H-compact $C \subset X$, such that, for each weakly H-convex set D with $C \subset D \subset X$, $\cap\{x \in D : f(x') - \langle T(x'), \eta(x,x')\rangle \notin -\operatorname{int}P\} \subset L$;*

(f) $\langle T(x'),\eta(x,x')\rangle \geq_P 0$ *for all* $x,x' \in K$, $x \neq x'$.

Then there exists $x_0 \in K$ *such that* $\{f(x)-\langle T(x),\eta(x_0,x)\rangle\}-\{f(x_0)-\langle T(x_0),\eta(x,x_0)\rangle\} \notin -\operatorname{int}P$ *for all* $x \in K$.

Proof. Theorem 4.9 gives the existence of $x_0 \in K$. Since f is T-η-invex in K $f(x)-f(x_0)-\langle T(x_0),\eta(x,x_0)\rangle \notin -\operatorname{int}P$ for all $x \in K$, that is, $f(x)-f(x_0)-\langle T(x_0),\eta(x,x_0)\rangle \geq_P 0$ for all $x \in K$. Thus $f(x)-\langle T(x),\eta(x_0,x)\rangle - f(x_0) \geq_P \langle T(x_0),\eta(x,x_0)\rangle - \langle T(x),\eta(x_0,x)\rangle$ for all $x \in K$. Again since T is η-monotone, we have $\langle T(x_0),\eta(x,x_0)\rangle + \langle T(x),\eta(x_0,x)\rangle \notin \operatorname{int}P$ for all $x \in K$, that is, $\langle T(x_0),\eta(x,x_0)\rangle + \langle T(x),\eta(x_0,x)\rangle \leq_P 0$ for all $x \in K$. Thus $-\langle T(x_0),\eta(x,x_0)\rangle - \langle T(x),\eta(x_0,x)\rangle \geq_P 0$ for all $x \in K$, that is, $-\langle T(x),\eta(x_0,x)\rangle \geq_P \langle T(x_0),\eta(x,x_0)\rangle$ for all $x \in K$. Since $\langle T(x_0),\eta(x,x_0)\rangle \geq_P 0$, we have, $-\langle T(x),\eta(x_0,x)\rangle \geq_P -\langle T(x_0),\eta(x,x_0)\rangle$ for all $x \in K$. From the above we conclude that $f(x)-\langle T(x),\eta(x_0,x)\rangle - f(x_0) \geq_P \langle T(x_0),\eta(x,x_0)\rangle - \langle T(x_0),\eta(x,x_0)\rangle$, that is, $f(x)-\langle T(x),\eta(x_0,x)\rangle - f(x_0) + \langle T(x_0),\eta(x,x_0)\rangle \geq_P \langle T(x_0),\eta(x,x_0)\rangle \geq_P 0$. Hence $\{f(x)-\langle T(x),\eta(x_0,x)\rangle\}-\{f(x_0)-\langle T(x_0),\eta(x,x_0)\rangle\} \geq_P 0$ for all $x \in K$, that is, $\{f(x)-\langle T(x),\eta(x_0,x)\rangle\}-\{f(x_0)-\langle T(x_0),\eta(x,x_0)\rangle\} \notin -\operatorname{int}P$ for all $x \in K$. □

Theorem 4.12. *Let* $(X,\{\Gamma_A\})$ *be an* H*-space and let* (Y,P) *be an ordered topological vector space equipped with a closed convex pointed cone* P *such that* $\operatorname{int}P \neq \emptyset$. *Let* $L(X,Y)$ *be the set of all linear functionals from* X *to* Y *and let* $T : K \to L(X,Y)$ *be a mapping. Let* $\eta : K \times K \to X$ *be a vector-valued function and let* $f : K \to Y$ *be continuous. Assume that the following conditions hold:*

(a) $f(x) \notin -\operatorname{int}P$ *for all* $x \in K$;

(b) $f : K \to Y$ *is* T-η*-invex in* K;

(c) *for each* $u \in K$ *the set* $B_u = \{x \in X : f(u)-f(x) \in -\operatorname{int}P\}$ *is either* H*-convex or empty;*

(d) *the mapping* $v \mapsto f(v)-\langle T(v),\eta(x,v)\rangle$ *of* K *into* Y *is continuous;*

(e) *there exists a compact set* $L \subset X$ *and an* H*-compact* $C \subset X$, *such that, for each weakly* H*-convex set* D *with* $C \subset D \subset X$, $\cap\{x \in D : f(x')-\langle T(x'),\eta(x,x')\rangle \notin -\operatorname{int}P\} \subset L$;

(f) $\langle T(x'),\eta(x,x')\rangle \geq_P 0$ *for all* $x,x' \in K$, $x \neq x'$.

