TIMELIKE TRAJECTORIES WITH FIXED ENERGY UNDER A POTENTIAL IN STATIC SPACETIMES

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By variational methods, we study the motions of a relativistic particle under the action of an external scalar potential. We consider standard static spacetimes whose metric coefficients grow at most quadratically at infinity. Fixing suitable values for the energy, we obtain timelike trajectories.

1. Introduction

In this paper, using variational methods, we study the connectedness of a certain class of Lorentzian manifolds (L,g) (see, e.g., [5, 12] for the basic notions of Lorentzian geometry) by trajectories z of the differential equation

$$D_s^L \dot{z} + \nabla_L V(z) = 0, \tag{1.1}$$

where D_s^L denotes the covariant derivative with respect to g, $V : L \to \mathbb{R}$ is a smooth potential, and ∇_L is the gradient of V with respect to g.

On Riemannian manifolds, (1.1) is the equation of classical Lagrangian systems and it has been widely studied. From a variational point of view, existence and multiplicity of solutions joining two fixed points can be easily proved when V has subquadratic growth. In [9] the same problem is analyzed when V has quadratic growth. In this case, the value of the arrival time is important. Indeed, existence and multiplicity results of solutions parametrized in $[0,T]$ are obtained if T satisfies a certain inequality.

As far as we know, solutions of (1.1) in the Lorentzian case have been studied only in [1, 3, 14]. In [14], the authors study the completeness of solutions for a class of differential equations including (1.1). In [3], periodic trajectories and connectedness by trajectories under V are studied, when V is bounded, on a class of orthogonal splitting Lorentzian manifolds. In [1], the authors prove the connectedness of standard static Lorentzian manifolds by trajectories of (1.1) (where $V = V(x,s)$ is time independent) with fixed arrival time T, when V and the coefficient β of the metric (see Definition 1.1) grow at most quadratically at infinity, and T satisfies the same condition of [9].

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Here we study the same equation under a different approach. Before presenting it, we need to recall the following facts (in the rest of the paper, for simplicity of notation, the Lorentzian metric $g(z)$ for $z \in L$ will be also denoted by $\langle \cdot , \cdot \rangle_L$).

As in the case of autonomous Lagrangian systems on Riemannian manifolds, it is easy to verify that for each solution $z : I \to L$, $I \subset \mathbb{R}$ interval, of (1.1), a constant $E_z \in \mathbb{R}$ exists such that

$$E_z = \frac{1}{2}\langle \dot{z}(s), \dot{z}(s)\rangle_L + V(z(s)) \quad \forall s \in I. \tag{1.2}$$

Throughout this paper, in analogy to the Riemannian case, E_z will be called energy.

Moreover, we recall that if L is a Lorentzian manifold, a vector $\zeta \in \mathrm{TL}$ is said to be timelike (resp., lightlike; spacelike) if $\langle \zeta, \zeta \rangle_L < 0$, (resp., $\langle \zeta, \zeta \rangle_L = 0$, $\zeta \neq 0$; $\langle \zeta, \zeta \rangle_L > 0$; or $\zeta = 0$). A curve z on L is said to be timelike, lightlike, or spacelike according to the causal character of $\dot{z}$.

Thus, it makes sense to study timelike solutions of (1.1) having a fixed value of the energy E. Timelike solutions are more interesting from a physical point of view because they represent the world lines of relativistic particles moving under the action of a gravitational field (described by the metric) and of an external scalar potential (described by V).

When one considers this kind of solutions for a fixed $E \in \mathbb{R}$, by (1.2) it is clear that the admissible region for the motion is the set

$$\Sigma = \{ z \in L \mid V(z) - E > 0 \} \tag{1.3}$$

which is an open subset of L (possibly equal to L).

Fixing $E \in \mathbb{R}$ such that Σ is not empty and $z_0, z_1 \in \Sigma$, our aim is to prove the existence of C^2 timelike curves $z : [0,T] \to L$ solutions of

$$\begin{gathered} D_s^L \dot{z} + \nabla_L V(z) = 0, \\ \frac{1}{2}\langle \dot{z}, \dot{z} \rangle_L + V(z) = E, \\ z(0) = z_0, \quad z(T) = z_1. \end{gathered} \tag{1.4}$$

We obtain two different results: the first one assuming that V is bounded from below and E is smaller than the infimum of V (in this case $\Sigma = L$) and the second one for a more general V possibly unbounded. In both cases, L is a standard static spacetime whose definition is recalled here.

Definition 1.1. A standard static spacetime is a connected Lorentzian manifold $(L, \langle \cdot , \cdot \rangle_L)$ with $L = M \times \mathbb{R}$ and

$$\langle \cdot , \cdot \rangle_L = \langle \cdot , \cdot \rangle - \beta(x) dt^2, \tag{1.5}$$

where $(M, \langle \cdot , \cdot \rangle)$ is a smooth, finite-dimensional Riemannian manifold, t is the natural coordinate of $\mathbb{R}$, and $\beta \in C^1(M, \mathbb{R})$ is a strictly positive function.

The manifold M satisfies the following assumptions:

(H_1) $(M,\langle\cdot,\cdot\rangle)$ is a complete, connected, at least C^3 Riemannian manifold. The coefficient β of the metric has at most quadratic growth. More precisely, we assume that

(H_2) there exist $\lambda \geq 0$, $k \in \mathbb{R}$, $p \in [0,2]$, and a point $y_0 \in M$ such that

$$0 < \beta(x) \leq \lambda d^p(x, y_0) + k \quad \forall x \in M, \tag{1.6}$$

where d denotes the distance canonically associated to the Riemannian metric in M. As the coefficient of the metric, V depends only on $x \in M$, that is,

(H_3) for all $z = (x,t) \in L = M \times \mathbb{R}$, we have $V(z) = V(x,t) = V(x,0) \equiv V(x)$.

