INVARIANT f-STRUCTURES ON THE FLAG MANIFOLDS $\mathrm{SO}(n)/\mathrm{SO}(2) \times \mathrm{SO}(n-3)$

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We consider manifolds of oriented flags $\mathrm{SO}(n)/\mathrm{SO}(2) \times \mathrm{SO}(n-3)$ ($n \geq 4$) as 4- and 6-symmetric spaces and indicate characteristic conditions for invariant Riemannian metrics under which the canonical f-structures on these homogeneous Φ-spaces belong to the classes **Kill f**, **NKf**, and **G₁f** of generalized Hermitian geometry.

1. Introduction

Invariant structures on homogeneous manifolds are of fundamental importance in differential geometry. Recall that an affinor structure F (i.e., a tensor field F of type $(1,1)$) on a homogeneous manifold G/H is called invariant (with respect to G) if for any $g \in G$ we have $d\tau(g) \circ F = F \circ d\tau(g)$, where $\tau(g)(xH) = (gx)H$. An important place among homogeneous manifolds is occupied by homogeneous Φ-spaces [8, 9] of order k (which are also referred to as k-symmetric spaces [17]), that is, the homogeneous spaces generated by Lie group automorphisms Φ such that $\Phi^k = \mathrm{id}$. Each k-symmetric space has an associated object, the commutative algebra $\mathcal{A}(\theta)$ of canonical affinor structures [7, 8], which is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on G/H. In its turn, $\mathcal{A}(\theta)$ contains well-known classical structures, in particular, f-structures in the sense of Yano [19] (i.e., affinor structures $F = f$ satisfying $f^3 + f = 0$). It should be mentioned that an f-structure compatible with a (pseudo-)Riemannian metric is known to be one of the central objects in the concept of generalized Hermitian geometry [14].

From this point of view it is interesting to consider manifolds of oriented flags of the form

$$\mathrm{SO}(n)/\mathrm{SO}(2) \times \mathrm{SO}(n-3) \quad (n \geq 4) \tag{1.1}$$

as they can be generated by automorphisms of any even finite order $k \geq 4$. At the same time, it can be proved that an arbitrary invariant Riemannian metric on these manifolds is (up to a positive coefficient) completely determined by the pair of positive numbers (s,t). Therefore, it is natural to try to find characteristic conditions imposed on s and t

under which canonical f-structures on homogeneous manifolds (1.1) belong to the main classes of f-structures in the generalized Hermitian geometry. This question is partly considered in the paper.

The paper is organized as follows. In Section 2, basic notions and results related to homogeneous regular Φ-spaces and canonical affinor structures on them are collected. In particular, this section includes a precise description of all canonical f-structures on homogeneous k-symmetric spaces.

In Section 3, we dwell on the main concepts of generalized Hermitian geometry and consider the special classes of metric f-structures such as **Kill f**, **NK f**, and **G$_1$f**.

In Section 4, we describe manifolds of oriented flags of the form

$$\mathrm{SO}(n)/\underbrace{\mathrm{SO}(2)\times\cdots\times\mathrm{SO}(2)}_{m}\times\mathrm{SO}(n-2m-1) \tag{1.2}$$

and construct inner automorphisms by which they can be generated.

In Section 5, we describe the action of the canonical f-structures on the flag manifolds of the form (1.1) considered as homogeneous Φ-spaces of orders 4 and 6.

Finally, in Section 6, we indicate characteristic conditions for invariant Riemannian metrics on the flag manifolds (1.1) under which the canonical f-structures on these homogeneous Φ-spaces belong to the classes **Kill f**, **NK f**, and **G$_1$f**.

2. Canonical structures on regular Φ-spaces

We start with some basic definitions and results related to homogeneous regular Φ-spaces and canonical affinor structures. More detailed information can be found in [6, 8, 9, 17, 18] and some others.

Let G be a connected Lie group, and let Φ be its automorphism. Denote by G^{Φ} the subgroup consisting of all fixed points of Φ and by G_0^{Φ} the identity component of G^{Φ}. Suppose a closed subgroup H of G satisfies the condition

$$G_0^{\Phi}\subset H\subset G^{\Phi}. \tag{2.1}$$

Then G/H is called a *homogeneous Φ-space* [8, 9].

Among homogeneous Φ-spaces a fundamental role is played by *homogeneous Φ-spaces of order k* ($\Phi^k=\mathrm{id}$) or, in the other terminology, *homogeneous k-symmetric spaces* (see [17]).

Note that there exist homogeneous Φ-spaces that are not reductive. That is why so-called regular Φ-spaces first introduced by Stepanov [18] are of fundamental importance.

Let G/H be a homogeneous Φ-space, let $\mathfrak{g}$ and $\mathfrak{h}$ be the corresponding Lie algebras for G and H, and let $\varphi=d\Phi_e$ be the automorphism of $\mathfrak{g}$. Consider the linear operator $A=\varphi-\mathrm{id}$ and the Fitting decomposition $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ with respect to A, where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ denote 0- and 1-component of the decomposition, respectively. Further, let $\varphi=\varphi_s\varphi_u$ be the Jordan decomposition, where φ_s and φ_u are semisimple and unipotent components of φ, respectively, $\varphi_s\varphi_u=\varphi_u\varphi_s$. Denote by $\mathfrak{g}^{\gamma}$ a subspace of all fixed points for a linear endomorphism γ in $\mathfrak{g}$. It is clear that $\mathfrak{h}=\mathfrak{g}^{\varphi}=\mathrm{Ker}\,A$, $\mathfrak{h}\subset\mathfrak{g}_0$, $\mathfrak{h}\subset\mathfrak{g}^{\varphi_s}$.

