# QUANTUM CURVE IN $q$-OSCILLATOR MODEL

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A lattice model of interacting $q$-oscillators, proposed by V. Bazhanov and S. Sergeev in 2005 is the quantum-mechanical integrable model in $2+1$ dimensional space-time. Its layer-to-layer transfer matrix is a polynomial of two spectral parameters, it may be regarded in terms of quantum groups both as a sum of $\mathrm{sl}(N)$ transfer matrices of a chain of length $M$ and as a sum of $\mathrm{sl}(M)$ transfer matrices of a chain of length $N$ for reducible representations. The aim of this paper is to derive the Bethe ansatz equations for the $q$-oscillator model entirely in the framework of $2+1$ integrability providing the evident rank-size duality.



## 1. Introduction

The $q$-oscillator lattice model was formulated recently in [5, 15]. It describes a system of interacting $q$-oscillators situated in the vertices of two-dimensional lattice, and therefore it is the quantum-mechanical system in $2+1$ dimensional space-time in the same way, as a chain of interacting particles (or spins) is regarded as a model in $1+1$ dimensional space-time. Formulation of the $q$-oscillator model provides a definition of a layer-to-layer transfer matrix as a polynomial of two spectral parameters. This transfer matrix may be interpreted in terms of quantum inverse scattering method and quantum groups, so that both sizes of the two-dimensional lattice may be interpreted as either a length of an effective chain or as symmetry group's rank. This was called in [5] the "rank-size" duality. The appearance of a complete set of fundamental transfer matrices for $\mathcal{U}_q(\widehat{\mathrm{sl}})$ series is a signal that the layer-to-layer transfer matrix of $q$-oscillator model is closely related to Bethe ansatz in the form of generalized Baxter's "$T$-$Q$" equations. The subject of this paper is the derivation of such equations in the framework of $2+1$ dimensional integrability.

Below in this introduction we formulate the answer, that is, we give an explicit form of "$T$-$Q$" equations in terms of a given layer-to-layer transfer matrix. To do this, we need

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to repeat the structure of the layer-to-layer transfer matrix for $q$-oscillator model in more details.

The $q$-oscillator model describes a system of interacting $q$-oscillators $\mathcal{H}_{\mathrm{v}}$,

$$\mathcal{H}_{\mathrm{v}} : \mathbf{x}_{\mathrm{v}}\mathbf{y}_{\mathrm{v}} = 1 - q^{2+2\mathbf{h}_{\mathrm{v}}}, \qquad \mathbf{y}_{\mathrm{v}}\mathbf{x}_{\mathrm{v}} = 1 - q^{\mathbf{h}_{\mathrm{v}}}, \qquad \mathbf{x}_{\mathrm{v}}q^{\mathbf{h}_{\mathrm{v}}} = q^{\mathbf{h}_{\mathrm{v}}+1}\mathbf{x}_{\mathrm{v}}, \qquad \mathbf{y}_{\mathrm{v}}q^{\mathbf{h}_{\mathrm{v}}} = q^{\mathbf{h}_{\mathrm{v}}-1}\mathbf{y}_{\mathrm{v}}, \tag{1.1}$$

situated in the vertices v of two-dimensional square lattice of sizes $N \times M$. Index v stands for a coordinate of a vertex. Oscillators from different vertices commute (what is called "locality"), the whole *algebra of observables* is thus $\mathcal{H}^{\otimes NM}$, and the vertex index v corresponds to the number of component of the tensor power. In this paper we imply mostly the Fock space representation $\mathcal{F}$ of $q$-oscillators.

The two-dimensional lattice may be identified with a layer (or a section) of three-dimensional cubic lattice, further we call it either the layer or the auxiliary lattice.

Auxiliary matrices $L_{\alpha,\beta}[\mathcal{H}_{\mathrm{v}}]$, acting in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathcal{F}_{\mathrm{v}}$, were introduced in [5]. The *layer* transfer matrix $\mathbf{T}(u,v)$ may be constructed as a trace of $2d$-ordered product of auxiliary matrices $L[\mathcal{H}_{\mathrm{v}}]$. The transfer matrix is a polynomial of two spectral parameters,

$$\mathbf{T}(u,v) = \sum_{n=0}^{N}\sum_{m=0}^{M} u^n v^m \mathbf{t}_{n,m}, \tag{1.2}$$

its coefficients $\mathbf{t}_{n,m} \in \mathcal{H}^{\otimes NM}$ form a complete commutative set. Matrices $L[\mathcal{H}_{\mathrm{v}}]$ depend on some extra $\mathbb{C}$-valued free parameters, for their generic values the model is inhomogeneous. The layer transfer matrix (1.2) may be identically rewritten in two ways,

$$\mathbf{T}(u,v) \equiv \sum_{n=0}^{N} u^n T^{(\mathrm{sl}_N)}_{\omega_n}(v) \equiv \sum_{m=0}^{M} v^m T^{(\mathrm{sl}_M)}_{\omega_m}(u), \tag{1.3}$$

where

$$T^{(\mathrm{sl}_N)}_{\omega_n}(v) = \sum_{m=0}^{M} v^m \mathbf{t}_{n,m} \tag{1.4}$$

is the $2d$ transfer matrix for $\mathcal{U}_q(\widehat{\mathrm{sl}}_N)$ *chain* of the length $M$, corresponding to the fundamental representation $\pi_{\omega_n}$ in the auxiliary space (here $\omega_n$ stand for the fundamental weights of $A_{N-1}$, $\pi_{\omega_0}$ and $\pi_{\omega_N}$ are two scalar representations, $T^{(\mathrm{sl}_N)}_0$ and $T^{(\mathrm{sl}_N)}_N$ may be written explicitly). The same layer transfer matrix $\mathbf{T}(u,v)$ may be rewritten as the sum of $\mathcal{U}_q(\widehat{\mathrm{sl}}_M)$ transfer matrices

$$T^{(\mathrm{sl}_M)}_{\omega_m}(u) = \sum_{n=0}^{N} u^n \mathbf{t}_{n,m} \tag{1.5}$$

for the length $N$ chain (the last part of (1.3)).

The result of this paper is the derivation of the dual Bethe ansatz equations for the $q$-oscillator model. They may be formulated as follows. Let normalized transfer matrices be

$$\tau_m^{(\mathrm{sl}_M)}(u) = T_{\omega_m}^{(\mathrm{sl}_M)}(-(-q)^m u), \qquad \tau_n^{(\mathrm{sl}_N)}(v) = T_{\omega_n}^{(\mathrm{sl}_n)}(-(-q)^n v), \tag{1.6}$$

and let $\mathbb{C}$-numerical parameters of $q$-oscillator lattice be inhomogeneous enough.

Then "$T$-$Q$" equation for $\mathrm{sl}_M$ is

$$\sum_{m=0}^{M} (-v)^m \tau_m^{(\mathrm{sl}_M)}(u) Q(q^{2m}u) = 0. \tag{1.7}$$

The statement is that if $\mathbf{t}_{n,m}$ take their eigenvalues, then there exist $M$ special values $v_1,\dots,v_M$ of $v$, such that corresponding $Q_1(u),\dots,Q_M(u)$ in (1.7) are polynomials. (Parameter $v$ in the "$u$-shift" equation (1.7) is irrelevant since a rescaling $Q(u) \to u^\nu Q(u)$ changes it.) Degrees of the polynomials are uniquely defined by certain occupation numbers of oscillators.

In its turn, equivalent "$T$-$Q$" equation for $\mathrm{sl}_N$ is

$$\sum_{n=0}^{N} (-u)^n \tau_n^{(\mathrm{sl}_N)}(v) \overline{Q}(q^{2n}v) = 0. \tag{1.8}$$

If $\mathbf{t}_{n,m}$ take their eigenvalues, then there exist $N$ special values $u_1,\dots,u_N$ of $u$, such that corresponding $\overline{Q}_1(v),\dots,\overline{Q}_N(v)$ in (1.8) are polynomials. All the other forms of nested Bethe ansatz equations follow from (1.7) or (1.8).

Polynomials $Q(u)$ and $\overline{Q}(v)$ may be denoted in the quantum-mechanical way as "wave functions" of states $\langle Q|$ and $|\overline{Q}\rangle$:

$$Q(u) = \langle Q \mid u\rangle, \qquad \overline{Q}(v) = \langle v \mid \overline{Q}\rangle, \tag{1.9}$$

where $|u\rangle$ and $\langle v|$ serve the simple Weyl algebra $\mathcal{W}$: $\mathbf{uv} = q^2\mathbf{vu}$,

$$\begin{aligned} \mathbf{u}|u\rangle &= |u\rangle u, & \mathbf{v}|u\rangle &= |q^2u\rangle v, \\ \langle v|\mathbf{u} &= u\langle q^2 v|, & \langle v|\mathbf{v} &= v\langle v|. \end{aligned} \tag{1.10}$$

Let

$$J(\mathbf{u},\mathbf{v}) = \sum_{n=0}^{N}\sum_{m=0}^{M} (-q)^{-nm}(-\mathbf{u})^n(-\mathbf{v})^m t_{n,m}. \tag{1.11}$$

Then (1.7) and (1.8) are correspondingly

$$\langle Q|J(\mathbf{u},\mathbf{v})|u\rangle = \langle v|J(\mathbf{u},\mathbf{v})|\overline{Q}\rangle = 0. \tag{1.12}$$

The formulation of the $q$-oscillator model and definition of $\mathbf{T}(u,v)$ are locally $3d$ invariant, the quantum group interpretation (1.3) is the secondary one. In this paper we will derive (1.11) without any quantum group technique.

To explain our method, we need to comment a little on the classical limit.

In the classical limit $q \to 1$, the local $q$-oscillator generators become the classical dynamical variables, the $q$-oscillator model becomes a model of classical mechanics, quantum evolution operators become Baecklund transformations for the dynamical variables. In particular, $T(u,v)$ may be understood as a partition function of a completely inhomogeneous free fermion six-vertex model on the square lattice (but it should not be regarded as a model of statistical mechanics). In its turn, $J(u,v)$ becomes a free fermion determinant (the sign $(-)^{nm+n+m}$ counts the number of fermionic loops). There exists a well-known formula in the theory of two-dimensional free fermion models, relating $T$ and $J$:

$$T(u,v) = \frac{1}{2}\big(J(-u,v) + J(u,-v) + J(-u,-v) - J(u,v)\big). \tag{1.13}$$

In the classical limit, equation $J(u,v) = 0$ defines the spectral curve. Dynamical variables may be expressed in terms of $\theta$-functions on the Jacobian of the spectral curve. The sequence of Baecklund transformations, which is the "discrete time" in the classical model, is a sequence of linear shifts of a point on the Jacobian. The classical model was formulated and solved by Korepanov [8].

Classical integrability is based on an auxiliary linear problem. Equation $J(u,v) = 0$ is the condition of the existence of a solution of the linear problem. Our point is that in quantum $q \neq 1$ case, the linear problem is still the basic concept of the solvability. Quantum $\mathbf{J}(\mathbf{u},\mathbf{v})$ is a well-defined determinant of an operator-valued matrix, and $\mathbf{J}(\mathbf{u},\mathbf{v})|\Psi\rangle = 0$ is again the condition of the existence of a solution of a quantum linear problem. The polynomial structure of, for example, $\langle v \mid \Psi\rangle$ follows from a more detailed consideration of the quantum linear problem in a special basis of diagonal "quantum Baker-Akhiezer function" (related to a quantum separation of variables).

The structure of the paper is the following. In the first section we recall briefly some basic notions of the classical model [8]: the linear problem, spectral curve, and details of the combinatorial representation of the spectral curve. In the second section we repeat the definition of the quantum model and its integrability [5, 15]. In particular, our definition of the spectral parameters differs from that of [5]. Quantum linear problem, derivation of (1.11), and properties of various forms of (1.12) are given in the third section. The fourth section includes an example.

## 2. The Korepanov model

We start with a short review of the integrable model of classical mechanics in discrete $2+1$ dimensional space-time [8]. The main purpose of this section is to recall the relation between Korepanov's linear problem, spectral determinant, and partition function for free fermion model. Another aim is to fix several useful definition and notations.

**2.1. Linear problem.** Consider a two-dimensional lattice formed by the intersection of straight lines enumerated by the Greek letters. Let the vertices of the lattice are enumerated in some way.

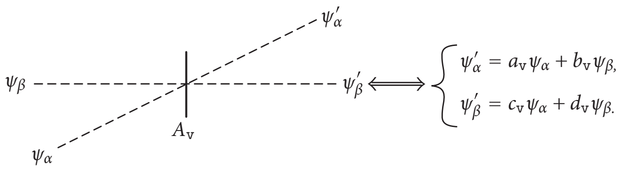


Figure 2.1. Vertex v is formed by intersection of $\alpha$- and $\beta$-lines of auxiliary lattice. Vertex linear problem is the pair of relations binding four edge variables.

Consider a particular vertex with a number v formed by the intersection of lines $\alpha$ and $\beta$, as it is shown in Figure 2.1. It was mentioned in the introduction that an auxiliary lattice is a section of three-dimensional lattice, the vertices on the auxiliary lattice are equivalent to the edges of the three-dimensional one. In Figure 2.1, the dashed lines are the lines of auxiliary lattice, while the solid line sprout from the vertex v is the edge of the three-dimensional lattice.