Our first result is the following theorem.

Theorem 1.2. *Let $(L,\langle\cdot,\cdot\rangle_L)$ be a standard static spacetime as in Definition 1.1 satisfying (H_1), (H_2). Let $V \in C^1(L,\mathbb{R})$ satisfy (H_3) and be bounded from below. Set*

$$\overline{E} = \inf_{x\in M} V(x). \tag{1.7}$$

Then, for any $E < \overline{E}$ and $z_0 = (x_0,t_0)$, $z_1 = (x_1,t_1) \in L$ with $z_0 \neq z_1$, setting $\Delta = t_1 - t_0$ and $N(E,z_0,z_1)$ the number of timelike solutions of (1.4), the following statements hold:

(a) *if $|\Delta|$ is sufficiently small and $x_0 \neq x_1$, then $N(E,z_0,z_1) = 0$;*

(b) *if $x_0 = x_1$, then $N(E,z_0,z_1) \geq 1$;*

(c) *if M is not contractible in itself, for any $m \in \mathbb{N}\setminus\{0\}$, $\gamma_m > 0$ exists such that if $|\Delta| > \gamma_m$, $N(E,z_0,z_1) \geq m$.*

Remark 1.3. Some comments about the assumptions of Theorem 1.2 are necessary, in order to compare it with previous results.

(1) Theorem 1.2 shows that if $x_0 = x_1$, problem (1.4) admits at least a solution. If $x_0 \neq x_1$, it has no solution if $|\Delta|$ is small, while the number of solutions goes to infinity as $|\Delta|$ goes to infinity. These results generalize previous ones obtained by variational methods for timelike geodesics (i.e, for (1.1) with $V = 0$) in standard static spacetimes (see, e.g., [6–8] and the textbook [11] if β has subquadratic asymptotic behavior and [2] if β has quadratic growth).

(2) The choice of fixing the energy E (instead of the arrival time T as in [1]) takes some advantages. It allows one to obtain timelike solutions, and no assumption on the asymptotic growth of V from above is necessary. In Theorem 1.2 we assume that V is bounded from below. The role of this assumption is to ensure that Σ coincides with the complete manifold L making easier the application of variational techniques. Nevertheless, it will be removed in the next theorem where we will deal with an unbounded potential both from below and from above.

In order to state our second result, we observe that if L is a standard static spacetime as in Definition 1.1 and a potential V satisfies (H_3), for any $E \in \mathbb{R}$, the set Σ defined in (1.3) is given by

$$\Sigma = \Lambda \times \mathbb{R} \quad \Lambda = \{x \in M \mid V(x) - E > 0\}. \tag{1.8}$$

As observed in Remark 1.3(2), when we deal with a more general V (possibly not bounded from below), we have to choose E such that the set Σ (and so Λ) is not empty. Note that this is true for any $E \in \mathbb{R}$ if V is unbounded from above, but Λ is different from M if V is unbounded from below. Hence, generally Λ is an open submanifold of M with topological boundary. This makes it necessary to consider more regular potentials V (at least C^2) such that for a certain $E \in \mathbb{R}$, the following assumption is satisfied:

(H_4) a positive number $\delta > 0$ exists such that for any $x \in M$, $E < V(x) < E + \delta$, we have

$$\nabla V(x) \neq 0, \tag{1.9}$$

$$H^V(x)[\xi,\xi] \leq 0 \quad \forall \xi \in T_xM \text{ such that } \langle \nabla V(x), \xi \rangle = 0, \tag{1.10}$$

$$\langle \nabla V(x), \nabla \beta(x) \rangle \geq 0, \tag{1.11}$$

where $H^V(x)[\xi,\xi]$ denotes the Hessian of V at x with respect to the Riemannian metric $\langle \cdot, \cdot \rangle$.

Remark 1.4. It is useful to discuss the meaning of (H_4). Condition (1.9) ensures that the level subsets $(V - E)^{-1}(a)$ are smooth hypersurfaces for a sufficiently small. By (1.10), each of these hypersurfaces is the convex boundary of $(V - E)^{-1}(]a, +\infty[)$ (see [4] for a detailed discussion of the different notions of convexity for the boundary of an open domain of a Riemannian manifold). As ∇V points out the interior of $(V - E)^{-1}(]a, +\infty[)$, condition (1.11) ensures that the same holds for $\nabla \beta$, for small a.

Our second result is the following theorem.