Definition 2.1 [6, 8, 9, 18]. A homogeneous Φ-space G/H is called a *regular Φ-space* if one of the following equivalent conditions is satisfied:

(1) $\mathfrak{h} = \mathfrak{g}_0$;

(2) $\mathfrak{g} = \mathfrak{h} \oplus A\mathfrak{g}$;

(3) the restriction of the operator A to $A\mathfrak{g}$ is nonsingular;

(4) $A^2 X = 0 \Rightarrow AX = 0$ for all $X \in \mathfrak{g}$;

(5) the matrix of the automorphism φ can be represented in the form $\left(\begin{smallmatrix} E & 0 \\ 0 & B \end{smallmatrix}\right)$, where the matrix B does not admit the eigenvalue 1;

(6) $\mathfrak{h} = \mathfrak{g}^{\varphi_s}$.

A distinguishing feature of a regular Φ-space G/H is that each such space is reductive, its reductive decomposition being $\mathfrak{g} = \mathfrak{h} \oplus A\mathfrak{g}$ (see [18]). $\mathfrak{g} = \mathfrak{h} \oplus A\mathfrak{g}$ is commonly referred to as the *canonical reductive decomposition* corresponding to a regular Φ-space G/H and $\mathfrak{m} = A\mathfrak{g}$ is the *canonical reductive complement*.

It should be mentioned that any homogeneous Φ-space G/H of order k is regular (see [18]), and, in particular, any k-symmetric space is reductive.

Let us now turn to canonical f-structures on regular Φ-spaces.

An *affinor structure* on a smooth manifold is a tensor field of type $(1,1)$ realized as a field of endomorphisms acting on its tangent bundle. An affinor structure F on a homogeneous manifold G/H is called *invariant* (with respect to G) if for any $g \in G$ we have $d\tau(g) \circ F = F \circ d\tau(g)$. It is known that any invariant affinor structure F on a homogeneous manifold G/H is completely determined by its value F_o at the point $o = H$, where F_o is invariant with respect to $\mathrm{Ad}(H)$. For simplicity, further we will not distinguish an invariant structure on G/H and its value at $o = H$ throughout the rest of the paper.

Let us denote by θ the restriction of φ to $\mathfrak{m}$.

Definition 2.2 [7, 8]. An invariant affinor structure F on a regular Φ-space G/H is called *canonical* if its value at the point $o = H$ is a polynomial in θ.

Remark that the set $\mathcal{A}(\theta)$ of all canonical structures on a regular Φ-space G/H is a commutative subalgebra of the algebra $\mathcal{A}$ of all invariant affinor structures on G/H. This subalgebra contains well-known classical structures such as *almost product structures* ($P^2 = \mathrm{id}$), *almost complex structures* ($J^2 = -\mathrm{id}$), *f-structures* ($f^3 + f = 0$).

The sets of all canonical structures of the above types were completely described in [7, 8]. In particular, for homogeneous k-symmetric spaces the precise computational formulae were indicated. For future reference we cite here the result pertinent to f-structures and almost product structures only. Put

$$u = \begin{cases} n & \text{if } k = 2n+1, \\ n-1 & \text{if } k = 2n. \end{cases} \tag{2.2}$$

THEOREM 2.3 [7, 8]. *Let G/H be a homogeneous Φ-space of order k ($k \geq 3$).*

(1) *All nontrivial canonical f-structures on G/H can be given by the operators*

$$f(\theta) = \frac{2}{k} \sum_{m=1}^{u} \left(\sum_{j=1}^{u} \zeta_j \sin \frac{2\pi m j}{k} \right) (\theta^m - \theta^{k-m}), \tag{2.3}$$

where $\zeta_j \in \{1, 0, -1\}$, $j = 1, 2, \ldots, u$, and not all ζ_j are equal to zero.

(2) *All canonical almost product structures P on G/H can be given by polynomials*
   $P(\theta) = \sum_{m=0}^{k-1} a_m \theta^m$, *where*
   (a) *if $k = 2n+1$, then*

$$a_m = a_{k-m} = \frac{2}{k} \sum_{j=1}^{u} \xi_j \cos \frac{2\pi m j}{k};$$
   (2.4)

(b) *if $k = 2n$, then*

$$a_m = a_{k-m} = \frac{1}{k} \left(2 \sum_{j=1}^{u} \xi_j \cos \frac{2\pi m j}{k} + (-1)^m \xi_n \right).$$
   (2.5)

Here the numbers ξ_j, $j = 1, 2, \ldots, u$, take their values from the set $\{-1, 1\}$.

The results mentioned above were particularized for homogeneous Φ-spaces of smaller orders 3, 4, and 5 (see [7, 8]). Note that there are no fundamental obstructions to considering of higher orders k. Specifically, for future consideration we need the description of canonical f-structures and almost product structures on homogeneous Φ-spaces of orders 4 and 6 only.