Let four free $\mathbb{C}$-valued variables

$$A_{\mathrm{v}} = (a_{\mathrm{v}}, b_{\mathrm{v}}, c_{\mathrm{v}}, d_{\mathrm{v}}) \tag{2.1}$$

be associated with vertex v. In addition, let $\mathbb{C}$-valued variables $\psi_\alpha$ and $\psi_\beta$ be associated with the ingoing edges, and let $\mathbb{C}$-valued variables $\psi'_\alpha$ and $\psi'_\beta$ be associated with the outgoing edges, as it is shown in Figure 2.1 (a certain orientation of auxiliary lines is implied). The local linear problem is a pair of linear relations binding the edge variables. Its standard form, the right-hand side of Figure 2.1 in matrix notations, is

$$\begin{pmatrix} \psi'_\alpha \\ \psi'_\beta \end{pmatrix} = X[A_{\mathrm{v}}]\begin{pmatrix} \psi_\alpha \\ \psi_\beta \end{pmatrix}, \quad \text{where } X[A_{\mathrm{v}}] \stackrel{\text{def}}{=} \begin{pmatrix} a_{\mathrm{v}} & b_{\mathrm{v}} \\ c_{\mathrm{v}} & d_{\mathrm{v}} \end{pmatrix}. \tag{2.2}$$

**2.2. Korepanov's equation.** Equations of motion in an integrable model arise as an associativity condition of its linear problem. To derive their local form, consider a vertex of three-dimensional lattice (not necessarily the cubic one), and sections of it by two auxiliary planes, as it is shown in Figure 2.2. From three-dimensional point of view, the auxiliary linear variables belong to the faces of $3d$ lattice, while the dynamical variables $A_{\mathrm{v}}$ belong to the edges of the $3d$ lattice. Therefore, $A_{\mathrm{v}}$ are distinguished from $A'_{\mathrm{v}}$, but linear variables on outer edges $\psi_\alpha, \ldots, \psi'_\gamma$, in both top and bottom auxiliary planes, are identified. Consider the bottom plane first. The linear problem rule (2.2) may be applied three times for excluding internal edges; as a result one obtains an expression of the "primed" linear variables in terms of "unprimed":

$$\begin{pmatrix} \psi'_\alpha \\ \psi'_\beta \\ \psi'_\gamma \end{pmatrix} = X_{\alpha,\beta}[A_1] \cdot X_{\alpha,\gamma}[A_2] \cdot X_{\beta,\gamma}[A_3] \cdot \begin{pmatrix} \psi_\alpha \\ \psi_\beta \\ \psi_\gamma \end{pmatrix}, \tag{2.3}$$

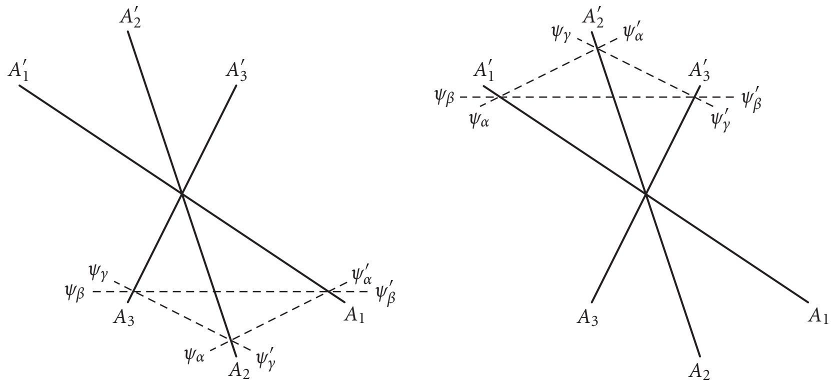


Figure 2.2. Left- and right-hand sides of Korepanov's equation.

where (cf. the ordering of $\alpha$, $\beta$, $\gamma$ in column vectors)

$$X_{\alpha,\beta}[A_1] = \begin{pmatrix} a_1 & b_1 & 0 \\ c_1 & d_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad X_{\alpha,\gamma}[A_2] = \begin{pmatrix} a_2 & 0 & b_2 \\ 0 & 1 & 0 \\ c_2 & 0 & d_2 \end{pmatrix}, \qquad X_{\beta,\gamma}[A_3] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_3 & b_3 \\ 0 & c_3 & d_3 \end{pmatrix}. \tag{2.4}$$

The top plane of Figure 2.2 may be considered in the same way,

$$\begin{pmatrix} \psi'_\alpha \\ \psi'_\beta \\ \psi'_\gamma \end{pmatrix} = X_{\beta,\gamma}[A'_3] \cdot X_{\alpha,\gamma}[A'_2] \cdot X_{\alpha,\beta}[A'_1] \cdot \begin{pmatrix} \psi_\alpha \\ \psi_\beta \\ \psi_\gamma \end{pmatrix}, \tag{2.5}$$

where the matrices $X_{\#,\#}$ are given by (2.4) with $A'_{\rm v} = (a'_{\rm v}, b'_{\rm v}, c'_{\rm v}, d'_{\rm v})$.

The associativity condition of linear problems (2.3) and (2.5) is the Korepanov equation

$$X_{\alpha,\beta}[A_1] \cdot X_{\alpha,\gamma}[A_2] \cdot X_{\beta,\gamma}[A_3] = X_{\beta,\gamma}[A'_3] \cdot X_{\alpha,\gamma}[A'_2] \cdot X_{\alpha,\beta}[A'_1], \tag{2.6}$$

relating the set of 12 variables $A_{\rm v}$ with the set of 12 variables $A'_{\rm v}$, ${\rm v} = 1,2,3$. Equation (2.6) describes a single $3d$ vertex. Equations of motion for three-dimensional integrable model are the collection of (2.6) for all vertices of the $3d$ lattice.

The Korepanov equation needs a very important comment. Matrices $X_{\#,\#}[A_{\rm v}]$ by definition (2.4) act in the direct sum of one-dimensional vector spaces labelled by the indices $\alpha$, $\beta$, $\gamma$, and so forth. The matrix $X_{\alpha,\beta}[A_1]$ in the block $(\alpha,\beta)$ coincides with $X[A_1]$ (2.2), and in the block $(\gamma,\ldots)$ it is the unity matrix. In what follows, such "direct sum" imbedding of $2\times 2$ matrices $X$ into higher-dimensional unity matrices will always be implied.

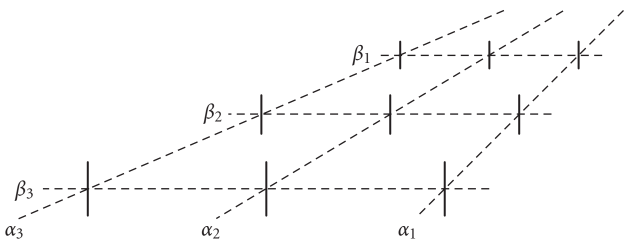


Figure 2.3. A fragment of the auxiliary square lattice.

**2.3. Linear problem with periodical boundary conditions.** Korepanov's solution of the equations of motion is based on the solution of the linear problem for the whole auxiliary lattice. Consider the square lattice with the sizes $N \times M$. Let the lines of the lattice be enumerated by

$$\alpha_n, \quad \beta_m, \quad n = 1,2,\ldots,N,\ m = 1,2,\ldots,M. \tag{2.7}$$

A fragment of the auxiliary lattice is shown in Figure 2.3. Notations for vertex and auxiliary variables for $(n,m)$th vertex of the plane are shown in Figure 2.4. The local auxiliary linear problem (2.2) for $(n,m)$th vertex takes the form

$$\begin{pmatrix} \psi_{\alpha_n}^{(m-1)} \\ \psi_{\beta_m}^{(n-1)} \end{pmatrix} = X[A_{n,m}] \cdot \begin{pmatrix} \psi_{\alpha_n}^{(m)} \\ \psi_{\beta_m}^{(n)} \end{pmatrix}, \quad n = 1,\ldots N,\ m = 1,\ldots,M. \tag{2.8}$$

The linearity of the whole set of (2.8) with respect to $\psi$'s makes it possible to define the quasi-periodical boundary conditions for them:

$$\psi_{\alpha_n}^{(m+M)} = u\psi_{\alpha_n}^{(m)}, \qquad \psi_{\beta_m}^{(n+N)} = v\psi_{\beta_m}^{(n)}, \tag{2.9}$$

where $u$ and $v$ are $\mathbb{C}$-valued spectral parameters.

Linear equations (2.8) may be iterated for the whole lattice as follows. Let

$$\psi_{\boldsymbol{\alpha}}^{(m)} = \begin{pmatrix} \psi_{\alpha_1}^{(m)} \\ \psi_{\alpha_2}^{(m)} \\ \vdots \\ \psi_{\alpha_N}^{(m)} \end{pmatrix}, \qquad \psi_{\boldsymbol{\beta}}^{(n)} = \begin{pmatrix} \psi_{\beta_1}^{(n)} \\ \psi_{\beta_2}^{(n)} \\ \vdots \\ \psi_{\beta_M}^{(n)} \end{pmatrix}. \tag{2.10}$$

Then the repeated use of (2.8) gives

$$\begin{pmatrix} \psi_{\boldsymbol{\alpha}}^{(0)} \\ \psi_{\boldsymbol{\beta}}^{(0)} \end{pmatrix} = \mathbb{X}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \begin{pmatrix} \psi_{\boldsymbol{\alpha}}^{(M)} \\ \psi_{\boldsymbol{\beta}}^{(N)} \end{pmatrix}, \tag{2.11}$$

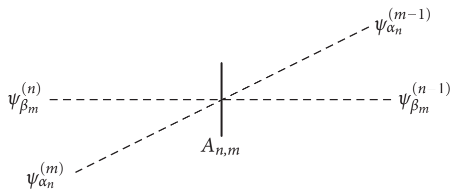


Figure 2.4. Notations for $(n,m)$th vertex of the auxiliary lattice.

where, in terms of matrix imbedding discussed right after (2.6), the $(N+M)\times(N+M)$ monodromy matrix $\mathbb{X}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ may be written as

$$\mathbb{X}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \prod_n^{\curvearrowright}\prod_m^{\curvearrowright} X_{\alpha_n,\beta_m}[A_{n,m}], \tag{2.12}$$

where

$$\prod_n^{\curvearrowright} f_n \stackrel{\text{def}}{=} f_1 f_2 \cdots f_{N-1} f_N, \qquad \prod_m^{\curvearrowright} f_m \stackrel{\text{def}}{=} f_1 f_2 \cdots f_{M-1} f_M. \tag{2.13}$$

The boundary conditions (2.9) give $\psi_{\boldsymbol{\alpha}}^{(M)} = u\psi_{\boldsymbol{\alpha}}^{(0)}$, $\psi_{\boldsymbol{\beta}}^{(N)} = v\psi_{\boldsymbol{\beta}}^{(0)}$, so that (2.11) becomes

$$\left(1 - \mathbb{X}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cdot \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right)\begin{pmatrix} \psi_{\boldsymbol{\alpha}}^{(0)} \\ \psi_{\boldsymbol{\beta}}^{(0)} \end{pmatrix} = 0. \tag{2.14}$$

The whole linear problem has a solution if and only if

$$J(u,v) \stackrel{\text{def}}{=} \det\left(1 - \mathbb{X}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cdot \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}\right) \tag{2.15}$$

is zero. Equation $J(u,v) = 0$ defines the spectral curve for the model, equations of motion (2.6) for the whole three-dimensional lattice have an exact solution in terms of $\theta$-functions on the Jacobian of the spectral curve [8].

**2.4. Free fermion model.** The determinant (2.15) has the very well-known combinatorial representation. Usual way to derive it is to define the determinant in terms of the Grassmanian integration and then to turn from normal symbols to matrix elements.

