ON n-FLAT MODULES AND n-VON NEUMANN REGULAR RINGS

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We show that each R-module is n-flat (resp., weakly n-flat) if and only if R is an $(n,n-1)$-ring (resp., a weakly $(n,n-1)$-ring). We also give a new characterization of n-von Neumann regular rings and a characterization of weak n-von Neumann regular rings for (CH)-rings and for local rings. Finally, we show that in a class of principal rings and a class of local Gaussian rings, a weak n-von Neumann regular ring is a (CH)-ring.

1. Introduction

All rings considered in this paper are commutative with identity elements and all modules are unital. For a nonnegative integer n, an R-module E is n-presented if there is an exact sequence $F_n \to F_{n-1} \to \cdots \to F_0 \to E \to 0$, in which each F_i is a finitely generated free R-module. In particular, "0-presented" means finitely generated and "1-presented" means finitely presented. Also, $\mathrm{pd}_R E$ will denote the projective dimension of E as an R-module. Costa [2] introduced a doubly filtered set of classes of rings throwing a brighter light on the structures of non-Noetherian rings. Namely, for nonnegative integers n and d, we say that a ring R is an (n,d)-ring if $\mathrm{pd}_R(E) \leq d$ for each n-presented R-module E (as usual, pd denotes projective dimension); and that R is a weak (n,d)-ring if $\mathrm{pd}_R(E) \leq d$ for each n-presented cyclic R-module E. The Noetherianness deflates the (n,d)-property to the notion of regular ring. However, outside Noetherian settings, the richness of this classification resides in its ability to unify classic concepts such as von Neumann regular, hereditary/Dedekind, and semi-hereditary/Prüfer rings. For instance, see [2–5, 8–10].

We say that R is n-von Neumann regular ring (resp., weak n-von Neumann regular ring) if it is $(n,0)$-ring (resp., weak $(n,0)$-ring). Hence, the 1-von Neumann regular rings and the weak 1-von Neumann regular rings are exactly the von Neumann regular ring (see [10, Theorem 2.1] for a characterization of n-von Neumann regular rings).

According to [1], an R-module E is said to be n-flat if $\mathrm{Tor}_R^n(E,G) = 0$ for each n-presented R-module G. Similarly, an R-module E is said to be weakly n-flat if $\mathrm{Tor}_R^n(E,G) = 0$ for each n-presented cyclic R-module G. Consequently, the 1-flat, weakly 1-flat, and flat

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properties are the same. Therefore, each R-module is 1-flat or weakly 1-flat if and only if R is a von Neumann regular ring.

In Section 2, we show that each R-module is n-flat (resp., weakly n-flat) if and only if R is an $(n,n-1)$-ring (resp., a weakly $(n,n-1)$-ring). Then we give a wide class of non weakly (n,d)-rings for each pair of positive integers n and d. In Section 3, we give a new characterization of n-von Neumann regular rings. Also, for (CH)-rings and local rings, a characterization of weak n-von Neumann regular rings is given. Finally, if R is a principal ring or a local Gaussian ring, we show that R is a weak n-von Neumann regular ring which implies that R is a (CH)-ring.

2. Rings such that each R-module is n-flat

Recall that an R-module E is said to be n-flat (resp., weakly n-flat) if $\mathrm{Tor}_R^n(E,G) = 0$ for each n-presented R-module G (resp., n-presented cyclic R-module G). It is clear but important to see that "all R-modules are n-flat" condition is equivalent to "every n-presented module has flat dimension at most $n-1$."

The following result gives us a characterization of those rings modules are n-flat (resp., weakly n-flat).

Theorem 2.1. *Let R be a commutative ring and let $n \geq 1$ be an integer. Then*

(1) *each R-module is n-flat if and only if R is an $(n,n-1)$-ring;*

(2) *each R-module is weakly n-flat if and only if R is a weak $(n,n-1)$-ring.*

Proof. (1) For $n = 1$, the result is well known. For $n \geq 2$, let R be an $(n,n-1)$-ring and N be an R-module. We claim that N is n-flat.

Indeed, if E is an n-presented R-module, then $\mathrm{pd}_R(E) \leq n-1$ since R is an $(n,n-1)$-ring. Hence, $fd_R(E) \leq n-1$ and so $\mathrm{Tor}_R^n(E,N) = 0$. Therefore, N is n-flat.

Conversely, assume that all R-modules are n-flat. Prove that R is an $(n,n-1)$-ring. Let E be an n-presented R-module and consider the exact sequence of R-modules

$$0 \longrightarrow Q \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow E \longrightarrow 0, \tag{2.1}$$

where F_i is a finitely generated free R-module for each i and Q an R-module. It follows that Q is a finitely presented R-module since E is an n-presented R-module. On the other hand, Q is a flat R-module since $fd_R(E) \leq n-1$ (since all R-modules are n-flat and E is n-presented). Therefore, Q is a projective R-module and so $\mathrm{pd}_R(E) \leq n-1$ which implies that R is an $(n,n-1)$-ring.