Then there exists $x_0 \in K$ *such that* $f(x)-f(x_0) \notin -\operatorname{int}P$ *for all* $x \in K$.

Proof. Theorem 4.9 gives the existence of $x_0 \in K$. Since f is T-η-invex in K $f(x)-f(x_0)-\langle T(x_0),\eta(x,x_0)\rangle \notin -\operatorname{int}P$ for all $x \in K$, that is, $f(x)-f(x_0)-\langle T(x_0),\eta(x,x_0)\rangle \geq_P 0$ for all $x \in K$. By (f), at $x' = x_0$, we have $\langle T(x_0),\eta(x,x_0)\rangle \geq_P 0$. Hence $f(x)-f(x_0) \geq_P 0$, that is, $f(x)-f(x_0) \notin -\operatorname{int}P$ for all $x \in K$. □

Theorem 4.13. *Let* $(X,\{\Gamma_A\})$ *be an* H*-space and let* (Y,P) *be an ordered topological vector space equipped with a closed convex pointed cone* P *such that* $\operatorname{int}P \neq \emptyset$. *Let* $L(X,Y)$ *be the set of all linear functionals from* X *to* Y *and let* $T : K \to L(X,Y)$ *be a mapping. Let* $\eta : K \times K \to X$ *be a vector-valued function and let* $f : K \to Y$ *be continuous. Assume that the following conditions hold:*

(a) $f(x) \notin -\operatorname{int}P$ *for all* $x \in K$;

(b) $f : K \to Y$ *is* T-η*-invex in* K;

(c) *for each* $u \in K$ *the set* $B_u = \{x \in X : f(u)-f(x) \in -\operatorname{int}P\}$ *is either* H*-convex or empty;*

(d) *the mapping* $v \mapsto f(v) - \langle T(v), \eta(x,v)\rangle$ *of* K *into* Y *is continuous;*
(e) *there exists a compact set* $L \subset X$ *and an* H*-compact* $C \subset X$*, such that, for each weakly* H*-convex set* D *with* $C \subset D \subset X$, $\cap\{x \in D : f(x') - \langle T(x'), \eta(x,x')\rangle \notin -\operatorname{int} P\} \subset L$;
(f) $\langle T(x'), \eta(x,x')\rangle \geq_P 0$ *for all* $x, x' \in K$, $x \neq x'$;
(g) $\langle T(x'), \eta(x,x')\rangle - \langle T(x), \eta(x',x)\rangle \geq_P 0$ *for all* $x, x' \in K$, $x \neq x'$.

Then there exists $x_0 \in K$ *such that* $f(x) - \langle T(x), \eta(x_0,x)\rangle \notin -\operatorname{int} P$ *for all* $x \in K$.

Proof. Theorem 4.9 gives the existence of $x_0 \in K$. By condition (g), at $x' = x_0$, $\langle T(x_0), \eta(x, x_0)\rangle - \langle T(x), \eta(x_0,x)\rangle \notin -\operatorname{int} P$ for all $x \in K$, that is, $\langle T(x_0), \eta(x,x_0)\rangle - \langle T(x), \eta(x_0,x)\rangle \geq_P 0$ for all $x \in K$. Since f is T-η-invex in K, we have $f(x) - f(x_0) - \langle T(x_0), \eta(x,x_0)\rangle \geq_P 0$ for all $x \in K$. Adding the above two inequalities, we get $f(x) - f(x_0) - \langle T(x), \eta(x_0,x)\rangle \geq_P 0$ for all $x \in K$, that is, $f(x) - \langle T(x), \eta(x_0,x)\rangle \geq_P f(x_0) \geq_P 0$ for all $x \in K$. Hence $f(x) - \langle T(x), \eta(x_0,x)\rangle \notin -\operatorname{int} P$ for all $x \in K$. □