Let in this subsection $\psi$ and $\overline{\psi}$ be the Grassmanian variables with the integration rules $\int d\psi = \int d\overline{\psi} = 0$ and $\int \psi d\psi = \int \overline{\psi} d\overline{\psi} = 1$. Then the determinant (2.15) may be written as

$$J(u,v) = \int e^{\mathcal{A}[\overline{\psi},\psi]} \mathcal{D}\overline{\psi}\mathcal{D}\psi, \tag{2.16}$$

where the "action" is

$$\mathscr{A} = \sum_{n,m=1}^{N,M} \left\{ \left(\overline{\psi}_{\alpha_n}^{(m-1)}, \overline{\psi}_{\beta_m}^{(n-1)}\right) \cdot X[A_{n,m}] \cdot \begin{pmatrix} \psi_{\alpha_n}^{(m)} \\ \psi_{\beta_m}^{(n)} \end{pmatrix} + \psi_{\alpha_n}^{(m)} \overline{\psi}_{\alpha_n}^{(m)} + \psi_{\beta_m}^{(n)} \overline{\psi}_{\beta_m}^{(n)} \right\}, \tag{2.17}$$

and the measure is

$$\mathscr{D}\overline{\psi}\mathscr{D}\psi = \prod_{n,m=1}^{N,M} d\overline{\psi}_{\alpha_n}^{(m)} d\psi_{\alpha_n}^{(m)} d\overline{\psi}_{\beta_m}^{(n)} d\psi_{\beta_m}^{(n)}. \tag{2.18}$$

Spectral parameters appear in (2.16) via

$$\overline{\psi}_{\alpha_n}^{(0)} = u\overline{\psi}_{\alpha_n}^{(M)}, \qquad \overline{\psi}_{\beta_m}^{(0)} = v\overline{\psi}_{\beta_m}^{(N)}. \tag{2.19}$$

In terms of the Grassmanian variables, the exponent of a quadratic form is a normal symbol of some operator $L$,

$$\exp\left\{ (\overline{\psi}_\alpha, \overline{\psi}_\beta) \cdot X[A_\mathrm{v}] \cdot \begin{pmatrix} \psi_\alpha \\ \psi_\beta \end{pmatrix} \right\} \stackrel{\text{def}}{=} \langle \overline{\psi}_\alpha, \overline{\psi}_\beta \,|\, L_{\alpha,\beta}[A_\mathrm{v}] \,|\, \psi_\alpha, \psi_\beta \rangle. \tag{2.20}$$

The fermionic coherent states are defined by

$$|\psi\rangle = |0\rangle + |1\rangle\psi, \qquad \langle\overline{\psi}| = \langle 0| + \overline{\psi}\langle 1|, \tag{2.21}$$

and the extra summands in (2.17) correspond to the unity operators

$$1 = \int |\psi\rangle e^{\psi\overline{\psi}} d\overline{\psi}\, d\psi \langle\overline{\psi}|. \tag{2.22}$$

It is important to note that the indices of the operator $L_{\alpha,\beta}[A_\mathrm{v}]$ (2.20) label copies of two-dimensional vector spaces (2.21) $\mathbb{C}^2 \ni x|0\rangle + y|1\rangle$. Thus, $L_{\alpha,\beta}$ acts in the tensor *product* of two-dimensional vector spaces, while $X_{\alpha,\beta}$ acts in the tensor *sum* of one-dimensional vector spaces. In the basis of the fermionic states

$$|n_\alpha, n_\beta\rangle = \left(|0,0\rangle, |1,0\rangle, |0,1\rangle, |1,1\rangle\right), \tag{2.23}$$

operator $L_{\alpha,\beta}$ (2.20) is $4\times 4$ matrix

$$L_{\alpha,\beta}[A_\mathrm{v}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a_\mathrm{v} & b_\mathrm{v} & 0 \\ 0 & c_\mathrm{v} & d_\mathrm{v} & 0 \\ 0 & 0 & 0 & z_\mathrm{v} \end{pmatrix}, \quad \text{where } z_\mathrm{v} \stackrel{\text{def}}{=} b_\mathrm{v}c_\mathrm{v} - a_\mathrm{v}d_\mathrm{v}. \tag{2.24}$$

Besides, the Korepanov equation (2.6) is the equality of the exponents of the normal symbol form of the local Yang-Baxter equation

$$L_{\alpha,\beta}[A_1] L_{\alpha,\gamma}[A_2] L_{\beta,\gamma}[A_3] = L_{\beta,\gamma}[A_3'] L_{\alpha,\gamma}[A_2'] L_{\alpha,\beta}[A_3'], \tag{2.25}$$

since

$$\langle\overline{\psi}|L_{\alpha,\beta}L_{\alpha,\gamma}L_{\beta,\gamma}|\psi\rangle = \exp\{\overline{\psi}\cdot X_{\alpha,\beta}X_{\alpha,\gamma}X_{\beta,\gamma}\cdot\psi\} \tag{2.26}$$

and so forth. Turn now to the expression of the determinant (2.15) in terms of operators $L$. Let $2^{N+M}\times 2^{N+M}$ matrix $\mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$ be the ordered product of local $L$'s:

$$\mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \prod_n^{\curvearrowright}\prod_m^{\curvearrowright} L_{\alpha_n,\beta_m}[A_{n,m}]. \tag{2.27}$$

This is related to the monodromy matrix (2.12) by means of (cf. (2.20))

$$\exp\left\{(\overline{\psi}_{\boldsymbol{\alpha}},\overline{\psi}_{\boldsymbol{\beta}})\cdot\mathbb{X}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\cdot\begin{pmatrix}\psi_{\boldsymbol{\alpha}}\\ \psi_{\boldsymbol{\beta}}\end{pmatrix}\right\} = \langle\overline{\psi}_{\boldsymbol{\alpha}},\overline{\psi}_{\boldsymbol{\beta}}\,|\,\mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\,|\,\psi_{\boldsymbol{\alpha}},\psi_{\boldsymbol{\beta}}\rangle. \tag{2.28}$$

Define now the boundary matrices for $\mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}$,

$$D(u)\stackrel{\text{def}}{=}\begin{pmatrix}1 & 0\\ 0 & u\end{pmatrix},\qquad D_{\boldsymbol{\alpha}}(u)=\prod_n D_{\alpha_n}(u),\qquad D_{\boldsymbol{\beta}}(v)=\prod_m D_{\beta_m}(v), \tag{2.29}$$

and let

$$T(u,v) = \operatorname*{Trace}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\left(D_{\boldsymbol{\alpha}}(u)D_{\boldsymbol{\beta}}(v)\mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\right). \tag{2.30}$$

By the construction, $T(u,v)$ is the partition function for a free-fermion lattice model with the inhomogeneous Boltzmann weights—matrix elements of $L_{\alpha_n,\beta_m}[A_{n,m}]$—and $u$, $v$-boundary conditions. It is the polynomial of $u$ and $v$:

$$T(u,v) = \sum_{n=0}^{N}\sum_{m=0}^{M} u^n v^m t_{n,m}. \tag{2.31}$$

Sometimes a pure combinatorial representation of the partition function is very useful. Any monomial in $T(u,v)$ (2.30) corresponds to a non-self-intersecting path on the toroidal lattice. A path may go through a vertex in one of five different ways as it is shown in Figure 2.5 (or do not go through at all). A factor $f_{\rm v}$ is associated with each variant, these factors are the matrix elements of $L_{\alpha,\beta}[A_{\rm v}]$ (2.24). A monomial in $T(u,v)$, corresponding to path $C$, is

$$t_C = \prod_{\text{along path } C} f_{\rm v}. \tag{2.32}$$

Any non-self-intersecting path on the toroidal lattice has a homotopy class

$$w(C) = nA + mB, \tag{2.33}$$

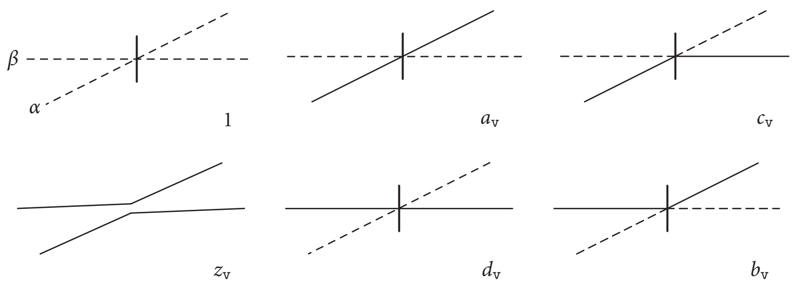


Figure 2.5. Six variants of bypassing the vertex. Vertex factors $f_v$ are matrix elements of $L$. Note that in the variant $z_v$ the path is not self-intersecting.

where $A$ is the toroidal cycle along the $\alpha$-lines, and $B$ is the toroidal cycle along the $\beta$-lines of Figure 2.3. Then the element $t_{n,m}$ of (2.31) is

$$t_{n,m} = \sum_{C:w(C)=nA+mB} t_C. \tag{2.34}$$

The determinant $J(u,v)$ (2.15) is related to $t_{n,m}$ via

$$J(u,v) = \sum_{n=0}^{N} \sum_{m=0}^{M} (-)^{nm+n+m} u^n v^m t_{n,m}, \tag{2.35}$$

where the sign $(-)^{nm+n+m}$ counts the number of fermionic loops on the toroidal square lattice. The determinant $J$ may be expressed in terms of $T$ and vice versa:

$$\begin{aligned} J(u,v) &= \frac{1}{2}\big(T(-u,-v) + T(-u,v) + T(u,-v) - T(u,v)\big), \\ T(u,v) &= \frac{1}{2}\big(J(-u,-v) + J(-u,v) + J(u,-v) - J(u,v)\big). \end{aligned} \tag{2.36}$$

The last equality is very well known in the two-dimensional free-fermion model as the formula relating the lattice partition function and fermionic determinant.

Now we may finish the collection of notions and definitions of the Korepanov model of classical integrable dynamics on three-dimensional lattice and proceed to the description of their quantum analogues.

## 3. Quantum model

In the previous section, we did not pay any attention to the structure of vertex variables $A_v$ (2.1), they were defined simply as the list of elements of $X$ (2.2). The key point for the quantization of the model is that it is possible do define a local Poisson structure on $A_v$ [5] such that the transformation

$$A_1 \otimes A_2 \otimes A_3 \longrightarrow A_1' \otimes A_2' \otimes A_3', \tag{3.1}$$

defined by the Korepanov equation (2.6), is a symplectic map. Symplectic structure admits an immediate quantization. We will skip here all the details and proceed directly to the ansatz for quantized $A_v$, $X[A_v]$ and $L[A_v]$. The aim of this section is just to give precise definition of $\mathbf{T}(u,v)$ (1.2).

**3.1. The quantum Korepanov and tetrahedron equations.** The local $q$-oscillator algebra $\mathcal{H}$ is defined by (1.1). The Fock space $\mathcal{F}$ representation for $q$-oscillators corresponds to

$$\mathrm{Spec}(\mathbf{h}_v) = 0,1,2,\ldots \quad \forall v. \tag{3.2}$$

Quantized dynamical variables (2.1) are the $q$-oscillator generators and a pair of $\mathbb{C}$-valued parameters, $\mathcal{A}_v \sim (\mathcal{H}_v;\lambda_v,\mu_v)$:

$$\mathcal{A}_v = \left(a_v = \lambda_v q^{\mathbf{h}_v},\ b_v = \mathbf{y}_v,\ c_v = -q^{-1}\lambda_v\mu_v\mathbf{x}_v,\ d_v = \mu_v q^{\mathbf{h}_v}\right). \tag{3.3}$$

Quantized $X$ (2.2) and $L$ (2.24) are given by

$$X[\mathcal{A}_v] = \begin{pmatrix} \lambda_v q^{\mathbf{h}_v} & \mathbf{y}_v \\ -q^{-1}\lambda_v\mu_v\mathbf{x}_v & \mu_v q^{\mathbf{h}_v} \end{pmatrix}, \tag{3.4}$$

$$L_{\alpha,\beta}[\mathcal{A}_v] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_v q^{\mathbf{h}_v} & \mathbf{y}_v & 0 \\ 0 & -q^{-1}\lambda_v\mu_v\mathbf{x}_v & \mu_v q^{\mathbf{h}_v} & 0 \\ 0 & 0 & 0 & -q^{-1}\lambda_v\mu_v \end{pmatrix}. \tag{3.5}$$

One may verify directly that the quantum Korepanov equation

$$X_{\alpha,\beta}[\mathcal{A}_1]X_{\alpha,\gamma}[\mathcal{A}_2]X_{\beta,\gamma}[\mathcal{A}_3]\mathbf{R} = \mathbf{R}X_{\beta,\gamma}[\mathcal{A}_3]X_{\alpha,\gamma}[\mathcal{A}_2]X_{\alpha,\beta}[\mathcal{A}_1] \tag{3.6}$$

is equivalent to the auxiliary tetrahedron equation (the quantum local Yang-Baxter equation)

$$L_{\alpha,\beta}[\mathcal{A}_1]L_{\alpha,\gamma}[\mathcal{A}_2]L_{\beta,\gamma}[\mathcal{A}_3]\mathbf{R} = \mathbf{R}L_{\beta,\gamma}[\mathcal{A}_3]L_{\alpha,\gamma}[\mathcal{A}_2]L_{\alpha,\beta}[\mathcal{A}_1]. \tag{3.7}$$

Here the intertwining operator $\mathbf{R} = \mathbf{R}_{123}$ acts in the tensor product of representation spaces $\mathcal{F}_1 \otimes \mathcal{F}_2 \otimes \mathcal{F}_3$ of three $q$-oscillators $\mathcal{H}_{1,2,3}$. Parameters $\lambda_v$, $\mu_v$ of $\mathcal{A}_v$ are the parameters of $\mathbf{R}$. Both (3.6) and (3.7) are equivalent to the following set of six equations:

$$\begin{gathered}
\mathbf{R}q^{\mathbf{h}_2}\mathbf{x}_1 = \frac{\lambda_2}{\lambda_3}\left(q^{\mathbf{h}_3}\mathbf{x}_1 - \frac{q}{\lambda_1\mu_3}q^{\mathbf{h}_1}\mathbf{x}_2\mathbf{y}_3\right)\mathbf{R}, \qquad \mathbf{R}\mathbf{x}_2 = \left(\mathbf{x}_1\mathbf{x}_3 + \frac{q^2}{\lambda_1\mu_3}q^{\mathbf{h}_1+\mathbf{h}_3}\mathbf{x}_2\right)\mathbf{R}, \\
\mathbf{R}q^{\mathbf{h}_2}\mathbf{x}_3 = \frac{\mu_2}{\mu_1}\left(q^{\mathbf{h}_1}\mathbf{x}_3 - \frac{q}{\lambda_1\mu_3}q^{\mathbf{h}_3}\mathbf{y}_1\mathbf{x}_2\right)\mathbf{R}, \qquad \mathbf{R}\mathbf{y}_2 = \left(\mathbf{y}_1\mathbf{y}_3 + \lambda_1\mu_3 q^{\mathbf{h}_1+\mathbf{h}_3}\mathbf{y}_2\right)\mathbf{R}, \\
\mathbf{R}q^{\mathbf{h}_1+\mathbf{h}_2} = q^{\mathbf{h}_1+\mathbf{h}_2}\mathbf{R}, \qquad \mathbf{R}q^{\mathbf{h}_2+\mathbf{h}_3} = q^{\mathbf{h}_2+\mathbf{h}_3}\mathbf{R}.
\end{gathered} \tag{3.8}$$

Classical equations (2.6) and (2.25) follow from (3.6) and (3.7) in the $q \to 1$ limit of the well-defined automorphism $\mathcal{A}'_v = \mathbf{R}\mathcal{A}_v\mathbf{R}^{-1}$. For irreducible representations of $\mathcal{H}_v$, $\mathbf{R}$ is defined uniquely, its matrix elements for the Fock space representation are given in [5]. Remarkably, Figure 2.2 may be used for the graphical representation of both classical and quantum equations, in the quantum case the solid $3d$ cross in Figure 2.2 stands for $\mathbf{R}$.