COROLLARY 2.4 [7, 8]. *Any homogeneous Φ-space of order 4 admits (up to sign) the only canonical f-structure*

$$f_0(\theta) = \frac{1}{2}(\theta - \theta^3)$$
   (2.6)

and the only almost product structure

$$P_0(\theta) = \theta^2.$$
   (2.7)

COROLLARY 2.5. *On any homogeneous Φ-space of order 6, there exist (up to sign) only the following canonical f-structures:*

$$f_1(\theta) = \frac{1}{\sqrt{3}}(\theta - \theta^5), \qquad f_2(\theta) = \frac{1}{2\sqrt{3}}(\theta - \theta^2 + \theta^4 - \theta^5),$$

$$f_3(\theta) = \frac{1}{2\sqrt{3}}(\theta + \theta^2 - \theta^4 - \theta^5), \qquad f_4(\theta) = \frac{1}{\sqrt{3}}(\theta^2 - \theta^4),$$
   (2.8)

and only the following almost product structures:

$$P_1(\theta) = -\operatorname{id}, \qquad P_2(\theta) = \frac{\theta}{3} + \theta^2 + \frac{\theta^3}{3} + \theta^4 + \frac{\theta^5}{3},$$

$$P_3(\theta) = \theta^3, \qquad P_4(\theta) = -\frac{2\theta^2}{3} + \frac{\theta^3}{3} - \frac{2\theta^5}{3}.$$
   (2.9)

3. Some important classes in generalized Hermitian geometry

The concept of generalized Hermitian geometry created in the 1980s (see [14]) is a natural consequence of the development of Hermitian geometry. One of its central objects

is a *metric f-structure*, that is, an f-structure compatible with a (pseudo-)Riemannian metric $g = \langle \cdot, \cdot \rangle$ in the following sense:

$$\langle fX, Y \rangle + \langle X, fY \rangle = 0 \quad \text{for any } X, Y \in \mathfrak{X}(M). \tag{3.1}$$

Evidently, this concept is a generalization of one of the fundamental notions in Hermitian geometry, namely, almost Hermitian structure J. It is also worth noticing that the main classes of generalized Hermitian geometry (see [5, 6, 12–14]) in the special case $f = J$ coincide with those of Hermitian geometry (see [11]).

In what follows, we will mainly concentrate on the classes **Killf**, **NKf**, and **G₁f** of metric f-structures defined below.

A fundamental role in generalized Hermitian geometry is played by a tensor T of type (2,1) which is called a *composition tensor* [14]. In [14] it was also shown that such a tensor exists on any metric f-manifold and it is possible to evaluate it explicitly:

$$T(X, Y) = \frac{1}{4} f\left(\nabla_{fX}(f)fY - \nabla_{f^2X}(f)f^2Y\right), \tag{3.2}$$

where ∇ is the Levi-Civita connection of a (pseudo-)Riemannian manifold (M, g), $X, Y \in \mathfrak{X}(M)$.

The structure of a so-called *adjoint Q-algebra* (see [14]) on $\mathfrak{X}(M)$ can be defined by the formula $X * Y = T(X, Y)$. It gives the opportunity to introduce some classes of metric f-structures in terms of natural properties of the adjoint Q-algebra. For example, if $T(X, X) = 0$ (i.e., $\mathfrak{X}(M)$ is an anticommutative Q-algebra), then f is referred to as a G_1f-structure. **G₁f** stands for the class of G_1f-structures.

A metric f-structure on (M, g) is said to be a *Killing f-structure* if

$$\nabla_X(f)X = 0 \quad \text{for any } X \in \mathfrak{X}(M) \tag{3.3}$$

(i.e., f is a Killing tensor) (see [12, 13]). The class of Killing f-structures is denoted by **Killf**. The defining property of *nearly Kähler f-structures* (or *NKf-structures*) is

$$\nabla_{fX}(f)fX = 0. \tag{3.4}$$

This class of metric f-structures, which is denoted by **NKf**, was determined in [5] (see also [2, 4]). It is easy to see that for $f = J$ the classes **Killf** and **NKf** coincide with the well-known class **NK** of *nearly Kähler structures* [10].

The following relations between the classes mentioned are evident:

$$\mathbf{Killf} \subset \mathbf{NKf} \subset \mathbf{G_1f}. \tag{3.5}$$

A special attention should be paid to the particular case of naturally reductive spaces. Recall that a homogeneous Riemannian manifold $(G/H, g)$ is known to be a *naturally reductive space* [15] with respect to the reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ if

$$g([X, Y]_{\mathfrak{m}}, Z) = g(X, [Y, Z]_{\mathfrak{m}}) \quad \text{for any } X, Y, Z \in \mathfrak{m}. \tag{3.6}$$

It should be mentioned that if G/H is a regular Φ-space, G a semisimple Lie group, then G/H is a naturally reductive space with respect to the (pseudo-)Riemannian metric g induced by the Killing form of the Lie algebra $\mathfrak{g}$ (see [18]). In [2–5] a number of results helpful in checking whether the particular f-structure on a naturally reductive space belongs to the main classes of generalized Hermitian geometry were obtained.

4. Manifolds of oriented flags

In linear algebra a *flag* is defined as a finite sequence $L_0,\ldots,L_m$ of subspaces of a vector space L such that

$$L_0 \subset L_1 \subset \cdots \subset L_m, \qquad (4.1)$$

$L_i \neq L_{i+1}$, $i = 0,\ldots,m-1$ (see [16]).

A flag (4.1) is known to be *full* if for any $i = 0,\ldots,n-1$, $\dim L_{i+1} = \dim L_i + 1$, where $n = \dim L$. It is readily seen that having fixed any basis $\{e_1,\ldots,e_n\}$ of L we can construct a full flag by setting $L_0 = \{0\}$, $L_i = \mathscr{L}(e_1,\ldots,e_i)$, $i = 1,\ldots,n$.