THEOREM 4.14. *Let* $(X, \{\Gamma_A\})$ *be an* H*-space and let* (Y,P) *be an ordered topological vector space equipped with a closed convex pointed cone* P *such that* $\operatorname{int} P \neq \varnothing$. *Let* $L(X,Y)$ *be the set of all linear functionals from* X *to* Y *and let* $T : K \to L(X,Y)$ *be a mapping. Let* $\eta : K \times K \to X$ *be a vector-valued function and let* $f : K \to Y$ *be continuous. Assume that the following conditions hold:*

(a) $f(x) \notin -\operatorname{int} P$ *for all* $x \in K$;
(b) $f : K \to Y$ *is* T*-*η*-invex in* K;
(c) *for each* $u \in K$ *the set* $B_u = \{x \in X : f(u) - f(x) \in -\operatorname{int} P\}$ *is either* H*-convex or empty;*
(d) *the mapping* $v \mapsto f(v) - \langle T(v), \eta(x,v)\rangle$ *of* K *into* Y *is continuous;*
(e) *there exists a compact set* $L \subset X$ *and an* H*-compact* $C \subset X$*, such that, for each weakly* H*-convex set* D *with* $C \subset D \subset X$, $\cap\{x \in D : f(x') - \langle T(x'), \eta(x,x')\rangle \notin -\operatorname{int} P\} \subset L$;
(f) $\langle T(x'), \eta(x,x')\rangle \geq_P 0$ *for all* $x, x' \in K$, $x \neq x'$.

Then there exists $x_0 \in K$ *such that*

(A) $\langle T(x_0), \eta(x,x_0)\rangle - \langle T(x), \eta(x_0,x)\rangle \notin -\operatorname{int} P$ *for all* $x \in K$;
(B) $\{f(x) - T(x),\ \eta(x_0,x)\} - \{f(x_0) - \langle T(x_0), \eta(x,x_0)\rangle\} \notin -\operatorname{int} P$ *for all* $x \in K$.

Proof. Theorem 4.12 gives the existence of $x_0 \in K$ with $f(x) - f(x_0) \notin -\operatorname{int} P$, that is, $f(x) - f(x_0) \geq_P 0$ for all $x \in K$.

(A) Since f is T-η-invex in K, $f(x_0) - f(x) - \langle T(x), \eta(x_0,x)\rangle \notin -\operatorname{int} P$ for all $x \in K$, that is, $f(x_0) - f(x) - \langle T(x), \eta(x_0,x)\rangle \geq_P 0$ for all $x \in K$; adding this with $f(x) - f(x_0) \geq_P 0$, we get $-\langle T(x), \eta(x_0,x)\rangle \geq_P 0$. By condition (f) at $x' = x_0$, we have $\langle T(x_0), \eta(x,x_0)\rangle \geq_P 0$. Hence $\langle T(x_0), \eta(x,x_0)\rangle - \langle T(x), \eta(x_0,x)\rangle \notin -\operatorname{int} P$ for all $x \in K$.
(B) Addition of (A) with $f(x) - f(x_0) \geq_P 0$ gives (B). □

5. Variational inequality problem in n-manifolds

In this section, we study the concept of T-η-invex functions and its application in generalized vector variational inequality problems (in short, GVVI) on the manifolds.

Let X and Y be differentiable manifolds with tangent bundles τX and τY, respectively. Let K be a closed convex cone in the manifold X and let P be a closed, convex, pointed ordered cone in Y with $\operatorname{int} P \neq \emptyset$. Let $\eta : K \times K \to \tau X$ be an application and let $T : K \to L(\tau X, \tau Y)$ be the linear application.

The *generalized vector variational inequality problem* $(\mathrm{GVVI})_X$ on the X can be formulated as follows:

$(\mathrm{GVVI})_X$ Find $x_0 \in K$ such that $\langle T(x_0), \eta(x, x_0)\rangle \notin -\operatorname{int} P$ for all $x \in K$.