Integrable model of quantum mechanics may be formulated purely in terms of matrices $L$ (3.5). The intertwiner $\mathbf{R}$ is related to evolution operators, it is another subject and we will not consider it here. Below we recall the definition of integrable model of quantum mechanics from [5, 15].

**3.2. Transfer matrix T.** For the square lattice $N \times M$ of the previous section, define the "monodromy" of quantized $L$'s literally by (2.27)

$$\mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}} = \overset{\curvearrowleft}{\prod_{n}} \overset{\curvearrowleft}{\prod_{m}} L_{\alpha_n,\beta_m}[\mathscr{A}_{n,m}], \tag{3.9}$$

and its trace (cf. (2.30))

$$\mathbf{T}(u,v) = \operatorname*{Trace}_{\boldsymbol{\alpha},\boldsymbol{\beta}} \left(D_{\boldsymbol{\alpha}}(u) D_{\boldsymbol{\beta}}(v) \mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\right), \tag{3.10}$$

where boundary matrices $D$ are defined by (2.29). To distinguish the classical and quantum cases, we use the boldface letters for the quantized $\mathbf{T}$ and its decomposition (2.31):

$$\mathbf{T}(u,v) = \sum_{n=0}^{N} \sum_{m=0}^{M} u^n v^m \mathbf{t}_{n,m}. \tag{3.11}$$

Since the arguments of $L$'s are local $q$-oscillator generators, $\mathbf{T}(u,v) \in \mathscr{H}^{\otimes NM}$ is by definition a layer-to-layer transfer matrix, its graphical representation is again the classical Figure 2.3. The model is integrable since

$$\mathbf{T}(u,v)\mathbf{T}(u',v') = \mathbf{T}(u',v')\mathbf{T}(u,v), \tag{3.12}$$

that is, the coefficients $\mathbf{t}_{n,m}$ in (3.11) form the set of the integrals of motion.

Now we give the technical proof of the commutativity (3.12). The commutativity of layer-to-layer transfer matrices follows from a proper tetrahedron equation [6]. In addition to $L_{\alpha,\beta}[\mathscr{A}_{\mathrm{v}}]$ (3.5), define

$$\widetilde{L}_{\alpha,\beta}[\mathscr{A}_0] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_0(-q)^{\mathbf{h}_0} & \mathbf{y}_0 & 0 \\ 0 & q^{-1}\lambda_0\mu_0\mathbf{x}_0 & \mu_0(-q)^{\mathbf{h}_0} & 0 \\ 0 & 0 & 0 & q^{-1}\lambda_0\mu_0 \end{pmatrix}, \tag{3.13}$$

where $\mathscr{A}_0 \sim (\mathscr{H}_0; \lambda_0, \mu_0)$. The *constant* tetrahedron equation for $L$ and $\widetilde{L}$,

$$\widetilde{L}_{\alpha,\alpha'}[\mathscr{A}_0]\widetilde{L}_{\beta,\beta'}[\mathscr{A}_0]L_{\alpha,\beta}[\mathscr{A}]L_{\alpha',\beta'}[\mathscr{A}] = L_{\alpha',\beta'}[\mathscr{A}]L_{\alpha,\beta}[\mathscr{A}]\widetilde{L}_{\beta,\beta'}[\mathscr{A}_0]\widetilde{L}_{\alpha,\alpha'}[\mathscr{A}_0], \tag{3.14}$$

may be verified directly, it is just $16\times 16$ matrix equation with the operator-valued entries. Another technical relation is

$$D_{\alpha}(u)D_{\alpha'}(u')\widetilde{L}_{\alpha,\alpha'}[\mathscr{A}_0]=\left(\frac{u}{u'}\right)^{\mathbf{h}_0}\widetilde{L}_{\alpha,\alpha'}[\mathscr{A}_0]\left(\frac{u'}{u}\right)^{\mathbf{h}_0}D_{\alpha}(u)D_{\alpha'}(u'),\tag{3.15}$$

where $D$ is given by (2.29). Combining (3.14) for the whole lattice, we come to

$$\widetilde{\mathbb{L}}_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}\widetilde{\mathbb{L}}_{\boldsymbol{\beta},\boldsymbol{\beta}'}\mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\mathbb{L}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}=\mathbb{L}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}\mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\widetilde{\mathbb{L}}_{\boldsymbol{\beta},\boldsymbol{\beta}'}\widetilde{\mathbb{L}}_{\boldsymbol{\alpha},\boldsymbol{\alpha}'},\tag{3.16}$$

where besides the "monodromies" (2.27) of $L[A_{\mathrm{v}}]$,

$$\mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}=\prod_{n,m}^{\curvearrowright}L_{\alpha_n,\beta_m}[\mathscr{A}_{n,m}],\qquad \mathbb{L}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}=\prod_{n,m}^{\curvearrowright}L_{\alpha'_n,\beta'_m}[\mathscr{A}_{n,m}],\tag{3.17}$$

we used

$$\widetilde{\mathbb{L}}_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}=\prod_{n}^{\curvearrowright}\widetilde{L}_{\alpha_n,\alpha'_n}[\mathscr{A}_0],\qquad \widetilde{\mathbb{L}}_{\boldsymbol{\beta},\boldsymbol{\beta}'}=\prod_{m}^{\curvearrowright}\widetilde{L}_{\beta_m,\beta'_m}[\mathscr{A}_0].\tag{3.18}$$

Multiplying (3.16) by $D_{\boldsymbol{\alpha}}(u)D_{\boldsymbol{\beta}}(v)D_{\boldsymbol{\alpha}'}(u')D_{\boldsymbol{\beta}'}(v')$ and taking into account (3.15), we get

$$\begin{aligned}&\widetilde{\mathbb{L}}_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}\left(\frac{u'v}{uv'}\right)^{\mathbf{h}_0}\widetilde{\mathbb{L}}_{\boldsymbol{\beta},\boldsymbol{\beta}'}\cdot D_{\boldsymbol{\alpha}}(u)D_{\boldsymbol{\beta}}(v)\mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\cdot D_{\boldsymbol{\alpha}'}(u')D_{\boldsymbol{\beta}'}(v')\mathbb{L}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}\\&\quad=D_{\boldsymbol{\alpha}'}(u')D_{\boldsymbol{\beta}'}(v')\mathbb{L}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}\cdot D_{\boldsymbol{\alpha}}(u)D_{\boldsymbol{\beta}}(v)\mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\cdot\left(\frac{u'}{u}\right)^{\mathbf{h}_0}\widetilde{\mathbb{L}}_{\boldsymbol{\beta},\boldsymbol{\beta}'}\widetilde{\mathbb{L}}_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}\left(\frac{v}{v'}\right)^{\mathbf{h}_0}.\end{aligned}\tag{3.19}$$

Now taking the trace over the representation space $\mathscr{F}_0$ of $\mathscr{H}_0$ and denoting

$$\mathbb{M}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}',\boldsymbol{\beta}'}=\underset{\mathscr{F}_0}{\mathrm{Trace}}\left(\widetilde{\mathbb{L}}_{\boldsymbol{\alpha},\boldsymbol{\alpha}'}\left(\frac{u'v}{uv'}\right)^{\mathbf{h}_0}\widetilde{\mathbb{L}}_{\boldsymbol{\beta},\boldsymbol{\beta}'}\right),\tag{3.20}$$

we come to the final similarity relation

$$\begin{aligned}&\mathbb{M}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}',\boldsymbol{\beta}'}\cdot D_{\boldsymbol{\alpha}}(u)D_{\boldsymbol{\beta}}(v)\mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\cdot D_{\boldsymbol{\alpha}'}(u')D_{\boldsymbol{\beta}'}(v')\mathbb{L}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}\\&\quad=D_{\boldsymbol{\alpha}'}(u')D_{\boldsymbol{\beta}'}(v')\mathbb{L}_{\boldsymbol{\alpha}',\boldsymbol{\beta}'}\cdot D_{\boldsymbol{\alpha}}(u)D_{\boldsymbol{\beta}}(v)\mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}}\cdot\mathbb{M}_{\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\alpha}',\boldsymbol{\beta}'},\end{aligned}\tag{3.21}$$

and therefore two transfer matrices $\mathbf{T}(u,v)=\mathrm{Trace}_{\boldsymbol{\alpha},\boldsymbol{\beta}}(D_{\boldsymbol{\alpha}}(u)D_{\boldsymbol{\beta}}(v)\mathbb{L}_{\boldsymbol{\alpha},\boldsymbol{\beta}})$ commute.

In the limit $q\to 1$ coefficients of (3.11) become the involutive moduli of the spectral curve (2.15). Since the moduli of the classical spectral curve are independent and the spectral curve defines completely the solution of the classical model [8], the set of integrals of motion $\mathbf{t}_{n,m}$ is complete.

The combinatorial representation for $\mathbf{T}(u,v)$ (3.10) is equivalent to the combinatorial representation for $T(u,v)$ (2.30). The vertex factors of $L[A_{\mathrm{v}}]$ (2.24) in Figure 2.5 are to be replaced by corresponding elements of $L[\mathscr{A}_{\mathrm{v}}]$ (3.5). Equations (2.32) and (2.34) remain unchanged. In particular, with the help of the combinatorial representation one may easily see that

$$\mathbf{T}(u,0)=\prod_n\left(1+u\prod_m\lambda_{n,m}q^{\mathbf{h}_{n,m}}\right),\qquad \mathbf{T}(0,v)=\prod_m\left(1+v\prod_n\mu_{n,m}q^{\mathbf{h}_{n,m}}\right),\tag{3.22}$$

so that the basic integrals of motion are $q^{\mathcal{J}_n}$ and $q^{\mathcal{K}_m}$,

$$\mathcal{J}_n=\sum_m\mathbf{h}_{n,m},\qquad \mathcal{K}_m=\sum_n\mathbf{h}_{n,m}.\tag{3.23}$$

Eigenvalues of $\mathcal{J}_n$ and $\mathcal{K}_m$ fix a subspace in the state space of the model.

**3.3. Transfer matrix (3.10) and $2d$ quantum inverse scattering method.** The transfer matrix (3.10) may be identically rewritten as the trace of $2d$ monodromy matrix

$$\mathbf{T}(u,v)=\operatorname*{Trace}_{\boldsymbol{\beta}}\left(D_{\boldsymbol{\beta}}(v)\overset{\curvearrowright}{\prod_n}\mathfrak{L}_{\boldsymbol{\beta}}^{(n)}(u)\right),\tag{3.24}$$

where

$$\mathfrak{L}_{\boldsymbol{\beta}}^{(n)}(u)=\operatorname*{Trace}_{\alpha_n}\left(D_{\alpha_n}(u)\overset{\curvearrowright}{\prod_m}L_{\alpha_n,\beta_m}[\mathcal{A}_{n,m}]\right).\tag{3.25}$$

Equations (3.14) and (3.15) provide in particular

$$\begin{aligned}&\widetilde{L}_{\alpha,\alpha'}^{(0)}\cdot\left(\frac{u'}{u}\right)^{\mathbf{h}_0}\widetilde{L}_{\beta_m,\beta_m'}^{(0)}\cdot D_\alpha(u)L_{\alpha,\beta_m}^{(m)}\cdot D_{\alpha'}(u')L_{\alpha',\beta_m'}^{(m)}\\&\quad=D_{\alpha'}(u')L_{\alpha',\beta_m'}^{(m)}\cdot D_\alpha(u)L_{\alpha,\beta_m}^{(m)}\cdot\left(\frac{u'}{u}\right)^{\mathbf{h}_0}\widetilde{L}_{\beta_m,\beta_m'}^{(0)}\cdot\widetilde{L}_{\alpha,\alpha'}^{(0)},\end{aligned}\tag{3.26}$$

and therefore

$$\mathfrak{R}_{\boldsymbol{\beta},\boldsymbol{\beta}'}\left(\frac{u}{u'}\right)\mathfrak{L}_{\boldsymbol{\beta}}(u)\mathfrak{L}_{\boldsymbol{\beta}'}(u')=\mathfrak{L}_{\boldsymbol{\beta}'}(u')\mathfrak{L}_{\boldsymbol{\beta}}(u)\mathfrak{R}_{\boldsymbol{\beta},\boldsymbol{\beta}'}\left(\frac{u}{u'}\right),\tag{3.27}$$

where

$$\mathfrak{R}_{\boldsymbol{\beta},\boldsymbol{\beta}'}\left(\frac{u}{u'}\right)=\operatorname*{Trace}_{\mathcal{F}_0}\left\{\left(\frac{u'}{u}\right)^{\mathbf{h}_0}\overset{\curvearrowright}{\prod_m}\widetilde{L}_{\beta_m,\beta_m'}[\mathcal{A}_0]\right\}.\tag{3.28}$$

In the same way, the other direction of the lattice may be chosen, and the dual Lax operator

$$\mathfrak{L}_{\boldsymbol{\alpha}}^{(m)}(v)=\operatorname*{Trace}_{\beta_m}\left(D_{\beta_m}(v)\overset{\curvearrowright}{\prod_n}L_{\alpha_n,\beta_m}[\mathcal{A}_{n,m}]\right)\tag{3.29}$$

may be considered.