We call a flag $L_{i_1} \subset L_{i_2} \subset \cdots \subset L_{i_n}$ (here and below the subscript denotes the dimension of the subspace) *oriented* if for any L_{i_j} and its two bases $\{e_1,\ldots,e_{i_j}\}$ and $\{e'_1,\ldots,e'_{i_j}\}$ $\det A > 0$, where $e'_t = A e_t$ for any $t = 1,\ldots,i_j$. Moreover, for any two subspaces $L_{i_k} \subset L_{i_j}$ their orientations should be set in accordance.

The notion of a flag manifold enjoys several interpretations (see, e.g., [1]). However, the most relevant to the case is the following.

Definition 4.1. For any vector space L and any fixed set $(i_1,i_2,\ldots,i_k)$ consider the set M of all (oriented) flags of L of the form $L_{i_1} \subset L_{i_2} \subset \cdots \subset L_{i_k}$ ($L_{i_j} \neq L_{i_{j+1}}$, $j = 1,\ldots,k-1$). Then M with a transitive action of a Lie group G is called a *flag manifold (manifold of oriented flags)*. Equivalently, a *flag manifold (manifold of oriented flags)* can be defined as a manifold of the form G/K, where G is a Lie group acting on M transitively and K is an isotropy subgroup at some point $L_{i_1}^0 \subset L_{i_2}^0 \subset \cdots \subset L_{i_k}^0$ ($L_{i_j}^0 \neq L_{i_{j+1}}^0$, $j = 1,\ldots,k-1$) of M.

And now let us turn to the manifold of oriented flags

$$\mathrm{SO}(n)/\underbrace{\mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2)}_{m} \times \mathrm{SO}(n - 2m - 1). \qquad (4.2)$$

PROPOSITION 4.2. *The set of all oriented flags*

$$L_1 \subset L_3 \subset \cdots \subset L_{2m+1} \subset L_n = L \qquad (4.3)$$

of a vector space L with respect to the action of $\mathrm{SO}(n)$ *is isomorphic to*

$$\mathrm{SO}(n)/\underbrace{\mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2)}_{m} \times \mathrm{SO}(n - 2m - 1). \qquad (4.4)$$

Proof. Fix some basis $\{e_1,\ldots,e_n\}$ in L_n. Consider the isotropy subgroup I_o at the point

$$o = (\mathscr{L}(e_1) \subset \mathscr{L}(e_1,e_2,e_3) \subset \cdots \subset \mathscr{L}(e_1,\ldots,e_{2m+1}) \subset \mathscr{L}(e_1,\ldots,e_n)). \qquad (4.5)$$

By the definition for any $A \in I_o$,

$$
\begin{aligned}
A &: \mathcal{L}(e_1) \longrightarrow \mathcal{L}(e_1), \\
A &: \mathcal{L}(e_1, e_2, e_3) \longrightarrow \mathcal{L}(e_1, e_2, e_3), \dots, \\
A &: \mathcal{L}(e_1, \dots, e_{2m+1}) \longrightarrow \mathcal{L}(e_1, \dots, e_{2m+1}), \\
A &: \mathcal{L}(e_1, \dots, e_n) \longrightarrow \mathcal{L}(e_1, \dots, e_n).
\end{aligned}
\tag{4.6}
$$

As $\{e_1, \dots, e_n\}$ is a basis, it immediately follows that

$$
\begin{aligned}
A &: \mathcal{L}(e_1) \longrightarrow \mathcal{L}(e_1), \\
A &: \mathcal{L}(e_2, e_3) \longrightarrow \mathcal{L}(e_2, e_3), \dots, \\
A &: \mathcal{L}(e_{2m}, e_{2m+1}) \longrightarrow \mathcal{L}(e_{2m}, e_{2m+1}), \\
A &: \mathcal{L}(e_{2m+2}, \dots, e_n) \longrightarrow \mathcal{L}(e_{2m+2}, \dots, e_n).
\end{aligned}
\tag{4.7}
$$

Thus $L = L_n$ can be decomposed into the sum of A-invariant subspaces

$$
L = \mathcal{L}(e_1) \oplus \mathcal{L}(e_2, e_3) \oplus \cdots \oplus \mathcal{L}(e_{2m}, e_{2m+1}) \oplus \mathcal{L}(e_{2m+2}, \dots, e_n).
\tag{4.8}
$$

The matrix of the operator A in the basis $\{e_1, \dots, e_n\}$ is cellwise-diagonal:

$$
A = \operatorname{diag}\{A^1_{1 \times 1}, A^3_{2 \times 2}, \dots, A^{2m+1}_{2 \times 2}, A^n_{(n-2m-1) \times (n-2m-1)}\}.
\tag{4.9}
$$

Since $A \in SO(n)$, its cells $A^1, A^3, \dots, A^{2m+1}, A^n$ are orthogonal matrices. All the flags we consider are oriented, thus for any $i \in \{1, 3, \dots, 2m + 1, n\}$, $\det A^i > 0$. This proves that $A^1 = (1)$, $A^3 \in SO(2), \dots, A^{2m+1} \in SO(2)$, $A^n \in SO(n - 2m - 1)$.