THEOREM 1.5. *Let $(L, \langle \cdot, \cdot \rangle_L)$ be a standard static spacetime as in (1.5) satisfying (H_1), (H_2). Let $V \in C^2(L, \mathbb{R})$ satisfy (H_3) and $E \in \mathbb{R}$ such that Σ (see (1.8)) is not empty. Assume also that (H_4) holds. Then, for any $z_0 = (x_0, t_0)$, $z_1 = (x_1, t_1) \in \Sigma$ with $z_0 \neq z_1$, setting $\Delta = t_1 - t_0$ and $N(E, z_0, z_1)$ the number of timelike solutions of (1.4), the following statements hold:*

(a) *if $|\Delta|$ is sufficiently small and $x_0 \neq x_1$, then $N(E, z_0, z_1) = 0$;*

(b) *if $x_0 = x_1$, then $N(E, z_0, z_1) \geq 1$;*

(c) *if Λ is not contractible in itself, for any $m \in \mathbb{N} \setminus \{0\}$, $\gamma_m > 0$ exists such that if $|\Delta| > \gamma_m$, $N(E, z_0, z_1) \geq m$.*

Remark 1.6. Parts (a) of Theorems 1.2 and 1.5 are known results for standard static spacetimes. Nevertheless, for completeness, we have listed their statement together with new results, providing a variational proof. Moreover, note that L is globally hyperbolic under our assumptions, thus, if $|\Delta|$ is small enough, z_0 and z_1 are not causally related, so they cannot be joined by any causal curve independent of the differential equation it may solve (see [2, 15] for more precise results).

Our variational approach is based on the study of the functional

$$f(z) = \frac{1}{2} \int_0^1 \langle \dot{z}, \dot{z} \rangle_L ds \int_0^1 (V(z) - E) ds \tag{1.12}$$

defined on a suitable manifold of curves joining z_0 and z_1. This is the Lorentzian version of the functional introduced in [16] for the study of brake orbits of a class of Hamiltonian systems. In the Riemannian case, f is essentially obtained by a modified version of the classical principle of least action. We will prove that solutions of (1.4) correspond to critical points of (1.12).

Unlike the Riemannian case, even if V is bounded, f is unbounded (both from below and from above) due to the fact that the Lorentzian metric is indefinite, so its critical points cannot be investigated by classical topological methods. In spite of this, when V depends only on the variable x and L is static, following the approach used in the pioneering paper [7], it is possible to deal with the Riemannian functional J defined by

$$J(x) = \frac{1}{2}\int_0^1 (V(x) - E)\,ds \left(\int_0^1 \langle \dot{x}, \dot{x} \rangle\, ds - \Delta^2 \left(\int_0^1 \frac{1}{\beta(x)}\, ds \right)^{-1} \right) \tag{1.13}$$

(see Proposition 2.3 for details). Critical points of J will be obtained by a minimization argument and Ljusternik-Schnirelmann category theory.

This paper is organized as follows. In Section 2 we state the variational setting both for Theorems 1.2 and 1.5 which will be proved, respectively, in Sections 3 and 4.

2. The variational setting

The natural setting for problem (1.4) on a standard static spacetime is the set Σ defined in (1.8) for a fixed $E \in \mathbb{R}$. From now on, we assume that Σ (and so Λ) is not empty for such E. Under the assumptions of Theorem 1.2, obviously we have $\Lambda = M$. Moreover, by (H_1), Λ is an open submanifold of M of class at least C^3.

Problem (1.4) has a variational structure, that is, its solutions are, up to reparameterizations, the critical points of the functional f defined in (1.12). In order to establish this property, we need to define some manifolds of curves with values in open subsets D of M (possibly $D = M$). We define $S = D \times \mathbb{R} \subset L$ and consider the set $H^1([0,1], S)$ of absolutely continuous curves on S whose derivatives are square summable. This is an infinite-dimensional Riemannian manifold diffeomorphic, when L is static, to the product manifold $H^1([0,1], D) \times H^1([0,1], \mathbb{R})$ with the Riemannian structure given by

$$\langle \zeta, \zeta \rangle_1 = \int_0^1 \langle \xi, \xi \rangle\, ds + \int_0^1 \langle D_s\xi, D_s\xi \rangle\, ds + \int_0^1 \tau^2\, ds + \int_0^1 \dot{\tau}^2\, ds \tag{2.1}$$

for any $z = (x,t) \in H^1([0,1], S)$ and $\zeta = (\xi, \tau) \in T_z H^1([0,1], S) \equiv T_x H^1([0,1], \Lambda) \times H^1([0,1], \mathbb{R})$ (where D_s denotes the covariant derivative induced by the Riemannian structure on M).

We point out that by the Nash embedding theorem, we can assume that D is a submanifold of an Euclidean space $\mathbb{R}^N$, $\langle \cdot, \cdot \rangle$ is the usual Euclidean metric, and d is the associated distance. Hence, the Sobolev space of curves $H^1([0,1], D)$ can be identified in this way:

$$H^1([0,1], D) \equiv \{x \in H^1([0,1], \mathbb{R}^N) \mid x([0,1]) \subset D\}. \tag{2.2}$$

We consider the submanifold of $H^1([0,1],S)$ of the curves joining two points $z_0 = (x_0,t_0)$ and $z_1 = (x_1,t_1)$ of S. More precisely, we define the following sets of curves:

$$\begin{aligned}\Omega^1(x_0,x_1,D) &= \{x \in H^1([0,1],D) \mid x(0) = x_0,\ x(1) = x_1\},\\ W(t_0,t_1) &= \{t \in H^1([0,1],\mathbb{R}) \mid t(0) = t_0,\ t(1) = t_1\},\\ Z(z_0,z_1,S) &= \Omega^1(x_0,x_1,D) \times W(t_0,t_1).\end{aligned} \tag{2.3}$$

The following properties are well known.