Further in this subsection we recall the structure of $2d$ Lax operator (3.25) and its $R$-matrix (3.28). Matrix $\mathfrak{L}_{\boldsymbol{\beta}}^{(n)}$, as the matrix in the auxiliary space $V_{\boldsymbol{\beta}} = (\mathbb{C}^2)^{\otimes M}$ with operator-valued matrix elements from $\mathcal{H}^{\otimes M}$, has a block-diagonal structure. Let, in the concordance with (2.21) and (2.23), $|0\rangle$ and $|1\rangle$ be the basis of $\mathbb{C}^2$. Define $\varphi_0$, $\varphi_j$, $\varphi_{j,j'}$, and so forth as the following elements of $V_{\boldsymbol{\beta}}$:

$$\begin{aligned}\varphi_0 &= |0\rangle \otimes \cdots \otimes |0\rangle,\\ \varphi_j &= |0\rangle \otimes \cdots |1\rangle_{j\text{th place}} \otimes \cdots \otimes |0\rangle,\end{aligned} \tag{3.30}$$

and so forth, in general $\varphi_{j_1,\dots,j_m}$ has $|1\rangle$ on $j_1$th,$\dots$, $j_m$th places, $j_1 < \cdots < j_m$. In the basis of $\varphi$,

$$\mathfrak{L}_{\boldsymbol{\beta}}^{(n)}(u)\varphi_0 = \varphi_0 L_0^{(n)}(u), \quad \text{where } L_0(u) = 1 + u\prod_m \lambda_{n,m} q^{\mathbf{h}_{n,m}}. \tag{3.31}$$

Next

$$\mathfrak{L}_{\boldsymbol{\beta}}^{(n)}(u)\varphi_k = \sum_{j=1}^{M} \varphi_j L_{j,k}^{(n)}(u), \tag{3.32}$$

where $L_{j,k}(u)$ are matrix elements of Lax operator for the vector representation of $\mathcal{U}_q(\widehat{\mathrm{sl}}_M)$, they are given in [5]. In general,

$$\mathfrak{L}_{\boldsymbol{\beta}}(u) = \bigoplus_{m=0}^{M} L_{\omega_m}(u), \tag{3.33}$$

where block $\omega_m$ corresponds to $M!/m!(M-m)!$-dimensional vector space $\pi_{\omega_m} \in V_{\boldsymbol{\beta}}$ with the basis $\varphi_{j_1,\dots,j_m}$, $j_1 < \cdots < j_m$.

Matrix $\mathfrak{R}_{\boldsymbol{\beta},\boldsymbol{\beta}'}$ (3.28) has the block structure as well,

$$\mathfrak{R}_{\boldsymbol{\beta},\boldsymbol{\beta}'}(u) = \bigoplus_{m,m'=0}^{M} \lambda_0^m \mu_0^{m'} R_{\omega_m,\omega_{m'}}(u), \tag{3.34}$$

where $\lambda_0$, $\mu_0$ are extra parameters of $\mathcal{A}_0$ (3.28), and $R_{\omega_m,\omega_{m'}}$ is the $\mathcal{U}_q(\widehat{\mathrm{sl}}_M)$ $R$-matrix for the representations $\pi_{\omega_m} \otimes \pi_{\omega_{m'}}$. In particular, in the sector $\pi_{\omega_1} \otimes \pi_{\omega_1}$ with the basis $\varphi_j$ (3.30), one can obtain the fundamental $R$-matrix (we used the Fock space (3.2) for $\mathcal{H}_0$ for the calculation of the trace in (3.28)):

$$\mathfrak{R}_{\boldsymbol{\beta},\boldsymbol{\beta}'}(u)\varphi_j \otimes \varphi_k = \lambda_0\mu_0 \frac{u}{(u-q^2)(u-1)} \sum_{j',k'} \varphi_{j'} \otimes \varphi_{k'} R_{j',k'}^{j,k}(u), \tag{3.35}$$

where

$$\begin{aligned} R_{j,k}^{j,k}(u) &= u - 1, \qquad & R_{j,j}^{j,j} &= q^{-1}(u - q^2),\\ R_{k,j}^{j,k}(u) &= q^{-1}(1-q^2) \quad \text{for } j<k, \qquad & R_{k,j}^{j,k}(u) &= q^{-1}u(1-q^2) \quad \text{for } j>k. \end{aligned} \tag{3.36}$$

Lax operator $\mathfrak{L}_{\boldsymbol{\beta}}^{(n)}(u)$ (3.25) has the center $q^{\mathcal{J}_n}$ (3.23), while $q^{\mathcal{K}_m}$ is the center of $\mathfrak{L}_{\boldsymbol{\alpha}}^{(m)}(v)$ (3.29). The quantum space of $\mathfrak{L}_{\boldsymbol{\beta}}$ is $\mathcal{F}^{\otimes M}$, it may be decomposed as

$$\mathcal{F}^{\otimes M} = \bigoplus_{J=0}^{\infty} \pi_{J\omega_1}, \tag{3.37}$$

where $J$ is the eigenvalue of $\mathcal{J}$, and $\pi_{J\omega_1}$ is the rank-$J$ symmetrical tensor representation of $\mathcal{U}_q(\widehat{\mathrm{sl}}_M)$ (dominant weight $J\omega_1$).

We would like to conclude this subsection by the example of $M = 2$ containing the six-vertex model. The block-diagonal structure of $\mathfrak{L}$ (3.25) is

$$\mathfrak{L}(u) = \begin{pmatrix} 1 + u\lambda_1\lambda_2 q^{\mathbf{h}_1+\mathbf{h}_2} & 0 & 0 \\ 0 & L(u) & 0 \\ 0 & 0 & \mu_1\mu_2\left(q^{\mathbf{h}_1+\mathbf{h}_2} + q^{-2}u\lambda_1\lambda_2\right) \end{pmatrix}, \tag{3.38}$$

where index $n$ of (3.25) is omitted, and $2 \times 2$ central block is

$$L(u) = \begin{pmatrix} \mu_1\left(q^{\mathbf{h}_1} - u\lambda_1\lambda_2 q^{\mathbf{h}_2-1}\right) & -q^{-1}u\lambda_1\mu_1\mathbf{x}_1\mathbf{y}_2 \\ -q^{-1}\mathbf{y}_1\lambda_2\mu_2\mathbf{x}_2 & \mu_2\left(q^{\mathbf{h}_2} - u\lambda_1\lambda_2 q^{\mathbf{h}_1-1}\right) \end{pmatrix}. \tag{3.39}$$

Fixed integer $\mathcal{J} = \mathbf{h}_1 + \mathbf{h}_2$ in the quantum space corresponds to spin $\mathcal{J}/2$ representation of $\mathrm{sl}_2$. For spin 1/2 representation ($\mathcal{J} = 1$), matrix elements of (3.39) may be presented by

$$q^{\mathbf{h}_1} = \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}, \qquad q^{\mathbf{h}_2} = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}, \qquad \mathbf{x}_1\mathbf{y}_2 = \begin{pmatrix} 0 & 0 \\ 1-q^2 & 0 \end{pmatrix}, \qquad \mathbf{y}_1\mathbf{x}_2 = \begin{pmatrix} 0 & 1-q^2 \\ 0 & 0 \end{pmatrix}, \tag{3.40}$$

and in the homogeneous case $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$ the matrix (3.39) becomes exactly the six-vertex $R$-matrix.

Let the chain of Lax matrices (3.25) be given, $n = 1, 2, \dots, N$. Six-vertex model corresponds to the choice $\mathcal{J}_n = 1$ for all $n$. The values of two extra integrals of motion, $\mathcal{K}_1$ and $\mathcal{K}_2$ of (3.23), are related to the total spin of the chain: $\mathcal{K}_1$ is the number of "spins up," $\mathcal{K}_2$ is the number of "spins down."

## 4. Quantum curve

Solution of the classical equations of motion is based on the notion of the spectral curve $J(u,v) = 0$ (2.15). In the previous section we succeeded in construction of the "quantum partition function" $\mathbf{T}(u,v)$ producing the set of the integrals of motion, but we did not answer the question: what is the quantum analogue of $J(u,v) = 0$?

The answer is the following (we are repeating the introduction). Let $\mathbf{u}$, $\mathbf{v}$ be an additional auxiliary Weyl pair,

$$\mathcal{W} : \mathbf{u}\mathbf{v} = q^2\mathbf{v}\mathbf{u} \tag{4.1}$$

serving two variants of "quantum-mechanical" notations

$$\langle Q \mid u\rangle = Q(u), \qquad \mathbf{u}|u\rangle = |u\rangle u, \qquad \mathbf{v}|u\rangle = |q^2u\rangle v, \tag{4.2}$$

or

$$\langle v \mid \overline{Q}\rangle = \overline{Q}(v), \qquad \langle v|\mathbf{v} = v\langle v|, \qquad \langle v|\mathbf{u} = u\langle q^2 v|. \tag{4.3}$$

For the given layer-to-layer transfer matrix $\mathbf{T}(u,v)$ (3.10), (3.11), define

$$\mathbf{J}(\mathbf{u},\mathbf{v}) = \sum_{n=0}^{N}\sum_{m=0}^{M}(-q)^{-nm}(-\mathbf{u})^n(-\mathbf{v})^m\mathbf{t}_{n,m}. \tag{4.4}$$

One may easily see that in the limit $q \to 1$ (4.4) becomes (2.35). Operator $\mathbf{J}(\mathbf{u},\mathbf{v})$ belongs to $\mathcal{H}^{\otimes NM} \otimes \mathcal{W}$. Let $|t\rangle \in \mathcal{F}^{\otimes NM}$ be an eigenvector of $\mathbf{T}(u,v)$,

$$\mathbf{t}_{n,n}|t\rangle = |t\rangle t_{n,m}, \tag{4.5}$$

so that

$$\mathbf{J}(\mathbf{u},\mathbf{v})\big(|t\rangle \otimes \mathrm{id}\big) = \big(|t\rangle \otimes \mathrm{id}\big)J(\mathbf{u},\mathbf{v}), \tag{4.6}$$

where $J(\mathbf{u},\mathbf{v}) \in \mathcal{W}$, (1.11). Define a linear space $\Psi$ by

$$\mathbf{J}(\mathbf{u},\mathbf{v})|\Psi\rangle = 0 \qquad \text{or} \qquad \langle\Psi|\mathbf{J}(\mathbf{u},\mathbf{v}) = 0. \tag{4.7}$$

The linear space may be decomposed with respect to the basis $|t\rangle$,

$$|\Psi\rangle = \bigoplus_t |t\rangle \otimes |\overline{Q}_t\rangle \qquad \text{or} \qquad \langle\Psi| = \bigoplus_t \langle t| \otimes \langle Q_t|. \tag{4.8}$$

The point is that $\langle Q_t|$ and $|\overline{Q}_t\rangle$ have a simple and predictable structure, in particular $\langle Q_t \mid u\rangle$ and $\langle v \mid \overline{Q}_t\rangle$ may be defined as polynomials.

Functions $\langle Q \mid u\rangle$ and $\langle v \mid \overline{Q}\rangle$ (index $t$ is usually omitted) obey the linear difference equations (4.7),

$$\langle\Psi|\mathbf{J}(\mathbf{u},\mathbf{v})|t\rangle \otimes |u\rangle = 0, \qquad \langle t| \otimes \langle v|\mathbf{J}(\mathbf{u},\mathbf{v})|\Psi\rangle = 0, \tag{4.9}$$

which are exactly (1.7) and (1.8).

In this section, we will derive (4.4) and (4.7) as a solution of quantized linear problem (2.2) for the whole auxiliary lattice. A nontrivial problem is how to define a quantized linear problem in such a way that each linear variable belongs to the same subspace (subspace (4.7) in the final solution). This is the problem of consistency of linear problem and invariance of quantum curve.