Therefore

$$
I_o = \underbrace{SO(2) \times \cdots \times SO(2)}_{m} \times SO(n - 2m - 1).
\tag{4.10}
$$

This completes the proof. $\qquad\square$

PROPOSITION 4.3. *The manifold of oriented flags*

$$
SO(n) / \underbrace{SO(2) \times \cdots \times SO(2)}_{m} \times SO(n - 2m - 1)
\tag{4.11}
$$

is a homogeneous Φ-space. It can be generated by inner automorphisms Φ of any finite order k, where k is even, $k > 2$ and $k \geq 2m + 2$:

$$
\Phi : SO(n) \longrightarrow SO(n), \qquad A \longrightarrow BAB^{-1},
\tag{4.12}
$$

where

$$B = \operatorname{diag}\{1, \varepsilon_1, \ldots, \varepsilon_m, -1, \ldots, -1\},$$

$$\varepsilon_t = \begin{pmatrix} \cos\dfrac{2\pi t}{k} & \sin\dfrac{2\pi t}{k} \\[2ex] -\sin\dfrac{2\pi t}{k} & \cos\dfrac{2\pi t}{k} \end{pmatrix}. \tag{4.13}$$

Proof. Here

$$G = SO(n), \qquad H = \underbrace{SO(2) \times \cdots \times SO(2)}_{m} \times SO(n - 2m - 1). \tag{4.14}$$

We need to prove that the group of all fixed points G^Φ satisfies the condition

$$G_0^\Phi \subset H \subset G^\Phi. \tag{4.15}$$

By definition $G^\Phi = \{A \mid BAB^{-1} = A\} = \{A \mid BA = AB\}$. Equating the correspondent elements of AB and BA and solving systems of linear equations it is possible to calculate that

$$G^\Phi = \{\pm 1\} \times \underbrace{SO(2) \times \cdots \times SO(2)}_{m} \times SO(n - 2m - 1). \tag{4.16}$$

$\square$

5. Canonical f-structures on 4- and 6-symmetric space $SO(n)/SO(2) \times SO(n-3)$

Let us consider $SO(n)/SO(2) \times SO(n-3)$ $(n \geq 4)$ as a homogeneous Φ-space of order 4. According to Proposition 4.3 it can be generated by the inner automorphism $\Phi : A \to BAB^{-1}$, where

$$B = \operatorname{diag}\left\{ 1, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \underbrace{-1, \ldots, -1}_{n-3} \right\}. \tag{5.1}$$

Therefore (1.1) is a reductive space. It is not difficult to check that the canonical reductive complement $\mathfrak{m}$ consists of matrices of the form

$$S = \begin{pmatrix} 0 & s_{12} & s_{13} & s_{14} & \cdots & s_{1n} \\ -s_{12} & 0 & 0 & s_{24} & \cdots & s_{2n} \\ -s_{13} & 0 & 0 & s_{34} & \cdots & s_{3n} \\ -s_{14} & -s_{24} & -s_{34} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -s_{1n} & -s_{2n} & -s_{3n} & 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{m}. \tag{5.2}$$

According to Corollary 2.4 the only canonical f-structure on this homogeneous Φ-space is determined by the formula

$$f_0(\theta) = \frac{1}{2}(\theta - \theta^3). \tag{5.3}$$

Its action can be written in the form:

$$
f_0 : S \longrightarrow
\begin{pmatrix}
0 & s_{13} & -s_{12} & 0 & \cdots & 0 \\
-s_{13} & 0 & 0 & -s_{34} & \cdots & -s_{3n} \\
s_{12} & 0 & 0 & s_{24} & \cdots & s_{2n} \\
0 & s_{34} & -s_{24} & 0 & \cdots & 0 \\
\hdotsfor{6} \\
0 & s_{3n} & -s_{2n} & 0 & \cdots & 0
\end{pmatrix}.
\tag{5.4}
$$

Now let us consider (1.1) as a 6-symmetric space generated by the inner automorphism $\Phi : A \to BAB^{-1}$, where

$$
B = \mathrm{diag}\left\{ 1, \begin{pmatrix} \dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \\[2mm] -\dfrac{\sqrt{3}}{2} & \dfrac{1}{2} \end{pmatrix}, \underbrace{-1,\ldots,-1}_{n-3} \right\}.
\tag{5.5}
$$

Taking Corollary 2.5 into account we can represent the action of the canonical f-structures on this homogeneous Φ-space as follows:

$$
f_1(\theta) = \frac{1}{\sqrt{3}}(\theta - \theta^5) : S \longrightarrow
\begin{pmatrix}
0 & s_{13} & -s_{12} & 0 & \cdots & 0 \\
-s_{13} & 0 & 0 & -s_{34} & \cdots & -s_{3n} \\
s_{12} & 0 & 0 & s_{24} & \cdots & s_{2n} \\
0 & s_{34} & -s_{24} & 0 & \cdots & 0 \\
\hdotsfor{6} \\
0 & s_{3n} & -s_{2n} & 0 & \cdots & 0
\end{pmatrix},
$$

$$
f_2(\theta) = \frac{1}{2\sqrt{3}}(\theta - \theta^2 + \theta^4 - \theta^5) : S \longrightarrow
\begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & -s_{34} & \cdots & -s_{3n} \\
0 & 0 & 0 & s_{24} & \cdots & s_{2n} \\
0 & s_{34} & -s_{24} & 0 & \cdots & 0 \\
\hdotsfor{6} \\
0 & s_{3n} & -s_{2n} & 0 & \cdots & 0
\end{pmatrix},
$$