Definition 5.1 [11]. Let $\varphi : X \to \mathbb{R}^n$ be a differential vector function. $d\varphi_u : \tau(X,u) \to \tau(\mathbb{R}^n, \varphi(u)) \equiv \mathbb{R}^n$ is called the differential of φ at $u \in X$, if $d\varphi_u(v) = d\varphi(u)(v)$ for all $v \in \tau(X,u)$.

Definition 5.2 [11]. A differential vector function $\varphi : X \to \mathbb{R}^n$ is said to be *invex* at $u \in X$ with respect to η (shortly, φ is η-invex) if there exists an application $\eta : K \times K \to \tau X$ such that $\varphi(x) - \varphi(u) \geq d\varphi_u(\eta(x,u))$ for all $x \in K$.

Definition 5.3. An H-space is called an *H-differentiable manifold* if it is also a differentiable manifold.

Example 5.4. $\mathbb{R}$ is an H-differentiable manifold.

In this section we prove some results in H-differentiable manifold $(X, \{\Gamma_A\})$.

Theorem 5.5. *Let $(X, \{\Gamma_A\})$ be an H-differentiable manifold with the tangent bundle τX and K a nonempty closed conex subset of X. Let $\varphi : X \to \mathbb{R}^n$ be a linear application. Let P be a closed, convex, and ordered pointed cone in $\mathbb{R}^n$ with $\operatorname{int} P \neq \emptyset$. Let $L(\tau X, \mathbb{R}^n)$ denote the set of linear maps from τX to $\tau \mathbb{R}^n \equiv \mathbb{R}^n$. Let $T : K \to L(\tau X, \mathbb{R}^n)$ and $\eta : K \times K \to \tau X$ be an application such that $\eta(x,u) \in \tau(X,u)$ for all $x, u \in K$. Suppose that*

(a) *$(-\varphi)$ is T-η-invex on K,*
(b) *for each $u \in K$, $U(u) = \{x \in X : \varphi(x) - \varphi(u) \in -\operatorname{int} P\}$ is either H-convex or empty,*
(c) *the application $u \mapsto \langle T(u), \eta(x,u)\rangle$ of K into $\mathbb{R}^n$ is continuous (or at least hemicontinuous) for all $x \in K$,*
(d) *there exists a compact set $L \subset X$ and an H-compact set $C \subset X$ such that for each weakly H-convex set D with $C \subset D \subset X$, $\cap\{\langle T(u), \eta(x,u)\rangle \notin -\operatorname{int} P : x \in D\} \subset L$.*

Then $(\mathrm{GVVI})_K$ is solvable, that is, there exists $x_0 \in K$ such that $\langle T(x_0), \eta(x,x_0)\rangle \notin -\operatorname{int} P$ for all $x \in K$.

Proof. Let $F : K \to 2^X$ be a set-valued application defined by the rule $F(x) = \{u \in K : \langle T(u), \eta(x,u)\rangle \notin -\operatorname{int} P\}$ for all $x \in X$. We prove that $\cap\{F(x) : x \in K\} \neq \emptyset$. It can be proved by showing that F is an *H-KKM*, mapping on the manifold X. If F is not an *H-KKM* then there exists a finite subset $A \subset X$ such that $\Gamma_A \not\subset \cup\{F(x) : x \in A\}$. Assume there exist $w \in \Gamma_A$ such that $w \notin \cup\{F(x) : x \in A\}$. This implies that $w \notin F(x)$ for all $x \in A$, that is, $\langle T(w), \eta(x,w)\rangle \in -\operatorname{int} P$ for all $x \in A$. Since $(-\varphi)$ is T-η-invex on K, $-\varphi(x) + \varphi(u) + \langle T(u), \eta(x,u)\rangle \notin -\operatorname{int} P$ for all $x, u \in K$; equivalently $\varphi(x) - \varphi(u) - \langle T(u), \eta(x,u)\rangle \notin \operatorname{int} P$ for all x, $u \in K$. At $w \in K$, we have $\varphi(x) - \varphi(w) - \langle T(w), \eta(x,w)\rangle \notin \operatorname{int} P$ for all $x \in K$. Hence $\varphi(x) - \varphi(w) - \langle T(w), \eta(x,w)\rangle \notin \operatorname{int} P$ for all $x \in A$ (since A is a nonempty finite subsets of X). Thus we get $\varphi(x) - \varphi(w) \in -\operatorname{int} P$ for all $x \in A$ and by assumption (b) we get $x \in U(w)$. Hence $A \subset B(w)$. By the H-convexity of $U(w)$ we get $\Gamma_A \subset U(w)$ for every finite subset $A \subset U(w)$. Since $w \in \Gamma_A$, $w \in U(w) \subset X$. Hence $0 = \varphi(w) - \varphi(w) \in -\operatorname{int} P$ which is a contradiction since $0 \notin -\operatorname{int} P$. Hence F is an *H-KKM* map.