**4.1. Generalized matrix of coefficients of linear problem.** Consider the linear problem (2.2) on an arbitrary lattice with $u,v$-periodical boundary conditions. The set of local linear problems

$$\begin{pmatrix}\psi'_\alpha\\ \psi'_\beta\end{pmatrix} = X_{\rm v}\cdot\begin{pmatrix}\psi_\alpha\\ \psi_\beta\end{pmatrix}, \qquad X_{\rm v} = \begin{pmatrix}a_{\rm v} & b_{\rm v}\\ c_{\rm v} & d_{\rm v}\end{pmatrix} \tag{4.10}$$

may be rewritten for the whole lattice in a matrix form,

$$\sum_j \ell_{k,j}\psi_j = 0. \tag{4.11}$$

The fragments in (4.11), corresponding to (4.10), are

$$\ell = \begin{pmatrix} \ddots & \vdots & \vdots & 0 & 0 & \ddots\\ 0 & 1 & 0 & -a_{\rm v} & -b_{\rm v} & 0\\ 0 & 0 & 1 & -c_{\rm v} & -d_{\rm v} & 0\\ \dots & 0 & 0 & 1 & 0 & \dots\\ \dots & 0 & 0 & 0 & 1 & \dots\\ \ddots & \vdots & \vdots & 0 & 0 & \ddots \end{pmatrix} \quad \text{for } \psi = \begin{pmatrix}\vdots\\ \psi'_\alpha\\ \psi'_\beta\\ \psi_\alpha\\ \psi_\beta\\ \vdots\end{pmatrix}. \tag{4.12}$$

In the quantum world, matrix elements $\ell_{k,j}$ are noncommutative operators $\boldsymbol{\ell}_{k,j}$. Therefore (4.11) may have two slightly different forms,

$$\text{(A)} \sum_j \boldsymbol{\ell}_{k,j}\,|\,\psi_j\rangle = 0 \qquad \text{or} \qquad \text{(B)} \sum_j \langle\psi_j\,|\,\boldsymbol{\ell}_{k,j} = 0. \tag{4.13}$$

Let the operators $\boldsymbol{\ell}_{k,j}$ obey the following exchange relations:

$$\begin{gathered}\boldsymbol{\ell}_{k,j}\boldsymbol{\ell}_{k',j'} - \boldsymbol{\ell}_{k',j}\boldsymbol{\ell}_{k,j'} = \boldsymbol{\ell}_{k',j'}\boldsymbol{\ell}_{k,j} - \boldsymbol{\ell}_{k,j'}\boldsymbol{\ell}_{k',j}, \quad k\neq k',\ j\neq j',\\ \boldsymbol{\ell}_{k,j}\boldsymbol{\ell}_{k',j} = \boldsymbol{\ell}_{k',j}\boldsymbol{\ell}_{k,j}.\end{gathered} \tag{4.14}$$

The aim of this subsection is to establish fundamental properties of such $\boldsymbol{\ell}$ and the form of solution of (4.13)(A) and (B).

Define

$$\det\boldsymbol{\ell} \overset{\text{def}}{=} \sum_\sigma (-)^\sigma \overset{\curvearrowright}{\prod_j} \boldsymbol{\ell}_{\sigma_j,j} \equiv \sum_\sigma (-)^\sigma \boldsymbol{\ell}_{\sigma_1,1}\boldsymbol{\ell}_{\sigma_2,2}\boldsymbol{\ell}_{\sigma_3,3}\dots, \tag{4.15}$$

where $\sigma$ are the permutations of the indices $1,2,3,\dots$. According to the first relation of (4.14), the definition (4.15) is invariant with respect to the ordering of $j$, for instance

$$\sum_\sigma (-)^\sigma \boldsymbol{\ell}_{\sigma_1,1}\boldsymbol{\ell}_{\sigma_2,2}\boldsymbol{\ell}_{\sigma_3,3}\cdots = \sum_\sigma (-)^\sigma \boldsymbol{\ell}_{\sigma_2,2}\boldsymbol{\ell}_{\sigma_1,2}\boldsymbol{\ell}_{\sigma_3,3}\cdots = \cdots, \tag{4.16}$$

and in general

$$\det\boldsymbol{\ell} = \sum_\sigma (-)^\sigma \overset{\curvearrowright}{\prod_j} \boldsymbol{\ell}_{\sigma_{\tau_j},\tau_j}, \tag{4.17}$$

where $\tau$ is any permutation of $1,2,3,\ldots$. Define next the algebraic supplements $\mathbf{A}_{j,k}$ of $\boldsymbol{\ell}$ (adjoint matrix) as (4.15)-determinants of the minors of $\boldsymbol{\ell}$,

$$\sum_k \mathbf{A}_{j,k}\boldsymbol{\ell}_{k,j} = \sum_k \boldsymbol{\ell}_{k,j}\mathbf{A}_{j,k} = \det\boldsymbol{\ell}. \tag{4.18}$$

The first equality here follows from (4.17), the last one in the definition of $\mathbf{A}_{j,k}$. The second line of (4.14) provides

$$\sum_k \mathbf{A}_{j,k}\boldsymbol{\ell}_{k,j'} = \sum_k \boldsymbol{\ell}_{k,j'}\mathbf{A}_{j,k} = 0 \quad \text{if } j \neq j'. \tag{4.19}$$

Therefore $(\det\boldsymbol{\ell})^{-1}\mathbf{A}_{j,k}$ and $\mathbf{A}_{j,k}(\det\boldsymbol{\ell})^{-1}$ are two variants of inverse matrices,

$$\sum_k (\det\boldsymbol{\ell})^{-1}\mathbf{A}_{j,k}\boldsymbol{\ell}_{k,j'} = \sum_k \boldsymbol{\ell}_{k,j'}\mathbf{A}_{j,k}(\det\boldsymbol{\ell})^{-1} = \delta_{j,j'}, \tag{4.20}$$

or since the inverse matrix must be both left-inverse and right-inverse,

$$\sum_j \boldsymbol{\ell}_{k,j}(\det\boldsymbol{\ell})^{-1}\mathbf{A}_{j,k_0} = \sum_j \mathbf{A}_{j,k_0}(\det\boldsymbol{\ell})^{-1}\boldsymbol{\ell}_{k,j} = \delta_{k,k_0}. \tag{4.21}$$

The main property of the elements of inverse matrices is the commutativity of their matrix elements with the same $k_0$, for example, for the variant (A),

$$\left[(\det\boldsymbol{\ell})^{-1}\mathbf{A}_{j,k_0},\, (\det\boldsymbol{\ell})^{-1}\mathbf{A}_{j',k_0}\right] = 0. \tag{4.22}$$

It is a particular case of the following statement:

$$\sum_j \boldsymbol{\ell}_{k,j}\mathbf{m}_j = \varepsilon_k, \quad \varepsilon_k \in \mathbb{C} \Longrightarrow \left[\mathbf{m}_j, \mathbf{m}_{j'}\right] = 0. \tag{4.23}$$

To prove (4.23), consider

$$c_{k,k'} = \sum_{j<j'} \left(\boldsymbol{\ell}_{k,j}\boldsymbol{\ell}_{k',j'} - \boldsymbol{\ell}_{k',j}\boldsymbol{\ell}_{k,j'}\right)\left(\mathbf{m}_j\mathbf{m}_{j'} - \mathbf{m}_{j'}\mathbf{m}_j\right), \quad k < k'. \tag{4.24}$$

Due to the first relation of (4.14) and definition (4.23), $c_{k,k'} = \varepsilon_k\varepsilon_{k'} - \varepsilon_{k'}\varepsilon_k = 0$. From the other side, let $\mathbf{A}^{(2)}_{j,j'|k,k'}$ be the matrix of the second algebraical supplements of $\boldsymbol{\ell}$:

$$\sum_{k<k'} \mathbf{A}^{(2)}_{i,i'|k,k'}\left(\boldsymbol{\ell}_{k,j}\boldsymbol{\ell}_{k',j'} - \boldsymbol{\ell}_{k',j}\boldsymbol{\ell}_{k,j'}\right) = \delta_{i,j}\delta_{i',j'}\det\boldsymbol{\ell}, \quad i < i',\ j < j'. \tag{4.25}$$

Then $0 = \sum_{k<k'} \mathbf{A}^{(2)}_{i,i'|k,k'} c_{k,k'} = \det\boldsymbol{\ell}\cdot(\mathbf{m}_j\mathbf{m}_{j'} - \mathbf{m}_{j'}\mathbf{m}_j)$, what proves the commutativity of $\mathbf{m}_j$. The commutativity (4.22) corresponds to $\varepsilon_k = \delta_{k,k_0}$.

For $\varepsilon_k = \delta_{k,k_0}$ with fixed $k_0$, let

$$\mathbf{m}_{j,j_0} = \left(\mathbf{m}_{j_0}\right)^{-1}\mathbf{m}_j = \mathbf{A}^{-1}_{j_0,k_0}\mathbf{A}_{j,k_0}, \qquad \mathbf{m}'_{j,j_0} = \mathbf{A}_{j,k_0}\mathbf{A}^{-1}_{j_0,k_0}. \tag{4.26}$$

Then the set $\mathbf{m}_{j,j_0}$, as well as the set $\mathbf{m}'_{j,j_0}$, is commutative ($k_0$ is fixed) and besides

$$\det \boldsymbol{\ell} \cdot \mathbf{m}_{j,j_0} = \mathbf{m}'_{j,j_0} \cdot \det \boldsymbol{\ell}. \tag{4.27}$$

Inverse relations (4.21) give

$$\sum_j \boldsymbol{\ell}_{k,j} \mathbf{m}_{j,j_0} = \delta_{k,k_0} \mathbf{A}^{-1}_{j_0,k_0} \det \boldsymbol{\ell}, \qquad \sum_j \mathbf{m}'_{j,j_0} \boldsymbol{\ell}_{k,j} = \delta_{k,k_0} \det \boldsymbol{\ell} \mathbf{A}^{-1}_{j_0,k_0}. \tag{4.28}$$

Equations (4.28) give the formal solution of the linear problem (4.13)(A) and (B): let $|\psi_{j_0}\rangle$ or $\langle\psi_{j_0}|$ for some $j_0$ defined by $\det \boldsymbol{\ell} \cdot |\psi_{j_0}\rangle = 0$ or $\langle\psi_{j_0}| \cdot \det \boldsymbol{\ell} = 0$. Then

$$\text{(A) } |\psi_j\rangle = \mathbf{m}_{j,j_0} |\psi_{j_0}\rangle \qquad \text{or} \qquad \text{(B) } \langle\psi_j| = \langle\psi_{j_0}| \mathbf{m}'_{j,j_0}. \tag{4.29}$$

A common feature of all $|\psi_j\rangle$ or $\langle\psi_j|$ is

$$\text{(A) } \det \boldsymbol{\ell} \cdot |\psi_j\rangle = 0 \qquad \text{or} \qquad \text{(B) } \langle\psi_j| \cdot \det \boldsymbol{\ell} = 0 \quad \forall j. \tag{4.30}$$

**4.2. Extended algebra of observables.** To establish the relation between the algebra of matrix elements (4.14) and our case of $X$ (3.4), we have to introduce the extended algebra of observables $\mathfrak{A}_{\mathrm{v}} = \mathcal{H}_{\mathrm{v}} \otimes \mathcal{W}_{\mathrm{v}}$. Define

$$X[\mathfrak{A}_{\mathrm{v}}] = \begin{pmatrix} \boldsymbol{\lambda}_{\mathrm{v}} q^{\mathbf{h}_{\mathrm{v}}} & \nu_{\mathrm{v}} \mathbf{y}_{\mathrm{v}} \\ \nu_{\mathrm{v}} \mathbf{x}_{\mathrm{v}} & \boldsymbol{\mu}_{\mathrm{v}} q^{\mathbf{h}_{\mathrm{v}}} \end{pmatrix}, \tag{4.31}$$

where $q^{\mathbf{h}_{\mathrm{v}}}$, $\mathbf{x}_{\mathrm{v}}$, $\mathbf{y}_{\mathrm{v}}$ are generators of $q$-oscillator $\mathcal{H}_{\mathrm{v}}$ (1.1) and $\boldsymbol{\lambda}_{\mathrm{v}}$, $\boldsymbol{\mu}_{\mathrm{v}}$, $\nu_{\mathrm{v}}$ are the generators of Weyl algebra $\mathcal{W}_{\mathrm{v}}$:

$$\boldsymbol{\lambda}_{\mathrm{v}} \boldsymbol{\mu}_{\mathrm{v}} = q^2 \boldsymbol{\mu}_{\mathrm{v}} \boldsymbol{\lambda}_{\mathrm{v}}, \qquad \nu_{\mathrm{v}}^2 = -\boldsymbol{\mu}_{\mathrm{v}} \boldsymbol{\lambda}_{\mathrm{v}}. \tag{4.32}$$

The $q$-oscillators and the Weyl algebra elements for different vertices always commute.