$$
f_3(\theta) = \frac{1}{2\sqrt{3}}(\theta + \theta^2 - \theta^4 - \theta^5) : S \longrightarrow
\begin{pmatrix}
0 & s_{13} & -s_{12} & 0 & \cdots & 0 \\
-s_{13} & 0 & 0 & 0 & \cdots & 0 \\
s_{12} & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 \\
\hdotsfor{6} \\
0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix},
$$

$$f_4(\theta) = \frac{1}{\sqrt{3}}(\theta^2 - \theta^4) : S \longrightarrow \begin{pmatrix} 0 & s_{13} & -s_{12} & 0 & \cdots & 0 \\ -s_{13} & 0 & 0 & s_{34} & \cdots & s_{3n} \\ s_{12} & 0 & 0 & -s_{24} & \cdots & -s_{2n} \\ 0 & -s_{34} & s_{24} & 0 & \cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & -s_{3n} & s_{2n} & 0 & \cdots & 0 \end{pmatrix}.$$

$$(5.6)$$

6. Canonical f-structures and invariant Riemannian metrics on $SO(n)/SO(2) \times SO(n-3)$

Let us consider manifolds of oriented flags of the form (1.1) as 4- and 6-symmetric spaces. Our task is to indicate characteristic conditions for invariant Riemannian metrics under which the canonical f-structures on these homogeneous Φ-spaces belong to the classes **Killf**, **NKf**, and **G_1f**.

We begin with some preliminary considerations.

PROPOSITION 6.1. *The reductive complement $\mathfrak{m}$ of the homogeneous space $SO(n)/SO(2) \times SO(n-3)$ admits the decomposition into the direct sum of $Ad(H)$-invariant irreducible subspaces $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$.*

Proof. The explicit form of the reductive complement of (1.1) was indicated in Section 5. Put

$$\mathfrak{m}_1 = \left\{ \begin{pmatrix} 0 & a_1 & a_2 & 0 & \cdots & 0 \\ -a_1 & 0 & 0 & 0 & \cdots & 0 \\ -a_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \middle| a_1, a_2 \in \mathbb{R} \right\},$$

$$\mathfrak{m}_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & c_1 & \cdots & c_{n-3} \\ 0 & 0 & 0 & d_1 & \cdots & d_{n-3} \\ 0 & -c_1 & -d_1 & 0 & \cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & -c_{n-3} & -d_{n-3} & 0 & \cdots & 0 \end{pmatrix} \middle| c_1,\ldots,c_{n-3} \in \mathbb{R},\ d_1,\ldots,d_{n-3} \in \mathbb{R} \right\},$$

$$\mathfrak{m}_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 & b_1 & \cdots & b_{n-3} \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ -b_1 & 0 & 0 & 0 & \cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -b_{n-3} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \middle| b_1,\ldots,b_{n-3} \in \mathbb{R} \right\}.$$

$$(6.1)$$

Since $SO(2) \times SO(n-3)$ is a connected Lie group, $\mathfrak{m}_i$ $(i = 1, 2, 3)$ is $Ad(H)$-invariant if and only if $[\mathfrak{h}, \mathfrak{m}_i] \subset \mathfrak{m}_i$. It can easily be shown that this condition holds.

We claim that for any $i \in \{1, 2, 3\}$ there exist no such nontrivial subspaces $\overline{\mathfrak{m}}_i$ and $\widehat{\mathfrak{m}}_i$ that $\mathfrak{m}_i = \overline{\mathfrak{m}}_i \oplus \widehat{\mathfrak{m}}_i$ and $[\mathfrak{h}, \overline{\mathfrak{m}}_i] \subset \overline{\mathfrak{m}}_i$, $[\mathfrak{h}, \widehat{\mathfrak{m}}_i] \subset \widehat{\mathfrak{m}}_i$.

To prove this we identify $\mathfrak{m}$ and $\{(a_1, a_2, b_1, \ldots, b_{n-3}, c_1, \ldots, c_{n-3}, d_1, \ldots, d_{n-3})\}$. In what follows we are going to represent any $H \in \mathfrak{h}$ in the form

$$H = \operatorname{diag}\{0, H_1, H_2\}, \tag{6.2}$$

where

$$H_1 = \begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix}, \tag{6.3}$$

$$H_2 = \begin{pmatrix} 0 & h_{12} & \cdots & h_{1\,n-3} \\ -h_{12} & 0 & \cdots & h_{2\,n-3} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -h_{1\,n-3} & -h_{2\,n-3} & \cdots & 0 \end{pmatrix}. \tag{6.4}$$

Put $F(H)(M) = [H, M]$ for any $H \in \mathfrak{h}$, $M \in \mathfrak{m}$. In the above notations we have

$$F(H)|_{\mathfrak{m}_1} : (a_1\ a_2)^T \longrightarrow H_1 (a_1\ a_2)^T,$$

$$F(H)|_{\mathfrak{m}_2} : (c_1 \cdots c_{n-3}\ d_1 \cdots d_{n-3})^T \longrightarrow \begin{pmatrix} H_2 & hE \\ -hE & H_2 \end{pmatrix} (c_1 \cdots c_{n-3}\ d_1 \cdots d_{n-3})^T, \tag{6.5}$$

$$F(H)|_{\mathfrak{m}_3} : (b_1 \cdots b_{n-3})^T \longrightarrow H_2 (b_1 \cdots b_{n-3})^T.$$

First, let us prove that $\mathfrak{m}_3$ cannot be decomposed into the direct sum of $Ad(H)$-invariant subspaces.