(i) $\Omega^1(x_0,x_1,D)$ is a smooth submanifold of $H^1([0,1],D)$ whose tangent space at $x \in \Omega^1(x_0,x_1,D)$ is given by

$$T_x\Omega^1(x_0,x_1,D) = \{\xi \in T_xH^1([0,1],D) \mid \xi(0) = 0 = \xi(1)\}. \tag{2.4}$$

(ii) $W(t_0,t_1)$ is a closed affine submanifold of $H^1([0,1],\mathbb{R})$, that is,

$$W(t_0,t_1) = H_0^1([0,1],\mathbb{R}) + t^*, \tag{2.5}$$

where

$$\begin{aligned}H_0^1([0,1],\mathbb{R}) &= \{t \in H^1([0,1],\mathbb{R}) \mid t(0) = 0 = t(1)\},\\ t^* : s \in [0,1] &\longmapsto t_0 + \Delta s \quad \Delta = t_1 - t_0.\end{aligned} \tag{2.6}$$

(iii) $Z(z_0,z_1,S)$ is a submanifold of $H^1([0,1],S)$ whose tangent space at $z = (x,t)$ is

$$T_xZ(z_0,z_1,S) = T_x\Omega^1(x_0,x_1,D) \times H_0^1([0,1],\mathbb{R}), \tag{2.7}$$

which can be equipped with the Riemannian metric (equivalent to $\langle\cdot,\cdot\rangle_1$)

$$\langle\zeta,\zeta\rangle_2 = \int_0^1 \langle D_s\xi, D_s\xi\rangle ds + \int_0^1 \dot{\tau}^2 ds \tag{2.8}$$

for any $z = (x,t) \in Z(z_0,z_1,S)$ and $\zeta = (\xi,\tau) \in T_zZ(z_0,z_1,S)$.

Remark 2.1. We point out that the manifolds $H^1([0,1],S)$ and $Z(z_0,z_1,S)$ are complete with respect to their Riemannian structure if D is complete. This is true in Theorem 1.2, where $D = M$ but not in Theorem 1.5 where, dealing with a more general class of potentials (possibly unbounded), D will be an open subset of M having a topological boundary.

A first variational principle can now be stated.

Proposition 2.2. *Let L be a standard static spacetime as in Definition 1.1 and let $V \in C^1(L,\mathbb{R})$ satisfy (H_3). Let D be an open subset of M and let $E \in \mathbb{R}$ be such that*

$$V(z) - E > 0 \quad \forall z = (x,t) \in S = D \times \mathbb{R}. \tag{2.9}$$

Consider $z_0 = (x_0, t_0)$, $z_1 = (x_1, t_1) \in S$ *and* $f : Z(z_0, z_1, S) \to \mathbb{R}$ *defined in (1.12). Then*

(a) *if* $z \in Z(z_0, z_1, S)$ *is a critical point of* f *such that* $f(z) < 0$, $y(s) = z(\omega s)$, $s \in [0, 1/\omega]$, *with*

$$\omega^2 = \frac{\int_0^1 (V(z) - E)\,ds}{-(1/2)\int_0^1 \langle \dot{z}, \dot{z}\rangle_L\,ds} \tag{2.10}$$

is a solution of (1.4);

(b) *if* $y : [0, T] \to S$ *is a solution of (1.4),* $z(s) = y(Ts)$, $s \in [0, 1]$, *is a critical point of* f *such that* $f(z) < 0$.

Proof. To prove (a), note that by (2.9), if $f(z) < 0$, then ω^2 is well defined. Each critical point z of f is a solution of

$$\omega^2 D_s^L \dot{z} + \nabla_L V(z) = 0 \tag{2.11}$$

so that y is a solution of (1.1) joining z_0 and z_1. By (2.11), we have

$$\frac{1}{2}\omega^2 \langle \dot{z}, \dot{z}\rangle_L + V(z) = c \tag{2.12}$$

for some $c \in \mathbb{R}$. Integrating (2.12) on $[0, 1]$ and substituting the value of ω^2, we obtain $c = E$, so (a) is proved.

Now, consider a solution y of problem (1.4) as in (b). The corresponding z satisfies

$$D_s^L \dot{z} + T^2 \nabla_L V(z) = 0. \tag{2.13}$$

Integrating on $[0, T]$ the second equation in (1.4) and taking into account that

$$\int_0^T \langle \dot{y}, \dot{y}\rangle_L\,ds = \frac{1}{T}\int_0^1 \langle \dot{z}, \dot{z}\rangle_L\,ds, \qquad \int_0^T V(y)\,ds = T\int_0^1 V(z)\,ds, \tag{2.14}$$

it is not difficult to prove that

$$T^2 = \frac{-(1/2)\int_0^1 \langle \dot{z}, \dot{z}\rangle_L\,ds}{\int_0^1 (V(z) - E)\,ds}, \tag{2.15}$$

which is well defined in our setting. Substituting this value in (2.13), we easily obtain that z is a critical point of f. As by (2.9) and (2.14)

$$\int_0^1 \langle \dot{z}, \dot{z}\rangle_L\,ds < 0, \tag{2.16}$$

also $f(z) < 0$ and the proof of (b) is complete. □

The previous proposition shows that solving problem (1.4) is equivalent to finding critical points of f. Due to the reasons already explained in Section 1, it is convenient to consider the functional $J : \Omega^1(x_0, x_1, D) \to \mathbb{R}$ defined in (1.13) whose critical points correspond to critical points of f. This is shown by the following proposition whose proof is a slight variant of that in [7, Theorem 2.1].