Matrix $\boldsymbol{\ell}$, defined according to (4.12) for vertex matrix $X[\mathfrak{A}_{\mathrm{v}}]$ (4.31), belongs exactly to the class (4.14) since

$$\begin{gathered} \nu \mathbf{x} \nu \mathbf{y} - \boldsymbol{\lambda} q^{\mathbf{h}} \boldsymbol{\mu} q^{\mathbf{h}} = \nu \mathbf{y} \nu \mathbf{x} - \boldsymbol{\mu} q^{\mathbf{h}} \boldsymbol{\lambda} q^{\mathbf{h}}, \\ [\nu \mathbf{y}, \boldsymbol{\mu} q^{\mathbf{h}}] = [\nu \mathbf{x}, \boldsymbol{\lambda} q^{\mathbf{h}}] = 0, \end{gathered} \tag{4.33}$$

and elements of $\mathfrak{A}_{\mathrm{v}}$ for different vertices commute. Determinant of $\boldsymbol{\ell}$ has a combinatorial representation of Figure 2.5, (2.32), (2.34), (2.35) with

$$L_{\alpha,\beta}[\mathfrak{A}_{\mathrm{v}}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \boldsymbol{\lambda}_{\mathrm{v}} q^{\mathbf{h}_{\mathrm{v}}} & \nu_{\mathrm{v}} \mathbf{y}_{\mathrm{v}} & 0 \\ 0 & \nu_{\mathrm{v}} \mathbf{x}_{\mathrm{v}} & \boldsymbol{\mu}_{\mathrm{v}} q^{\mathbf{h}_{\mathrm{v}}} & 0 \\ 0 & 0 & 0 & \nu_{\mathrm{v}}^2 \end{pmatrix}. \tag{4.34}$$

Let

$$\det \boldsymbol{\ell} \stackrel{\text{def}}{=} \mathbf{J}[\mathfrak{A}] = \sum (-)^{nm+n+m} u^n v^m \mathbf{J}_{n,m}[\mathfrak{A}], \tag{4.35}$$

where $u$ and $v$ are $\mathbb{C}$-valued spectral parameters introduced according to (2.9). Let locally

$$\boldsymbol{\lambda}_{\mathrm{v}} = \lambda_{\mathrm{v}} e^{Q_{\mathrm{v}}}, \qquad \boldsymbol{\mu}_{\mathrm{v}} = \mu_{\mathrm{v}} e^{P_{\mathrm{v}}} \quad \text{and therefore } \nu_{\mathrm{v}}^2 = -q^{-1}\lambda_{\mathrm{v}}\mu_{\mathrm{v}} e^{P_{\mathrm{v}}+Q_{\mathrm{v}}}, \qquad [Q_{\mathrm{v}}, P_{\mathrm{v}}] = \log q^2. \tag{4.36}$$

In the combinatorial representation of the determinant (Figure 2.5, (2.32), (2.34), (2.35)), consider a path $C_{n,m}$ of the homotopy class $n\mathcal{A} + m\mathcal{B}$. A monomial summand, corresponding to this path, may be factorized as

$$\mathbf{J}_{C_{n,m}} = \mathbf{t}_{C_{n,m}} e^{\boldsymbol{\phi}(C_{n,m})}. \tag{4.37}$$

Monomial $\mathbf{t}_{C_{n,m}}$ gathers all the $q$-oscillators and $\mathbb{C}$-valued parameters $\lambda_{\mathrm{v}}$, $\mu_{\mathrm{v}}$ and $-q^{-1}\lambda_{\mathrm{v}}\mu_{\mathrm{v}}$, and therefore it is exactly the $C_{n,m}$-monomial of $\mathbf{T}(u,v)$ (up to unessential renormalization of $\mathbf{x}_{\mathrm{v}}$ and $\mathbf{y}_{\mathrm{v}}$). Operator $\boldsymbol{\phi}$ is a sum of local $Q_{\mathrm{v}}$ and $P_{\mathrm{v}}$. One may easily see,

$$[\boldsymbol{\phi}(C_{n,m}), \boldsymbol{\phi}(C'_{n',m'})] = (nm' - mn')\log q^2 \quad \forall C, C'. \tag{4.38}$$

Let $C_{1,0}$ and $C_{0,1}$ be two particular fixed paths. Due to (4.38), all

$$\widetilde{\boldsymbol{\phi}}(C_{n,m}) = \boldsymbol{\phi}(C_{n,m}) - n\boldsymbol{\phi}(C_{1,0}) - m\boldsymbol{\phi}(C_{0,1}) \tag{4.39}$$

commute. Therefore one may diagonalize them simultaneously and without lost of generality (since $\lambda_{\mathrm{v}}$, $\mu_{\mathrm{v}}$ are free) put

$$\widetilde{\boldsymbol{\phi}}(C_{n,m}) \equiv 0. \tag{4.40}$$

Under this condition,

$$\boldsymbol{\phi}(C_{n,m}) \longrightarrow n\boldsymbol{\phi}(C_{1,0}) + m\boldsymbol{\phi}(C_{0,1}) \equiv nQ_0 + mP_0, \tag{4.41}$$

and exponent of $\boldsymbol{\phi}(C_{n,m})$, together with the combinatorial factor $(-)^{nm+n+m}u^n v^m$, become

$$(-)^{nm+n+m} e^{nQ_0+mP_0} u^n v^m = (-q)^{-nm}(-\mathbf{u})^n(-\mathbf{v})^m, \quad \mathbf{u} = e^{Q_0}u,\ \mathbf{v} = e^{P_0}v. \tag{4.42}$$

Therefore, on the subspace (4.40), $\mathbf{J}[\mathfrak{A}]$ becomes exactly $\mathbf{J}(\mathbf{u},\mathbf{v})$ (4.4).

Definition of $\mathbf{u}$, $\mathbf{v}$ (4.42) on the subspace (4.40) may be written in terms of $\boldsymbol{\lambda}_{\mathrm{v}}$, $\boldsymbol{\mu}_{\mathrm{v}}$ as

$$u\prod_m \boldsymbol{\lambda}_{nm} \overset{\widetilde{\phi}=0}{\longmapsto} \mathbf{u}\prod_m \lambda_{nm}, \qquad v\prod_n \boldsymbol{\mu}_{nm} \overset{\widetilde{\phi}=0}{\longmapsto} \mathbf{v}\prod_n \mu_{nm}. \tag{4.43}$$

**4.3. Structure of a solution of linear problem.** Turn now to the structure of linear spaces $|\Psi\rangle$ and $\langle\Psi|$ defined by

$$\text{(A) } \mathbf{J}[\mathfrak{A}]|\Psi\rangle = 0, \qquad \text{(B) } \langle\Psi|\mathbf{J}[\mathfrak{A}] = 0. \tag{4.44}$$

According to (4.22), (4.26), (4.27), each of $|\psi_j\rangle$ and $\langle\psi_j|$ belongs to these subspaces, and there exist sets of commutative operators $\mathbf{m}_j$ and $\mathbf{m}'_j$ (4.29) such that $|\psi_j\rangle = \mathbf{m}_j|\Psi\rangle$ and

$\langle\psi_j| = \langle\Psi|\mathbf{m}'_j$. In particular, one may consider the linear spaces $|\Psi\rangle$ and $\langle\Psi|$ in the basis of diagonal $\mathbf{m}_j$, $\mathbf{m}'_j$:

$$\text{(A) } |\psi_j\rangle = |\Psi\rangle m_j \qquad \text{or} \qquad \text{(B) } \langle\psi_j| = m_j\langle\Psi|, \tag{4.45}$$

where $m_j$ are eigenvalues of $\mathbf{m}_j$, and so forth. Every local pair of linear equations (4.10) takes the form

$$\text{(A)} \begin{array}{l}(1-\boldsymbol{\lambda}_{\rm v} q^{\mathbf{h}_{\rm v}} A_{\rm v} - \nu_{\rm v}\mathbf{y}_{\rm v} B_{\rm v})|\Psi\rangle = 0,\\ (1-\nu_{\rm v}\mathbf{x}_{\rm v} C_{\rm v} - \boldsymbol{\mu}_{\rm v} q^{\mathbf{h}_{\rm v}} D_{\rm v})|\Psi\rangle = 0,\end{array} \qquad \text{or} \qquad \text{(B)} \begin{array}{l}\langle\Psi|(1-A_{\rm v}\boldsymbol{\lambda}_{\rm v} q^{\mathbf{h}_{\rm v}} - B_{\rm v}\nu_{\rm v}\mathbf{y}_{\rm v}) = 0,\\ \langle\Psi|(1-C_{\rm v}\nu_{\rm v}\mathbf{x}_{\rm v} - D_{\rm v}\boldsymbol{\mu}_{\rm v} q^{\mathbf{h}_{\rm v}}) = 0,\end{array} \tag{4.46}$$

where in the notations of (4.10) $A_{\rm v} = m_\alpha/m_{\alpha'}$, $B_{\rm v} = m_\beta/m_{\alpha'}$, $C_{\rm v} = m_\alpha/m_{\beta'}$, and $D_{\rm v} = m_\beta/m_{\beta'}$.

The linear spaces $|\Psi\rangle$ and $\langle\Psi|$ belong to a module of $\mathcal{H}^{\otimes NM} \otimes \mathcal{W}^{\otimes NM}$. Its "physical" part is $\mathcal{F}^{\otimes NM}$, the artificial Weyl part may be defined in many inequivalent ways. Let $|n\rangle$ be the state of $\mathcal{F}^{\otimes NM}$ with $n_{\rm v}$ bosons in the vertex v, $n = \{n_{\rm v}\}$. We will focus on the formal states of $\mathcal{W}^{\otimes NM}$ for (4.44) on the basis of (4.45):

$$\text{(A) } |\Psi_n\rangle = (\langle n| \otimes \mathrm{id})|\Psi\rangle \qquad \text{or} \qquad \langle\Psi_n| = \langle\Psi|(|n\rangle \otimes \mathrm{id}). \tag{4.47}$$

Equations (4.46) give for (4.47)

$$\begin{gathered}\text{(A)} \begin{array}{l}(1-\boldsymbol{\lambda}_{\rm v} A_{\rm v})|\Psi_0\rangle = 0 \quad \forall {\rm v},\\ |\Psi_n\rangle = \displaystyle\prod_{\rm v} \frac{\nu_{\rm v}^{-n_{\rm v}}(\boldsymbol{\mu}_{\rm v} D_{\rm v}; q^2)_n}{C_{\rm v}^{n_{\rm v}}\sqrt{(q^2;q^2)_{n_{\rm v}}}}|\Psi_0\rangle\end{array}\\ \text{or}\\ \text{(B)} \begin{array}{l}\langle\Psi_0|(1-\boldsymbol{\mu}_{\rm v} D_{\rm v}) = 0 \quad \forall {\rm v},\\ \langle\Psi_n| = \langle\Psi_0|\displaystyle\prod_{\rm v}\frac{(\boldsymbol{\lambda}_{\rm v} A_{\rm v}; q^2)_{n_{\rm v}}\nu_{\rm v}^{-n_{\rm v}}}{B_{\rm v}^{n_{\rm v}}\sqrt{(q^2;q^2)_{n_{\rm v}}}}.\end{array}\end{gathered} \tag{4.48}$$

Here we used the representation

$$\mathbf{x}|n\rangle = |n-1\rangle\sqrt{1-q^{2n}}, \qquad \mathbf{y}|n\rangle = |n+1\rangle\sqrt{1-q^{2+2n}}. \tag{4.49}$$

Turn first to the conditions for $\Psi_0$ in (4.48). Since the total Fock vacuum is the eigenstate of $q$-oscillator counterpart of $\mathbf{J}[\mathfrak{A}]$, one may consider directly

$$\begin{gathered}\text{(A) } 0 = (\langle 0| \otimes \mathrm{id})\mathbf{J}[\mathfrak{A}]|\Psi\rangle = \mathbf{J}_0[\mathcal{W}^{\otimes NM}]|\Psi_0\rangle\\ \text{or}\\ \text{(B) } 0 = \langle\Psi|\mathbf{J}[\mathfrak{A}](|0\rangle \otimes \mathrm{id}) = \langle\Psi_0|\mathbf{J}_0[\mathcal{W}^{\otimes NM}],\end{gathered} \tag{4.50}$$

where

$$\mathbf{J}_0[\mathcal{W}^{\otimes NM}] = \prod_m \Big(1 - v\prod_n \boldsymbol{\mu}_{nm}\Big)\prod_n\Big(1 - u\prod_m \boldsymbol{\lambda}_{nm}\Big). \tag{4.51}$$

Evidently, $\mathbf{J}_0$ commutes with all $\widetilde{\boldsymbol{\phi}}$ (4.39) and therefore allows the projection $\mathcal{W}^{\otimes NM} \to \mathcal{W}$ (4.40). Simply applying (4.43),

$$\mathbf{J}_0[\mathcal{W}^{\otimes NM}] \overset{\widetilde{\phi}=0}{\longmapsto} J_0(\mathbf{u},\mathbf{v}) = \prod_m \Big(1 - \mathbf{v}\prod_n \mu_{nm}\Big)\prod_n\Big(1 - \mathbf{u}\prod_m \lambda_{nm}\Big). \tag{4.52}$$

Now the state of the Weyl algebra $\mathbf{u}$, $\mathbf{v}$ may be chosen in the most convenient way. For the case (A) the proper basis is $\langle v|$ of (4.2)-(4.3), and for the case (B) the proper basis is $|u\rangle$. If $u = u_n$ for the case (A) or $v = v_m$ for the case (B),

$$u_n = \prod_m \lambda_{nm}^{-1}, \qquad v_m = \prod_n \mu_{nm}^{-1}, \tag{4.53}$$

then

$$\text{(A) } \langle v \mid \Psi_0\rangle = 1 \qquad \text{or} \qquad \text{(B) } \langle \Psi_0 \mid u\rangle = 1. \tag{4.54}$$

Let further $|t\rangle$ and $\langle t|$ be the eigenstates of $\mathbf{t}_{nm}$, (4.5). Analogously to (4.47) let

$$\text{(A) } |\Psi_t\rangle = \big(\langle t|\otimes \mathrm{id}\big)|\Psi\rangle \qquad \text{or} \qquad \text{(B) } \langle\Psi_t| = \langle\Psi|\big(|t\rangle\otimes\mathrm{id}\big). \tag{4.55}$$

It follows from the second lines of (4.48)

$$\text{(A) } |\Psi_t\rangle = \mathcal{P}_t(\nu,\boldsymbol{\mu})|\Psi_0\rangle \qquad \text{or} \qquad \text{(B) } \langle\Psi_t| = \langle\Psi_0|\mathcal{P}'_t(\nu,\boldsymbol{\lambda}), \tag{4.56}$$

where, for example, $\mathcal{P}_t$ is a polynomial of $\boldsymbol{\mu}_{\mathrm{v}}$ of a total power not more that the number of bosons in $|t\rangle$, its structure with respect to $\nu_{\mathrm{v}}$ is rather simple. In addition, since we consider the eigenstates in the Fock counterpart of $\mathfrak{A}^{\otimes NM}$, polynomials $\mathcal{P}_t$ and $\mathcal{P}'_t$ must commute with all $\widetilde{\boldsymbol{\phi}}$ (4.39) and therefore must allow the projection $\mathcal{W}^{\otimes NM} \to \mathcal{W}$ (4.40):

$$\mathcal{P}_t \longrightarrow \mathbf{w}^{-J}P_t(\mathbf{v}), \qquad \mathcal{P}'_t \longrightarrow P'_t(\mathbf{u})\mathbf{w}^{-K}, \tag{4.57}$$

where $\mathbf{w}^2 = -\mathbf{vu}$, $\mathbf{w}$-factor comes from $\nu_{\mathrm{v}}$ factors; $J$ and $K$ are integers; $P_t$ and $P'_t$ are polynomials of a power not higher than the total number of bosons in the state $|t\rangle$.