Next we prove that F is closed. Let $\{y_n\}$ be a sequence in $F(x)$ such that $y_n \to y$. We need to show that $y \in F(x)$. Since the application $y_n \mapsto \langle T(y_n), \eta(x, y_n)\rangle$ is continuous, $y_n \to y$ gives $\langle T(y_n), \eta(x, y_n)\rangle \to \langle T(y), \eta(x, y)\rangle$. Also $y_n \in F(x)$ gives $\langle T(y), \eta(x, y)\rangle \notin -\operatorname{int} P$ for all $x \in (\mathbb{R}^n - \{-\operatorname{int} P\})$. Since $(\mathbb{R}^n - \{-\operatorname{int} P\})$ is a closed set, $\langle T(y_n), \eta(x, y_n)\rangle \in (\mathbb{R}^n - \{-\operatorname{int} P\})$ for all $x \in K$, showing $\langle T(y), \eta(x, y)\rangle \notin -\operatorname{int} P$ for all $x \in \mathbb{R}^n$. Hence $y \in F(x)$. By [1, Theorem 1, page 486], we get $\cap\{F(x) : x \in K\} \neq \emptyset$. Hence there exists an $x_0 \in K$, such that $\langle T(x_0), \eta(x, x_0)\rangle \notin -\operatorname{int} P$ for all $x \in K$. □

Theorem 5.6. *Let $(X, \{\Gamma_A\})$ and $(Y, \{\Gamma_B\})$ be H-differentiable manifolds with the tangent bundle τX and τY, respectively, and K a nonempty closed convex subset of X. Let P be a closed, convex, and pointed ordered cone Y in such that* $\operatorname{int} P \neq \emptyset$. *Let $L(\tau X, \tau Y)$ denote the set of linear maps from τX to τY and $\eta : K \times K \to \tau X$ an application. Let $\varphi : X \to Y$ be a linear application and let $e_u = \tau\varphi_u : \tau X \to \tau Y$ be the corresponding bundle map defined by $e_u(v) \in \tau(Y, \varphi(u))^-$ for all $v \in \tau(X, u)$, where $\tau(Y, \varphi(u))^- = \{w \in \tau(Y, \varphi(u)) : w \notin \operatorname{int} P\}$. Let $T : K \to L(\tau X, \tau Y)$ be defined by the rule $T(u)$, $v = (d\varphi_u - e_u)(v)$ for all $v \in \tau(X, u)$. Let*

(a) *$(-\varphi)$ be a differentiable η-invex on K,*
(b) *for each $u \in K$, $U(u) = \{x \in X : \varphi(x) - \varphi(u) \in -\operatorname{int} P\}$ is either H-convex or empty,*
(c) *the application $u \mapsto d\varphi_u - e_u$, $\eta(x, u)$ of K into τY is continuous (or at least hemi-continuous) for all $x \in X$.*
(d) *there exists a compact set $L \subset X$ and an H-compact set $V \subset X$ such that for each weakly H-compact set D with $V \subset D \subset X$, $\cap\{\langle d\varphi_u - e_u, \eta(x, u)\rangle \notin -\operatorname{int} P$, $x \in D\} \subset L$.*

Then $(\mathrm{GVVI})_K$ *is solvable, that is, there exists $x_0 \in K$ such that $\langle d\varphi_{x_0} - e_{x_0}, \eta(x, x_0)\rangle \notin -\operatorname{int} P$ for all $x \in K$.*