Proposition 2.3. *Under the assumptions of Proposition 2.2, the following are equivalent:*

(a) $z = (x,t) \in Z(z_0,z_1,S)$ *is a critical point of* f*;*

(b) x *is a critical point of* $J : \Omega^1(x_0,x_1,D) \to \mathbb{R}$ *defined in (1.13) and* $t = \Psi(x)$ *with* $\Psi : \Omega^1(x_0,x_1,D) \to W(t_0,t_1)$ *such that*

$$\Psi(x)(s) = t_0 + \Delta\left(\int_0^1 \frac{1}{\beta(x)}ds\right)^{-1}\int_0^s \frac{1}{\beta(x(\sigma))}d\sigma \tag{2.17}$$

for any $x \in \Omega^1(x_0,x_1,D)$ *and* $s \in [0,1]$.

Moreover, $f(z) = J(x)$.

In the next section, we will prove Theorem 1.2 directly showing the existence of critical points of J defined on the complete manifold $\Omega^1(x_0,x_1,M)$ (see Remark 2.1), whereas in Theorem 1.5 some difficulties arise. In fact J may not satisfy the Palais-Smale condition for the following reasons.

(a) As J is null on the boundary of Λ, Palais-Smale sequences may be not bounded.

(b) Due to possible incompleteness of $\Omega^1(x_0,x_1,\Lambda)$, bounded Palais-Smale sequences may not converge to a curve in $\Omega^1(x_0,x_1,\Lambda)$.

To deal with these problems, we consider a subset of Λ where the first term of the product in (1.13) is bounded from below. More precisely, define

$$\Lambda_a = \{x \in M \mid V(x) - E > a\} \tag{2.18}$$

for any $a \in]0,\delta[$ (see (H_4)). It is clear that

$$\Lambda_{a_2} \subset \Lambda_{a_1} \quad \text{if } a_1 \le a_2, \qquad \bigcup_{a\in]0,\delta[} \Lambda_a = \Lambda. \tag{2.19}$$

So, fixing $x_0,x_1 \in \Lambda$, we can choose $a \in]0,\delta[$ such that $x_1,x_2 \in \Lambda_a$ and find critical points of J on $\Omega^1(x_0,x_1,\Lambda_a)$.

As Λ_a has a boundary, we need also to use a penalization argument. By (H_1) and (1.9) in (H_4), the boundary of Λ_a is given by

$$\partial\Lambda_a = \{x \in M \mid V(x) - E = a\} \tag{2.20}$$

and it is a smooth submanifold of M.

Consider the function $\phi : \Lambda_a \cup \partial\Lambda_a \to [0,+\infty[$ defined by

$$\phi(x) = V(x) - E - a. \tag{2.21}$$

Again by (1.9) in (H_4), ϕ verifies that

$$\begin{gathered} \phi^{-1}(0) = \partial\Lambda_a, \\ \phi > 0 \quad \text{on } \Lambda_a, \\ \nabla\phi(x) \neq 0, \quad \text{for any } x \in \partial\Lambda_a. \end{gathered} \tag{2.22}$$

We point out that by properties (2.22), the following lemma holds (for the proof see, e.g., [8, Lemma 2.3]).

Lemma 2.4. *Let $(x_m)_m$ be a sequence in $\Omega^1(x_0,x_1,\Lambda_a)$ such that*

$$\sup_{m\in\mathbb{N}}\int_0^1 \langle \dot{x}_m,\dot{x}_m\rangle ds<\infty \tag{2.23}$$

and assume the existence of a sequence $(s_m)_m$ in $[0,1]$ such that

$$\lim_{m\to\infty}\phi(x_m(s_m))=0. \tag{2.24}$$

Then

$$\lim_{m\to\infty}\int_0^1 \frac{1}{\phi^2(x_m(s))}ds=+\infty. \tag{2.25}$$

Now we can introduce the penalization term. For any $\epsilon>0$, consider the functions $\psi_\epsilon:[0,+\infty[\to\mathbb{R}$ defined by

$$\psi_\epsilon(s)=\begin{cases}0 & 0\le s\le \dfrac{1}{\epsilon},\\ \displaystyle\sum_{n=3}^{+\infty}\frac{1}{n!}\left(s-\frac{1}{\epsilon}\right)^n & s>\dfrac{1}{\epsilon}.\end{cases} \tag{2.26}$$

Note that ψ_ϵ are smooth and verify the following properties. For any $\epsilon>0$, two positive constants a_ϵ, b_ϵ exist such that

$$\psi_\epsilon(s)\ge a_\epsilon s-b_\epsilon \quad \forall s\ge 0, \tag{2.27}$$

and if $0<\epsilon\le\epsilon'$,

$$\psi_\epsilon(s)\le\psi_{\epsilon'}(s)\quad \forall s\ge 0. \tag{2.28}$$

For any $\epsilon\in]0,1]$, we consider the following family of penalized functionals $J_\epsilon:\Omega^1(x_0,x_1,\Lambda_a)\to\mathbb{R}$:

$$J_\epsilon(x)=J(x)+\int_0^1 (V(x)-E)ds\int_0^1 \psi_\epsilon\left(\frac{1}{\phi^2(x)}\right)ds, \tag{2.29}$$

where J has been defined in (1.13). In Section 4 we will prove that for ϵ sufficiently small, each critical point of J_ϵ is also a critical point of J.

Since we will find critical points of J and J_ϵ by means of the Ljusternik-Schnirelman theory, we conclude this section recalling some basic facts about this theory (for more details see, e.g., [13]).