(2) Mimic the proof of (1), when E is a cyclic n-presented replace, E is an n-presented. □

Note that, even if all R-modules are 2-flat, there may exist an R-module which is not flat. An illustration of this situation is shown in the following example.

Example 2.2. Let R be a Prüfer domain which is not a field. Then all R-modules are 2-flat by [10, Corollary 2.2] since each Prüfer domain is a (2,1)-domain. But, there exists an R-module which is not flat since R is not a von Neumann regular ring (since R is a domain which is not a field).

Let A be a ring, let E be an A-module, and $R = A \propto E$ be the set of pairs (a,e) with pairwise addition and multiplication defined by

$$(a,e)(a',e') = (aa', ae' + a'e). \tag{2.2}$$

R is called the trivial ring extension of A by E. For instance, see [7, 9, 11].

It is clear that every Noetherian nonregular ring is an example of a ring which is not a weak (n,d)-ring for any n, d. Now, we give a wide class of rings which are not a weak (n,d)-ring (and so not an (n,d)-ring) for each pair of positive integers n and d.

Proposition 2.3. *Let A be a commutative ring and let $R = A \propto A$ be the trivial ring extension of A by A. Then, for each pair of positive integers n and d, R is not a weak (n,d)-ring. In particular, it is not an (n,d)-ring.*

Proof. Let $I := R(0,1)(= 0 \propto A)$. Consider the exact sequence of R-modules

$$0 \longrightarrow \text{Ker}(u) \longrightarrow R \xrightarrow{u} I \longrightarrow 0, \tag{2.3}$$

where $u(a,e) = (a,e)(0,1) = (0,a)$. Clearly, $\text{Ker}(u) = 0 \propto A = R(0,1) = I$. Therefore, I is m-presented for each positive integer m by the above exact sequence. It remains to show that $\text{pd}_R(I) = \infty$.

We claim that I is not projective. Deny. Then the above exact sequence splits. Hence, I is generated by an idempotent element $(0,a)$, where $a \in A$. Then $(0,a) = (0,a)(0,a) = (0,0)$. So, $a = 0$ and $I = 0$, the desired contradiction (since $I \neq 0$). It follows from the above exact sequence that $\text{pd}_R(I) = 1 + \text{pd}_R(I)$ since $\text{Ker}(u) = I$. Therefore, $\text{pd}_R(I) = \infty$ and then R is not a weak (n,d)-ring for each pair of positive integers n and d. □

Remark 2.4. Let A be a commutative ring and let $R = A \propto A$ be the trivial ring extension of A by A. Then, for each positive integer n, there exists an R-module which is not a weakly n-flat, in particular it is not n-flat, by Theorem 2.1 and Proposition 2.3.

3. Characterization of (weak) n-von Neumann regular rings

In [10, Theorem 2.1], the author gives a characterization of n-von Neumann regular rings ($(n,0)$-rings). In the sequel, we give a new characterization of n-von Neumann regular rings. Recall first that R is a (CH)-ring if each finitely generated proper ideal has a nonzero annihilator.

Theorem 3.1. *Let R be a commutative ring. Then R is an n-von Neumann regular ring if and only if R is a (CH)-ring and all R-modules are n-flat.*

Proof. Assume that R is n-von Neumann regular. Then R is a (CH)-ring by [10, Theorem 2.1]. On the other hand, R is obviously an $(n,n-1)$-ring since it is an $(n,0)$-ring. So, all R-modules are n-flat by Theorem 2.1.

Conversely, suppose that R is a (CH)-ring and all R-modules are n-flat. Then, R is an $(n,n-1)$-ring by Theorem 2.1 and hence R is an n-von Neumann regular ring by [10, Corollary 2.3] since R is a (CH)-ring. □

The "(CH)" and "all modules are n-flat" properties in Theorem 3.1 are not comparable via inclusion as the following two examples show.

Example 3.2. Let R be a Prüfer domain which is not a field. Then

(1) all R-modules are n-flat for each integer $n \geq 2$ by Theorem 2.1 since each Prüfer domain is an $(n, n-1)$-domain;

(2) R is not a (CH)-ring since R is a domain which is not a field.

Example 3.3. Let A be a (CH)-ring and let $R = A \propto A$ be the trivial ring extension of A by A. Then

(1) R is a (CH)-ring by [11, Lemma 2.6(1)] since A is a (CH)-ring;

(2) R is not an (n, d)-ring for each pair of positive integers n and d by Proposition 2.3. In particular, R does not satisfy the property that "all R-modules are n-flat" by Theorem 2.1.

Now, we give two characterizations of weak n-von Neumann regular rings under some hypothesis.