Proof. First we show that $(-\varphi)$ is T-η-invex on K. Since $-\varphi : X \to Y$ is a differentiable η-invex on K, we have $(-\varphi)(x) - (-\varphi)(u) - d(-\varphi)_u(\eta(x, u)) \notin -\operatorname{int} P$ for all $x, u \in K$, that is, $(-\varphi)(x) - (-\varphi)(u) + d\varphi_u(\eta(x, u)) \notin -\operatorname{int} P$ and this implies that $-\varphi(x) + \varphi(u) + d\varphi_u(\eta(x, u)) \notin -\operatorname{int} P$. The definition of e_u, $e_u(\eta(x, u)) \in T(Y, \varphi(u))^-$ implies $e_u(\eta(x, u)) \notin \operatorname{int} P$, that is, $-e_u(\eta(x, u)) \notin -\operatorname{int} P$. Hence $-\varphi(x) + \varphi(u) + d\varphi_u(\eta(x, u)) - e_u(\eta(x, u)) \notin -\operatorname{int} P$ for all $x, u \in K$, that is, $-\varphi(x) + \varphi(u) + \langle T(u), \eta(x, u)\rangle \notin -\operatorname{int} P$ for all $x, u \in K$. Thus $-\varphi$ is T-η-invex on K.

Construct a set-valued mapping $G : K \to 2^X$ by the rule $G(x) = \{z \in K : (d\varphi_z - e_z)(\eta(x, z)) \notin -\operatorname{int} P\}$. We show that G is an *H-KKM* mapping. Suppose to the contrary that G is not an *H-KKM* application. Then there exists a finite subset $A \subset X$ such that $\Gamma_A \not\subset \cup\{G(x) : x \in A\}$. Let $w \in \Gamma_A$ be such that $w \notin \cup\{G(x) : x \in A\}$. So $w \notin G(x)$ for all $x \in A$, that is, $(d\varphi_w - e_w)(\eta(x, w)) \in -\operatorname{int} P$ for all $x \in A$. Since $-\varphi$ is $S - \eta -$ invex on X, we have $-\varphi(x) + \varphi(u) + \langle S(u), \eta(x, u)\rangle \notin -\operatorname{int} P$ for all x, $u \in K$, equivalently $\varphi(x) - \varphi(u) - \langle S(u), \eta(x, u)\rangle \notin \operatorname{int} P$ for all $x, u \in X$ (by Remark 3.2(F)). At point w in X, we get $\varphi(x) - \varphi(w) - \langle S(w), \eta(x, w)\rangle \notin \operatorname{int} P$ for all $x \in X$ (since Γ_A is a nonempty contractible subset of X). Thus $\varphi(x) - \varphi(w) - \langle S(w), \eta(x, w)\rangle \notin \operatorname{int} P$ for all $x \in A$ (since A is a nonempty finite subsets of K), that is, $\varphi(x) - \varphi(w) - (d\varphi_w - e_w)(\eta(x, w)) \notin \operatorname{int} P$ for all $x \in A$. Therefore $\varphi(x) - \varphi(w) \in -\operatorname{int} P$ for all $x \in A$, giving $x \in B(w)$. Hence $A \subset B(w)$. By H-convexity of $B(w)$, we get $\Gamma_A \subset B(w)$ for every finite subset $A \subset B(w)$. Since $w \in \Gamma_A$, $w \in B(w) \subset X$.

Hence $0 = \varphi(w) - \varphi(w) \in -\operatorname{int}P$ gives a contradiction that $0 \notin -\operatorname{int}P$. Hence G is an H-KKM.

Next we prove that G is closed. Let $\{z_n\}$ be a sequence in $G(x)$ such that $z_n \to z$, then we show that $z \in G(x)$. Again since the application $z_n \mapsto (d\varphi_{x_n} - e_{z_n})(\eta(x,z_n))$ is finite dimensional, we have $(d\varphi_{x_n} - e_{z_n})(\eta(x,z_n)) \to (d\varphi_z - e_{z_n})(\eta(x,z))$ as $z_n \to z$. As we have assumed $z_n \in G(x)$, so that $(d\varphi_{x_n} - e_{z_n})(\eta(x,z_n)) \notin -\operatorname{int}P$ for all $x \in (Y - \{-\operatorname{int}P\})$. But $(Y - \{-\operatorname{int}P\})$ is a closed set; therefore, $(d\varphi_z - e_{z_n})(\eta(x,z)) \in (Y - \{-\operatorname{int}P\})$ for all $x \in X$ implying that $(d\varphi_z - e_z)(\eta(x,z)) \notin -\operatorname{int}P$ for all $x \in X$. Hence $z \in G(x)$. Thus G is closed.