Definition 2.5. Let X be a topological space. The Ljusternik-Schnirelman category of a subset A of X, briefly $\mathrm{cat}_X(A)$, is the least number of closed and contractible subsets of X covering A. If A cannot be covered by a finite number of such sets, $\mathrm{cat}_X(A)=+\infty$.

In the sequel we will use this notation:

$$\mathrm{cat}(X)=\mathrm{cat}_X(X). \tag{2.30}$$

Theorem 2.6. *Let Ω be a Riemannian manifold and J a C^1 functional on Ω satisfying the Palais-Smale condition, that is, any $(x_m)_m \subset \Omega$ such that*

$$(J(x_m))_m \text{ is bounded,} \qquad \lim_{m\to+\infty} J'(x_m) = 0 \tag{2.31}$$

converges in Ω up to subsequences. For any $m \in \mathbb{N}\setminus\{0\}$, define

$$c_m = \inf_{A\in\Gamma_m} \sup_{x\in A} J(x), \qquad \Gamma_m = \{A \subset \Omega \mid \operatorname{cat}_\Omega(A) \geq m\}. \tag{2.32}$$

If Ω is complete or if each sublevel of J is complete, then

(a) *for each m such that $\Gamma_m \neq \emptyset$ and $c_m \in \mathbb{R}$, c_m is a critical value of J;*

(b) *if $c_i = \cdots = c_{i+j} = c$ for some i and j and c is finite, there are at least $j+1$ critical points at level c;*

(c) *if J is bounded from below, it has at least* $\operatorname{cat}(\Omega)$ *critical points, and if* $\operatorname{cat}(\Omega) = +\infty$, *a sequence $(x_m)_m$ of critical points of J exists such that*

$$\lim_{m\to+\infty} J(x_m) = +\infty. \tag{2.33}$$

In order to prove the multiplicity results, we need the following estimate on the category of the space $\Omega^1(x_0,x_1,D)$ (see [10] for the proof).

Proposition 2.7. *Let D be a Riemannian manifold. If D is noncontractible in itself, for any $x_0,x_1 \in D$,*

$$\operatorname{cat}\left(\Omega^1(x_0,x_1,D)\right) = +\infty, \tag{2.34}$$

and $\Omega^1(x_0,x_1,D)$ contains compact subsets of arbitrary large category.

3. Proof of Theorem 1.2

Assume that all the assumptions of Theorem 1.2 hold. In this case, as $\Lambda = M$, we study functional $J : \Omega^1(x_0,x_1,M) \to \mathbb{R}$ defined in (1.13) and prove existence and multiplicity of its critical points. We begin by investigating under which conditions it admits a negative minimum. To this aim, we will prove that J is bounded from below and is coercive, that is,

$$\lim_{\|\dot{x}\|\to+\infty} J(x) = +\infty \tag{3.1}$$

(here $\|\dot{x}\|$ denotes the L^2-norm, i.e.,

$$\|\dot{x}\|^2 = \int_0^1 \langle \dot{x},\dot{x}\rangle\, ds \tag{3.2}$$

for any $x \in \Omega^1(x_0,x_1,M)$).

Functional J is the product of J_1 and J_2, where

$$\begin{aligned} J_1(x) &= \int_0^1 (V(x) - E)\,ds, \\ J_2(x) &= \frac{1}{2}\int_0^1 \langle \dot{x}, \dot{x}\rangle\,ds - \frac{\Delta^2}{2}\left(\int_0^1 \frac{1}{\beta(x)}\,ds\right)^{-1} \end{aligned} \tag{3.3}$$

for $x \in \Omega^1(x_0, x_1, M)$. We observe that

(i) if V is bounded from below, J_1 is bounded from below and

$$J_1(x) \geq \inf_{x\in M} V(x) - E \equiv A > 0 \quad \forall x \in \Omega^1(x_0, x_1, M); \tag{3.4}$$

(ii) if (H_2) holds, J_2 is coercive and has minimum.

The proof of the last statement is quite simple if $p \in [0,2[$ in (1.6) (i.e., if β has subquadratic growth). If $p = 2$ (i.e., if β has quadratic growth), this is a result obtained in the study of geodesic connectedness of standard static spacetimes (for the proof, see [2, Proposition 4.1]).

By the previous properties, it is straightforward to prove the following lemmas.

Lemma 3.1. *Under the assumptions of Theorem 1.2, J is coercive and bounded from below.*

Lemma 3.2. *Under the assumptions of Theorem 1.2, J satisfies the Palais-Smale condition.*

Proof. Let $(x_m)_m$ be a sequence in $\Omega^1(x_0, x_1, M)$ satisfying (2.31). By Lemma 3.1, $(\|\dot{x}_m\|)_m$ is bounded, whence

$$\sup\{d(x_m(s), x_0) \mid s \in [0,1],\ m \in \mathbb{N}\} < +\infty. \tag{3.5}$$

Thus $(x_m)_m$ is bounded in $H^1([0,1], \mathbb{R}^N)$ and converges, up to subsequences, to a curve $x \in H^1([0,1], \mathbb{R}^N)$ weakly and uniformly. Reasoning as in [6, Lemma 2.1] and applying standard arguments, $x \in \Omega^1(x_0, x_1, M)$ and the convergence is strong. □

Now we are able to prove Theorem 1.2.