The proof is by reductio ad absurdum. Suppose there exists an $Ad(H)$-invariant subspace $W \subset \mathfrak{m}_3$. This implies that for any H_2 of the form (6.4) and $x = (x_1 \cdots x_{n-3})^T \in W$, $H_2 x$ belongs to W.

It is possible to choose a vector $v_1 = (\alpha_1 \cdots \alpha_{n-3})^T \in W$ such that $\alpha_1 \neq 0$. Indeed, the nonexistence of such a vector yields that for any $w = (w_1 \cdots w_{n-3})^T \in W$, $w_1 = 0$. Take such $w \in W$ that, for some $1 < i \leq n-3$, $w_i \neq 0$ and the skew-symmetric matrix $K = \{k_{ij}\}$ with all elements except $k_{1i} = -k_{i1} = 1$ equal to zero. Then $Kw = (w_i * \cdots *) \notin W$.

Consider the following system of vectors $\{v_1, \ldots, v_{n-3}\}$, where

$$v_2 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} v_1 = (\alpha_2 - \alpha_1 0 \cdots 0)^T,$$

$$v_3 = \begin{pmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 0 & \cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} v_1 = (\alpha_3 0 - \alpha_1 \cdots 0)^T, \ldots,$$

$$v_{n-3} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix} v_1 = (\alpha_n 0 \cdots 0 - \alpha_1)^T.$$

(6.6)

Obviously,

$$\dim \mathscr{L}(v_1, \ldots, v_{n-3}) = \mathrm{rank} \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_{n-3} \\ \alpha_2 & -\alpha_1 & 0 & \cdots & 0 \\ \alpha_3 & 0 & -\alpha_1 & \cdots & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \alpha_{n-3} & 0 & 0 & \cdots & -\alpha_1 \end{pmatrix} = n - 3.$$

(6.7)

This contradicts our assumption.

Continuing the same line of reasoning, we see that neither $\mathfrak{m}_1$ nor $\mathfrak{m}_2$ can be decomposed into the sum of $\mathrm{Ad}(H)$-invariant summands. $\square$

It is not difficult to check that the space in question possesses the following property.

PROPOSITION 6.2.

$$[\mathfrak{m}_i, \mathfrak{m}_{i+1}] \subset \mathfrak{m}_{i+2} \quad (modulo\ 3).$$

(6.8)

Denote by g_0 the naturally reductive metric generated by the Killing form B: $g_0 = -B|_{\mathfrak{m}\times\mathfrak{m}}$. In our case $B = -(n-1)\,\mathrm{Tr}\,X^T Y$, $X, Y \in \mathfrak{so}(n)$.

PROPOSITION 6.3. *The decomposition $\mathfrak{h} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ is B-orthogonal.*

Proof. For the explicit form of $\mathfrak{m}$ and $\mathfrak{h}$ see Sections 5 and 6. It can easily be seen that for any $X \in \mathfrak{m}$, $Y \in \mathfrak{h}$, $\mathrm{Tr}\,X^T Y = 0$. It should also be noted that it was proved in [18] that $\mathfrak{h}$ is orthogonal to $\mathfrak{m}$ with respect to B.

For any almost product structure P put

$$\mathfrak{m}^- = \{X \in \mathfrak{m} | P(X) = -X\}, \qquad \mathfrak{m}^+ = \{X \in \mathfrak{m} | P(X) = X\}.$$

(6.9)

Suppose that P is compatible with g_0, that is, $g_0(X, Y) = g_0(PX, PY)$ (e.g., this is true for any canonical almost product structure P [6]). Clearly, $\mathfrak{m}^-$ and $\mathfrak{m}^+$ are orthogonal with respect to g_0, since for any $X \in \mathfrak{m}^+$, $Y \in \mathfrak{m}^-$,

$$g_0(X, Y) = g_0(P(X), P(Y)) = g_0(X, -Y) = -g_0(X, Y).$$

(6.10)

Let us consider the action of the canonical almost product structures on the 6-symmetric space (1.1). Here we use notations of Corollary 2.5.

For $P_2(\theta) = (1/3)\theta + \theta^2 + (1/3)\theta^3 + \theta^4 + (1/3)\theta^5$ $\mathfrak{m}^- = \mathfrak{m}_1 \cup \mathfrak{m}_2$, $\mathfrak{m}^+ = \mathfrak{m}_3$, therefore $\mathfrak{m}_3 \perp \mathfrak{m}_1$, $\mathfrak{m}_3 \perp \mathfrak{m}_2$.

For $P_3(\theta) = \theta^3$ $\mathfrak{m}^- = \mathfrak{m}_1 \cup \mathfrak{m}_3$, $\mathfrak{m}^+ = \mathfrak{m}_2$, thus $\mathfrak{m}_2 \perp \mathfrak{m}_1$. The statement is proved.   □

It can be deduced from Propositions 6.1 and 6.3 that any invariant Riemannian metric g on (1.1) is (up to a positive coefficient) uniquely defined by the two positive numbers (s, t). It means that

$$g = g_0|_{\mathfrak{m}_1} + s g_0|_{\mathfrak{m}_2} + t g_0|_{\mathfrak{m}_3}. \tag{6.11}$$

Definition 6.4. (s, t) are called the *characteristic numbers* of the metric (6.11).

It should be pointed out that the canonical f-structures on the homogeneous Φ-space (1.1) of the orders 4 and 6 are metric f-structures with respect to all invariant Riemannian metrics, which are proved by direct calculations.