By [1, Theorem 1, page 486], we get $\cap\{G(x) : x \in X\} \neq \emptyset$. Hence there exists an $x_0 \in K$ such that $(d\varphi_{x_0} - e_{x_0})(\eta(x,x_0)) \notin -\operatorname{int}P$ for all $x \in K$, that is, x_0 solves the problem. □

6. Complementarity problem solution using fixed point theorems in manifolds

In this section we are interested in studying the behavior of continuous functions on manifolds with particular interest in finding the solutions of complementarity problems in the presence of fixed points or coincidences. Though the complementarity problem is a classical problem, the techniques involved in this section may prove enlightening to take a brief look at some development of the problems.

Let M be a closed manifold and let $f : M \to M$ be a map. Then for each k there is the induced homomorphism on homology with rational coefficients $f_k : H_k(M;\mathbb{Q}) \to H_k(M;\mathbb{Q})$. For each k we may choose a basis for the finite-dimensional rational vector space $H_k(M;\mathbb{Q})$ and write f_k as a matrix with respect to this basis. Denote by $\operatorname{tr}(f_k)$ the trace of the matrix. If we define the *Lefschetz number* by $L(f) = \sum_{k=0}^{\infty}(-1)^k \operatorname{tr}(f_k)$, then $L(f)$ is independent of the choices involved and hence is a well-defined, rational-valued function of f and $L(f)$ depends only on the homotopy class of f [15].

Let X be a closed, convex, and oriented Riemannian n-manifold, modeled on the Hilbert space H with Riemannian metric g. It is well known that the tangent bundle $\tau(X)$ can be identified with the cotangent bundle $\tau^*(X)$ by the Riemannian metric, because H^*, the dual of H, can be identified with H [10]. If v, $w \in \tau_x(X)$, then we write $g_x(v,w) = \langle v,w\rangle_x$. Let $F : X \to H$ be an operator. The complementarity problem is to find $x_0 \in X$ such that $F(x_0) \in \tau^*(X)$ and $g_{x_0}(Fx_0,x_0) = \langle Fx_0,x_0\rangle_{x_0} = 0$.

Theorem 6.1. *Let X be a closed, convex, and oriented Riemannian n-manifold, modeled on the Hilbert space H with Riemannian metric g and let $f : X \to X$ be a map with Lefschetz number $L(f)$. Let $F : X \to H$ be an operator. Then there exists a unique $x_0 \in X$ such that $F(x_0) \in \tau^*(X)$ and $g_{x_0}(Fx_0,x_0) = \langle Fx_0,x_0\rangle_{x_0} = 0$.*

Proof. Since X is nonempty, closed, and convex endowed with the Riemannian metric g, for every $y \in X$, there is an unique $x \in X$ closest to $y - F(y)$. Thus $\|x - y + F(y)\| \leq \|z - y + F(y)\|$ for every $z \in X$, that is, $\langle x, z - x\rangle_x \geq \langle y - F(y), z - x\rangle_x$ for all $z \in X$. Let $f : X \to X$ be defined by $f(y) = y - F(y) + x$ for every $y \in X$, where x is the unique element corresponding to y. Now for every $y \in X$, $(1_X - f)(y) = 1_X(y) - f(y) = F(y) - x$ and at the unique $x \in X$, we have $(1_X - f)(x) = F(x) - x = (F - 1_X)(x)$, that is, $1_X - f =$

$F-1_X$ at the unique $x \in X$. Define $G: X \times I \to X$ by the rule

$$G(y,t)=\begin{cases}(1_X-f)\big((1-2t)y+2tx\big), & 0 \le t \le \dfrac{1}{2},\\[2mm] (F-1_X)\big((2t-1)y+2(1-t)x\big), & \dfrac{1}{2} \le t \le 1.\end{cases} \tag{6.1}$$