Proof of Theorem 1.2. Statement (a) is a consequence of the properties of the spacetime L (see Remark 1.6). Nevertheless, a variational proof follows by (3.4) and showing that J_2 is positive for $|\Delta|$ sufficiently small. This is trivial if in (H_2) $p \in [0,2[$, while if $p = 2$ this is a consequence of the results in [2], where the authors prove that

$$\lim_{\|\dot{x}\|\to+\infty} \int_0^1 \frac{\|\dot{x}\|^2}{\beta(x)}\,ds = +\infty. \tag{3.6}$$

So, if $x_0 \neq x_1$, for any $x \in \Omega^1(x_0, x_1, M)$,

$$\begin{aligned} 2J_2(x) &= \|\dot{x}\|^2\left(1 - \Delta^2\left(\int_0^1 \frac{\|\dot{x}\|^2}{\beta(x)}\,ds\right)^{-1}\right) \\ &\geq \|\dot{x}\|^2\left(1 - \Delta^2 L\right) \end{aligned} \tag{3.7}$$

for some positive constant L.

By Lemmas 3.1, 3.2, it is easy to prove that J admits a minimum point $\overline{x}$. If $x_0 = x_1$ and we consider the constant curve $y(s) \equiv x_0$ in $\Omega^1(x_0,x_1,M)$, we have

$$J(\overline{x}) \leq J(y) = \big(V(x_0) - E\big)\Big(- \frac{1}{2}\Delta^2\beta(x_0)\Big) < 0. \tag{3.8}$$

By Propositions 2.2 and 2.3, after a reparametrization, we obtain a solution of (1.4) which is timelike by (1.7), so (b) is proved.

In order to prove (c), we observe that by Lemma 3.2, J verifies all the assumptions of Theorem 2.6. Let $m \in \mathbb{N}$, $m \geq 1$. By Proposition 2.7, a compact subset B_m of $\Omega^1(x_0,x_1,M)$ exists with category larger that m. By Theorem 2.6, there are at least m distinct critical points of J at levels $c_1,\dots,c_m$ defined in (2.32) and

$$c_1 \leq c_2 \leq \cdots \leq c_m \leq \max_{x\in B_m} J(x). \tag{3.9}$$

For any $x \in B_m$, we have

$$J_2(x) \leq \frac{1}{2}\Big(a_m - \frac{\Delta^2}{b_m}\Big), \tag{3.10}$$

where

$$\begin{aligned} a_m &= \max_{x\in B_m}\int_0^1 \langle \dot{x},\dot{x}\rangle\, ds, \\ b_m &= \max_{x\in B_m}\int_0^1 \frac{1}{\beta(x)} ds \end{aligned} \tag{3.11}$$

so that $J_2(x) < 0$ if

$$|\Delta| > \sqrt{a_m b_m} \equiv \gamma_m. \tag{3.12}$$

Recalling that J_1 is positive,

$$\max_{x\in B_m} J(x) < 0, \tag{3.13}$$

so by (3.9), $c_1, c_2,\dots,c_m$ are negative and again using the variational principles, we obtain m distinct solutions of (1.4).

Again, all the solutions obtained are timelike because they have fixed energy E verifying (1.7). □

4. Proof of Theorem 1.5

Assume that all the assumptions of Theorem 1.5 hold and fix $a \in]0,\delta[$ such that x_0, x_1 are in Λ_a (see Section 2). To prove Theorem 1.5, it is necessary to state some properties of the penalized functionals $J_\epsilon : \Omega^1(x_0,x_1,\Lambda_a) \to \mathbb{R}$ (see (2.29)) and their critical points. Observe that by the definition of J_ϵ,

$$J_\epsilon(x) \geq J(x) \quad \forall x \in \Omega^1(x_0,x_1,\Lambda_a),\ \epsilon \in]0,1], \tag{4.1}$$

hence by (4.1) and reasoning as in Section 3, it is clear that

(i) J_ϵ is coercive and bounded from below for any $\epsilon \in]0,1[$.

An immediate consequence (using also (2.27) and Lemma 2.4) is the following lemma.

Lemma 4.1. *Let $(x_m)_m$ be a sequence in $\Omega^1(x_0,x_1,\Lambda_a)$ such that for some $K>0$,*

$$J_\epsilon(x_m) \le K \quad \forall m \in \mathbb{N}. \tag{4.2}$$

Then

$$\inf\{\phi(x_m(s)) \mid s \in [0,1],\ m \in \mathbb{N}\} > 0 \tag{4.3}$$

(where ϕ is as in (2.21)).

Moreover, by Lemma 4.1, the following properties hold.

(i) For any $\epsilon \in]0,1]$ and $c \in \mathbb{R}$, the sublevels

$$J_\epsilon^c = \{x \in \Omega^1(x_0,x_1,\Lambda_a) \mid J_\epsilon(x) \le c\} \tag{4.4}$$

are complete metric subspaces of $\Omega^1(x_0,x_1,\Lambda_a)$.

(ii) For any $\epsilon \in]0,1]$, J_ϵ satisfies the Palais-Smale condition.

Thus, it is straightforward to prove that

(iii) J_ϵ admits a minimum for any $\epsilon \in]0,1[$.

Our next aim is to show that if ϵ is sufficiently small, the critical points of J_ϵ are uniformly far from the boundary of Λ (and so are critical points of J). Firstly, we observe that if x is a critical point of J_ϵ, x is a smooth curve satisfying the following equation:

$$J_{2,\epsilon}(x)\nabla V(x) = J_1(x)\left(D_s\dot{x} + \frac{\Delta^2}{2\beta^2(x)}\left(\int_0^1 \frac{1}{\beta(x)}ds\right)^{-2}\nabla\beta(x) + \frac{2}{\phi^3(x)}\psi_\epsilon'\left(\frac{1}{\phi^2(x)}\right)\nabla\phi(x)\right), \tag{4.5}$$

where

$$J_{2,\epsilon}(x) = J_2(x) + \int_0^1 \psi_\epsilon\left(\frac{1}{\phi^2(x)}\right)ds \tag{4.6}$$

and J_1, J_2 have been defined, respectively, in (3.3).