Theorem 3.4. *Let R be a commutative ring and let n be a positive integer.*

(1) *If R is a (CH)-ring, then R is a weak n-von Neumann regular ring if and only if all R-modules are weakly n-flat.*

(2) *If R is a local ring, then R is a weak n-von Neumann regular ring if and only if each nonzero proper ideal of R is not $(n-1)$-presented.*

Proof. (1) Let R be a (CH)-ring. If R is a weak $(n,0)$-ring, then R is obviously a weak $(n, n-1)$-ring and so each R-module is a weakly n-flat by Theorem 2.1(2). Conversely, assume that each R-module is a weakly n-flat. Then, R is a weak $(n, n-1)$-ring by Theorem 2.1(2). Our purpose is to show that R is a weak $(n,0)$-ring. Let E be a cyclic n-presented R-module and consider the exact sequence of R-module

$$0 \longrightarrow Q \longrightarrow F_{n-2} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow E \longrightarrow 0, \tag{3.1}$$

where F_i is a finitely generated free R-module for each i and Q an R-module. Hence, Q is a finitely generated projective R-module by the same proof of Theorem 2.1(1). Therefore, E is m-presented for each positive integer m and so E is a projective R-module by mimicking the end of the proof of [10, Theorem 2.1] since R is a (CH)-ring.

(2) If each proper ideal of R is not $(n-1)$-presented, then R is obviously a weak $(n,0)$-ring. Conversely, assume that R is a local weak $(n,0)$-ring. We must show that each proper ideal is not $(n-1)$-presented. Assume to the contrary that I is a proper $(n-1)$-presented ideal of R. Then, R/I is an n-presented cyclic R-module, so R/I is a projective R-module since R is a weak $(n,0)$-ring. Hence, the exact sequence of R-modules

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0 \tag{3.2}$$

splits. So, I is generated by an idempotent, that is, there exists $e \in R$ such that $I = Re$ and $e(e-1) = 0$. But R is a local ring, so I is a free R-module (since I is a finitely generated projective R-module) and then $e(e-1) = 0$ implies that $e - 1 = 0$. So, $I = Re = R$ and then I is not a proper ideal, a desired contradiction. Hence, each proper ideal of R is not $(n-1)$-presented. □

Remark 3.5. In Theorem 3.4(2), the condition R local is necessary. In fact, let R be a von Neumann regular ring (i.e., $(1,0)$-ring) which is not a field. Then, R is a weak $(1,0)$-ring and there exist many finitely generated proper ideals of R.

If R is an $(n,0)$-ring, then R is a (CH)-ring by [10, Theorem 2.1]. The $(1,0)$-ring is a (CH)-ring. So we are led to ask the following question.

Question 1. If R is a weak $(n,0)$-ring for a positive integer $n \geq 2$, does this imply that R is a (CH)-ring?

If R is a principal ring (i.e., each finitely generated ideal of R is principal) or a local Gaussian ring, we give an affirmative answer to this question.

For a polynomial $f \in R[X]$, denote by $C(f)$—the content of f—the ideal of R generated by the coefficients of f. For two polynomials f and g in $R[X]$, $C(fg) \subseteq C(f)C(g)$. A polynomial f is called a Gaussian polynomial if this containment becomes equality for every polynomial g in $R[X]$. A ring R is called a Gaussian ring if every polynomial with coefficients in R is a Gaussian polynomial. For instance, see [6].

Proposition 3.6. *Let R be a weak $(n,0)$-ring for a positive integer $n \geq 2$. Then*

(1) *R is a total ring;*

(2) *if R is a principal ring, then R is a (CH)-ring;*

(3) *if R is a local Gaussian ring, then R is a (CH)-ring.*

Proof. (1) Let $a(\neq 0)$ be a regular element of R. Our aim is to show that a is unit. The ideal Ra is n-presented for each positive integer n since $Ra \cong R$ (since a is regular), so R/Ra is n-presented for each positive integer n by the exact sequence of R-modules $0 \to Ra \to R \to R/Ra \to 0$. Hence, R/Ra is a projective R-module (since R is a weak $(n,0)$-ring) and so the above exact sequence splits. Then Ra is generated by an idempotent, that is, there exists $e \in R$ such that $Ra = Re$ and $e(e-1) = 0$. But e is regular since so is a (since $Ra = Re$). Hence, $e(e-1) = 0$ implies that $e - 1 = 0$ and so $Ra = R$, that is, a is unit.

(2) Argue by (1) and since R is principal.

(3) Let (R,M) be a local Gaussian weak $(n,0)$-ring. By the proof (case 1) of [6, Theorem 3.2], it suffices to show that each $a \in M$ is zero divisor. But R is a total ring by (1). Therefore, each $a \in M$ is a zero divisor and this completes the proof of Proposition 3.6. □

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