$G(y,0)=(1_X-f)(y)$ and $G(y,1)=(F-1_X)$ for each $y \in X$. At $t=1/2$, $G(y,1/2)=(1_X-f)(x)=(F-1_X)(x)$. So G is continuous by Pasting lemma. Thus $G:(1_X-f)\simeq(F-1_X)$. Thus, the coincidence index set of f is given by $I_f=(1_X-f)*0_X=(F-1_X)*0_X$ and $I_f \neq 0$. Since $f: X \to X$ is a mapping with $L(f)=I_f \neq 0$, by [15, Theorem 7.16, Lefschitz fixed-point theorem], f has a fixed point. Let the fixed point be y_0 in X, that is, $f(y_0)=y_0$. Let x_0 be the unique element that corresponds to y_0. Thus we have $\langle x_0, z-x_0\rangle_{x_0} \ge \langle y_0-F(y_0), z-x_0\rangle_{x_0}$ for all $z \in X$, giving $\langle x_0, z-x_0\rangle_{x_0} \ge \langle f(y_0)-y_0, z-x_0\rangle_{x_0}$ for all $z \in X$, that is, $\langle 2x_0-y_0, z-x_0\rangle_{x_0} \ge 0$ for all $z \in X$. At $y=y_0$, we get $f(y_0)=y_0-F(y_0)+x_0$, that is, $x_0=F(y_0)$. We show that $F(y_0)\in \tau^*(X)$. By definition of coincidence index set, we have $I_f=(1_X-f)*0_X=(F-1_X)*0_X=I_F$, which means that f and F have the same fixed point in X, that is, $f(y_0)=y_0=F(y_0)$. So, we get $x_0=y_0$. Putting $x_0=y_0$ in $\langle 2F(y_0)-y_0, z-x_0\rangle_{x_0}\ge 0$, we get $\langle F(y_0), z-y_0\rangle_{y_0} \ge 0$ for all $z \in X$, that is, $F(y_0)\in T^*(X)$. Again putting $z=0$ and $z=2y_0$ in $\langle F(y_0), z-y_0\rangle_{y_0}\ge 0$, respectively, we get $\langle F(y_0), y_0\rangle_{y_0} \le 0$ and $\langle F(y_0), y_0\rangle_{y_0}\ge 0$. Hence $\langle F(y_0), y_0\rangle_{y_0}=0$. □

Theorem 6.2. *Let $f: S^n \to S^n$, $n \ge 1$, be a map of degree $m \neq (-1)^{n+1}$. Let $T: S^n \to S^n$ be any operator. Then there exists a $y_0 \in S^n T(y_0)$, $z-y_0 \ge 0$ for all $z \in S^n$.*

Proof. Since $S^n \subset R^{n+1}$ is closed, for each $y \in S^n$, there exists a unique $x \in S^n$ closest to $y-T(y)$, that is, $\|x-y+T(y)\| \le \|z-y+T(y)\|$ for all $z \in S^n$. Thus $\langle x, z-x\rangle \ge \langle y-T(y), z-x\rangle$ for all $z \in S^n$, that is, $\langle x-y+T(y), z-x\rangle \ge 0$ for all $z \in S^n$. Define $f: S^n \to S^n$ by the rule $f(y)=x$. This is well defined since x is unique in S^n corresponding to each $y \in S^n$. Obviously f is a homeomorphism. Replacing x by $f(y)$ in the inequality $\langle x-y+T(y), z-x\rangle \ge 0$ we get $\langle f(y)-y+T(y), z-f(y)\rangle \ge 0$ for all $z \in S^n$. Since S^n is a closed n-manifold, and $f: S^n \to S^n$, $n \ge 1$, is a map of degree $m \neq (-1)^{n+1}$, f has a fixed point and is homotopic to the identity map $i: S^n \to S^n$, that is, there is an element $y_0 \in S^n$ such that $f(y_0)=y_0$. Taking $y=y_0$ in the inequality $\langle f(y)-y+T(y), z-f(y)\rangle \ge 0$, we get $\langle T(y_0), z-y_0\rangle \ge 0$ for all $z \in S^n$. □

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