Now we can prove the following proposition.

Proposition 4.2. *Under assumption (H_4), let x_ϵ be a critical point of J_ϵ such that*

$$J_\epsilon(x_\epsilon) < 0. \tag{4.7}$$

Then, $\epsilon_0 \in]0,1]$ exists such that

$$\min_{s\in[0,1]} \phi(x_\epsilon(s)) \ge \sqrt{\epsilon} \quad \forall \epsilon \in]0,\epsilon_0]. \tag{4.8}$$

Proof. Assume by contradiction that (4.8) does not hold, so a decreasing sequence $(\epsilon_m)_m$ in $]0,1]$ and a sequence $(x_m)_m$ of critical points of $J_{\epsilon_m} \equiv J_m$ exist such that

$$J_m(x_m) < 0, \tag{4.9}$$

$$\phi(x_m(s_m)) = \min_{s\in[0,1]} \phi(x_m(s)) < \sqrt{\epsilon_m}. \tag{4.10}$$

We set $g_m(s) = \phi(x_m(s))$, $s \in [0,1]$. Differentiating g_m twice, using (4.5), and recalling the definition of ϕ (see (2.26)), we have

$$\begin{aligned} g_m''(s_m) &= H^V(x_m(s_m))[\dot{x}_m(s_m), \dot{x}_m(s_m)] + \langle \nabla V(x_m(s_m)), D_s\dot{x}_m(s_m)\rangle \\ &= H^V(x_m(s_m))[\dot{x}_m(s_m), \dot{x}_m(s_m)] + \frac{J_{2,m}(x_m)}{J_1(x_m)} |\nabla V(x_m(s_m))|^2 \\ &\quad - \frac{\Delta^2}{2\beta^2(x_m(s_m))}\left(\int_0^1 \frac{1}{\beta(x_m)}ds\right)^{-2} \langle \nabla\beta(x_m(s_m)), \nabla V(x_m(s_m))\rangle \\ &\quad - \frac{2}{\phi^3(x_m(s_m))}\psi_m'\left(\frac{1}{\phi^2(x_m(s_m))}\right)|\nabla V(x_m(s_m))|^2, \end{aligned} \tag{4.11}$$

where $J_{2,\epsilon_m} \equiv J_{2,m}$ and $\psi_{\epsilon_m} \equiv \psi_m$. Observe now that by (4.9), $J_{2,m}(x_m) < 0$ and by (4.10), for m sufficiently large, we have

$$g_m(s_m) < \delta, \tag{4.12}$$

where δ is as in (H_4). Then, by (1.10), (1.11), and (4.11),

$$g_m''(s_m) \le -\frac{2}{\phi^3(x_m(s_m))}\psi_m'\left(\frac{1}{\phi^2(x_m(s_m))}\right)|\nabla V(x_m(s_m))|^2. \tag{4.13}$$

By the definition of ψ_ϵ and (4.10),

$$\psi_m'\left(\frac{1}{\phi^2(x_m(s_m))}\right) > 0, \tag{4.14}$$

so by (1.9), we have

$$g_m''(s_m) < 0, \tag{4.15}$$

which is a contradiction since s_m is a minimum point of g_m.

Finally, we can end our proof. □

Proof of Theorem 1.5. For the proof of (a), see Theorem 1.2 and Remark 1.6.

To prove (b), let x_ϵ be a minimum point of J_ϵ and observe that

$$J_\epsilon(x_\epsilon) < 0 \tag{4.16}$$

if $\epsilon < \phi^2(x_0)$ (using the constant curve x_0 as in the proof of Theorem 1.2). So by Proposition 4.2 and the variational principles, if ϵ is small, we get a solution of (1.4).

In order to prove (c), we observe that J_ϵ verifies all the assumptions of Theorem 2.6 and that if Λ is not contractible, the same holds for Λ_a if a is small. Then, for any $m \in \mathbb{N}$, $m \geq 1$, there are at least m distinct critical points of J_ϵ at levels

$$c_{1,\epsilon} \leq \cdots \leq c_{m,\epsilon} \leq \max_{x \in B_m} J_\epsilon(x), \tag{4.17}$$

where $c_{1,\epsilon}, \ldots, c_{m,\epsilon}$ are the critical values defined as in (2.32) and B_m is a compact subset of $\Omega^1(x_0, x_1, \Lambda_a)$ with category greater than m (see Proposition 2.7). By Proposition 4.2, if we choose $\epsilon < \epsilon_0$, we obtain m critical points of J with negative critical values. In fact, by (2.28), for any $x \in B_m$,

$$J_{2,\epsilon}(x) \leq \frac{1}{2}\left(a_m - \frac{\Delta^2}{b_m}\right) + c_m, \tag{4.18}$$

where a_m, b_m are defined as in (3.11),

$$c_m = \max_{x \in B_m} \int_0^1 \psi_1\left(\frac{1}{\phi^2(x)}\right), \tag{4.19}$$

then we can reason as in Theorem 1.2(c).

The solutions obtained in the proof of (b) and (c) are timelike because their spatial components lie in Λ_a. $\square$

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