Recall that in case of an arbitrary Riemannian metric g the Levi-Civita connection has its Nomizu function defined by the formula (see [15])

$$\alpha(X, Y) = \frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y), \tag{6.12}$$

where $X, Y \in \mathfrak{m}$, the symmetric bilinear mapping U is determined by means of the formula

$$2g(U(X, Y), Z) = g(X, [Z, Y]_{\mathfrak{m}}) + g([Z, X]_{\mathfrak{m}}, Y), \quad X, Y, Z \in \mathfrak{m}. \tag{6.13}$$

Suppose g is an invariant Riemannian metric on the homogeneous Φ-space (1.1) with the characteristic numbers (s, t) $(s, t > 0)$. The following statement is true.

PROPOSITION 6.5.

$$U(X, Y) = \frac{t - s}{2}\left([X_{\mathfrak{m}_2}, Y_{\mathfrak{m}_3}] + [Y_{\mathfrak{m}_2}, X_{\mathfrak{m}_3}]\right) + \frac{t - 1}{2s}\left([X_{\mathfrak{m}_1}, Y_{\mathfrak{m}_3}] + [Y_{\mathfrak{m}_1}, X_{\mathfrak{m}_3}]\right)$$
$$+ \frac{s - 1}{2t}\left([X_{\mathfrak{m}_1}, Y_{\mathfrak{m}_2}] + [Y_{\mathfrak{m}_1}, X_{\mathfrak{m}_2}]\right). \tag{6.14}$$

Outline of the proof. First we apply (6.11) and the definition of g_0 to (6.13). We take four matrices $X = \{x_{ij}\}$, $Y = \{y_{ij}\}$, $Z = \{z_{ij}\}$, and $U = \{u_{ij}\}$ and calculate the right-hand and left-hand sides of the equality obtained. After that we can represent it in the form

$$c_{12}z_{12} + c_{13}z_{13} + \sum_{i=1}^{n} c_{1i}z_{1i} + \sum_{i=1}^{n} c_{2i}z_{2i} + \sum_{i=1}^{n} c_{3i}z_{3i} = 0, \tag{6.15}$$

where $c_{12}, c_{13}, c_{1i}, c_{2i}, c_{3i}$ $(i = 1, \dots, n)$ depend on elements of the matrices X, Y, and U. As (6.15) holds for any $Z \in \mathfrak{m}$, it follows in the standard way that

$$c_{12} = c_{13} = c_{1i} = c_{2i} = c_{3i} = 0, \quad (i = 1, \dots, n). \tag{6.16}$$

Using (6.16), we calculate $u_{ij} = u_{ij}(X, Y)$. To conclude the proof, it remains to transform the formula for $U(X, Y)$ into (6.14), which is quite simple. □

In the notations of Section 2 we have the following statement.

Theorem 6.6. *Consider* $\mathrm{SO}(n)/\mathrm{SO}(2) \times \mathrm{SO}(n-3)$ *as a 4-symmetric Φ-space. Then the only canonical f-structure f_0 on this space is*

(1) *a Killing f-structure if and only if the characteristic numbers of a Riemannian metric are* $(1, 4/3)$;

(2) *a nearly Kähler f-structure if and only if the characteristic numbers of a Riemannian metric are* $(1, t)$, $t > 0$;

(3) *a $G_1 f$-structure with respect to any invariant Riemannian metric.*

Proof. Application of (6.12) to the definitions of the classes **Kill f**, **NKf**, and **G₁f** yields that

(1) $f \in$ **Kill f** if and only if $(1/2)[X, fX]_{\mathfrak{m}} + U(X, fX) - f(U(X, X)) = 0$;

(2) $f \in$ **NKf** if and only if $(1/2)[fX, f^2X]_{\mathfrak{m}} + U(fX, f^2X) - f(U(fX, fX)) = 0$;

(3) $f \in$ **G₁f** if and only if $f(2U(fX, f^2X) - f(U(fX, fX)) + f(U(f^2X, f^2X))) = 0$.

The proof is straightforward. For example, it is known that f_0 is a nearly Kähler f-structure in the naturally reductive case, which means that $(1/2)[f_0 X, f_0^2 X]_{\mathfrak{m}} = 0$ for any $X \in \mathfrak{m}$ (see [5]). Making use of Propositions 6.2 and 6.5, we obtain $U(f_0 X, f_0 X) \in \mathrm{Ker}\, f_0$ for any $X \in \mathfrak{m}$, $U(f_0 X, f_0^2 X) = 0$ for any $X \in \mathfrak{m}$ if and only if $s = 1$. Thus we have (2). Other statements are proved in the same manner. □

The similar technique is used to prove the following.

Theorem 6.7. *Consider* $\mathrm{SO}(n)/\mathrm{SO}(2) \times \mathrm{SO}(n-3)$ *as a 6-symmetric space. Let (s, t) be the characteristic numbers of an invariant Riemannian metric. Then*

(1) *f_1 is a Killing f-structure if and only if $s = 1$, $t = 4/3$; f_2, f_3, f_4 do not belong to* **Kill f** *for any s and t;*

(2) *f_1 is an NKf-structure if and only if $s = 1$; f_2 and f_3 are NKf-structures for any s and t; f_4 is not an NKf-structure for any s and t;*

(3) *f_1, f_2, f_3, f_4 are $G_1 f$-structures for any s and t.*

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