Taking now into account (4.54), we come to the final statement: in the restricted algebra $\mathcal{H}^{\otimes NM}\otimes\mathcal{W}$ the equations

$$\text{(A) } \big(\langle t|\otimes\langle v|\big)\mathbf{J}(\mathbf{u},\mathbf{v})|\Psi\rangle = 0, \qquad \text{(B) } \langle\Psi|\mathbf{J}(\mathbf{u},\mathbf{v})\big(|t\rangle\otimes|u\rangle\big) = 0 \tag{4.58}$$

have the solutions

$$\begin{aligned} &\text{(A) if } u = u_n, \quad \text{then } \langle v \mid \Psi\rangle = v^{-J_n/2}\overline{Q}_n(v),\\ &\text{(B) if } v = v_m, \quad \text{then } \langle \Psi \mid u\rangle = u^{-K_m/2}Q_m(u), \end{aligned} \tag{4.59}$$

where $u_n$, $v_m$ are given by (4.53), and the integers $J_n$, $K_m$ and degrees of polynomials $Q_n$, $\overline{Q}_m$ are not higher than the total number of bosons.

In the next section, considering the examples, we will see that $J_n$ and $K_m$ are the eigenvalues of $\mathcal{J}_n$ and $\mathcal{K}_m$ (3.23) and, moreover, the degrees of $\overline{Q}_m$ and $Q_n$ are exactly $K_m$ and $J_n$.

In the next section we will add $q^{-\mathcal{J}_n}$ and $q^{-\mathcal{K}_m}$ to the definition of $u_n$ and $v_m$. This allows one to cancel the half-integer prefactors $v^{-J_n/2}$ of $\overline{Q}_n(v)$ and $u^{-K_m/2}$ of $Q_m(u)$. The corresponding values of $u_n$ and $v_m$ may be obtained via conditions $\overline{Q}(0) = 1$ and $Q_n(0) = 1$.

## 5. Examples

Let us illustrate (4.9) for the six-vertex model first, and then for an arbitrary square lattice.

**5.1. Six-vertex chain.** Consider the lattice with $M = 2$ and arbitrary $N$. This is the case of six-vertex model, where (4.9) becomes the Baxter equation. It follows from (3.38),

$$\begin{gathered}\sum_n u^n \mathbf{t}_{n,0} = \prod_n \left(1 + u\lambda_{n,1}\lambda_{n,2} q^{\mathbf{h}_{n,1}+\mathbf{h}_{n,2}}\right), \\ \sum_n u^n \mathbf{t}_{n,1} = \mathbf{t}(u), \\ \sum_n u^n \mathbf{t}_{n,2} = \prod_n \mu_{n,1}\mu_{n,2}\left(q^{\mathbf{h}_{n,1}+\mathbf{h}_{n,2}} + q^{-2}u\lambda_{n,1}\lambda_{n,2}\right),\end{gathered} \tag{5.1}$$

where $\mathbf{t}(u)$ is the transfer matrix for the Lax operator (3.39). Its opposite elements are

$$\begin{gathered}\mathbf{t}_{0,1} = \prod_n \left(\mu_{n,1} q^{\mathbf{h}_{n,1}}\right) + \prod_n \left(\mu_{n,2} q^{\mathbf{h}_{n,2}}\right), \\ \mathbf{t}_{N,1} = \prod_n \left(-q^{-1}\lambda_{n,1}\lambda_{n,2}\mu_{n,2} q^{\mathbf{h}_{n,1}}\right) + \prod_n \left(-q^{-1}\lambda_{n,2}\lambda_{n,1}\mu_{n,1} q^{\mathbf{h}_{n,2}}\right).\end{gathered} \tag{5.2}$$

Applying the rule (4.4) $u^n v^m \mapsto (-q)^{-nm}(-\mathbf{u})^n(-\mathbf{v})^m$, we come to

$$\begin{aligned}\mathbf{J}(q\mathbf{u},\mathbf{v})\mathbf{v}^{-1} = {} & -\mathbf{t}(\mathbf{u}) + \mathbf{v}^{-1}\prod_n \left(1 - q^{-1}\mathbf{u}\lambda_{n,1}\lambda_{n,2} q^{\mathbf{h}_{n,1}+\mathbf{h}_{n,2}}\right) \\ & + \mathbf{v}\prod_n \mu_{n,1}\mu_{n,2}\left(q^{\mathbf{h}_{n,1}+\mathbf{h}_{n,2}} - q^{-1}\mathbf{u}\lambda_{n,1}\lambda_{n,2}\right).\end{aligned} \tag{5.3}$$

Equation $\langle Q|J(q\mathbf{u},\mathbf{v})\mathbf{v}^{-1}|u\rangle = 0$ is exactly Baxter's equation for $\mathcal{U}_q(\widehat{\mathrm{sl}}_2)$,

$$Q(u)t(u) = Q(q^{-2}u)\phi(u) + Q(q^2u)\phi'(u), \tag{5.4}$$

where

$$\begin{aligned}\phi(u) &= v^{-1}\prod_n \left(1 - q^{-1}u\lambda_{n,1}\lambda_{n,2} q^{\mathbf{h}_{n,1}+\mathbf{h}_{n,2}}\right), \\ \phi'(u) &= v\prod_n \mu_{n,1}\mu_{n,2}\left(q^{\mathbf{h}_{n,1}+\mathbf{h}_{n,2}} - q^{-1}u\lambda_{n,1}\lambda_{n,2}\right).\end{aligned} \tag{5.5}$$

Condition $Q(0) = 1$ gives (see (5.2))

$$v^{-1} + v\prod_n \mu_{n,1}\mu_{n,2}q^{\mathbf{h}_{n,1}+\mathbf{h}_{n,2}} = \prod_n \mu_{n,1}q^{\mathbf{h}_{n,1}} + \prod_n \mu_{n,2}q^{\mathbf{h}_{n,2}}, \tag{5.6}$$

which has two solutions corresponding to two Baxter's functions $Q$: the first one is

$$v = v_1 \stackrel{\text{def}}{=} \left(\prod_n \mu_{n,1}q^{\mathbf{h}_{n,1}}\right)^{-1}, \tag{5.7}$$

so that $Q = Q_1(u)$ is a polynomial of the degree $\sum_n \mathbf{h}_{n,1} = \mathcal{K}_1$ (see (3.23)), this value of the degree follows from the second line of (5.2); the second solution is

$$v = v_2 \stackrel{\text{def}}{=} \left(\prod_n \mu_{n,2}q^{\mathbf{h}_{n,2}}\right)^{-1}, \tag{5.8}$$

and corresponding $Q = Q_2(u)$ is the polynomial of the power $\sum_n \mathbf{h}_{n,2} = \mathcal{K}_2$. In some sense, two functions $Q$ correspond to two sheets of classical $q = 1$ spectral hyperelliptic curve $v^{-1}\phi(u) + v\phi'(u) = t(u)$. Note that we consider now the Bethe ansatz equations for $\mathcal{U}_q(\widehat{\mathrm{sl}}_2)$ chain with arbitrary $\mathcal{J}_n$, this is the inhomogeneity of highest spin. Six-vertex case corresponds to $\mathcal{J}_n = 1$ for all $n$.

In the *inhomogeneous* case the spectrum of $t(u)$ follows from (5.4) and just the condition $Q \neq 0$ (without fixing $v$ and considering the polynomial structure of $Q$). Let us fix $\mathcal{J}_n = 1$, the six-vertex case, and a priori the Fock space representation. Simply substituting $u = (\lambda_{n,1}\lambda_{n,2})^{-1}$ and $u = q^2(\lambda_{n,1}\lambda_{n,2})^{-1}$ into (5.4) and excluding $Q$, one comes to

$$t\left(\frac{1}{\lambda_{n,1}\lambda_{n,2}}\right)t\left(\frac{q^2}{\lambda_{n,1}\lambda_{n,2}}\right) = \prod_k q\mu_{k,1}\mu_{k,2}\left(1 - q^2\frac{\lambda_{k,1}\lambda_{k,2}}{\lambda_{n,1}\lambda_{n,2}}\right)\left(1 - q^{-2}\frac{\lambda_{k,1}\lambda_{k,2}}{\lambda_{n,1}\lambda_{n,2}}\right) \tag{5.9}$$

with $n = 1,\ldots,N$. These $N$ equations and conditions (5.2) do produce the whole spectrum of $t(u)$ for inhomogeneous six-vertex model.

Equations (5.9) may be obtained in the other way—via $\overline{Q}(v) = \langle v|\overline{Q}\rangle$. Condition $\overline{Q}(0) = 1$ fixes $u = u_n$,

$$u_n = \left(\lambda_{n,1}\lambda_{n,2}q^{\mathbf{h}_{n,1}+\mathbf{h}_{n,2}}\right)^{-1}, \tag{5.10}$$

corresponding $\overline{Q} = \overline{Q}_n(v)$ is a polynomial of the power $\mathcal{J}_n = \mathbf{h}_{n,1} + \mathbf{h}_{n,2}$. In the six-vertex case $\mathcal{J}_n = 1$, therefore $\overline{Q}_n(v) = v - \zeta_n$, and equating to zero each $v$-term of the right-hand side of (4.9), one comes to

$$\zeta_n = -\frac{t(q^3u_n)}{\prod_k q\mu_{k,1}\mu_{k,2}(1 - q^{-2}u_n/u_k)} = -\frac{\prod_k(1 - q^2u_n/u_k)}{t(qu_n)}. \tag{5.11}$$

The second equality gives (5.9).

**5.2. $\mathcal{U}_q(\widehat{\mathrm{sl}}_M)$ equations.** Let the lattice have arbitrary $N$ and $M$. We will consider (4.9) for $\langle Q \mid u\rangle = Q(u)$ and for $\langle u \mid \overline{Q}\rangle = \overline{Q}(u)$—the last function was not mentioned before, but it is interesting to discuss what it is.

Equations $\langle Q|J(\mathbf{u},\mathbf{v})|u\rangle = 0$ and $\langle u|J(\mathbf{u},\mathbf{v})|\overline{Q}\rangle = 0$ read correspondingly

$$\sum_{m=0}^{M} Q(q^{2m}u)(-v)^m\tau_m(u) = 0, \qquad \sum_{m=0}^{M} \tau(q^{-2m}u)(-v)^m\overline{Q}(q^{-2m}u) = 0, \tag{5.12}$$

where

$$\tau_m(u) = \sum_{n=0}^{N}(-q)^{nm}(-u)^n\mathbf{t}_{nm} \tag{5.13}$$

(cf. (1.5), (1.6)). Combinatorially, one can obtain the following summation formulae: analogue of (5.1)

$$\begin{gathered}\sum_n u^n t_{n,0} = \prod_n\left(1+u\prod_m \lambda_{n,m}q^{\mathbf{h}_{n,m}}\right),\\ \sum_n u^n t_{n,M} = \prod_n\left(\prod_m \mu_{n,m}q^{\mathbf{h}_{n,m}} + u\prod_m -q^{-1}\lambda_{n,m}\mu_{n,m}\right),\end{gathered} \tag{5.14}$$

and analogue of (5.2)

$$\begin{gathered}\sum_m v^m t_{0,m} = \prod_m\left(1+v\prod_n \mu_{n,m}q^{\mathbf{h}_{n,m}}\right),\\ \sum_m v^m t_{N,m} = \prod_m\left(\prod_n \lambda_{n,m}q^{\mathbf{h}_{n,m}} + v\prod_n -q^{-1}\lambda_{n,m}\mu_{n,m}\right).\end{gathered} \tag{5.15}$$

Let us fix notations for the inhomogeneities (cf. (4.53))

$$u_n = \left(\prod_m \lambda_{nm}q^{\mathbf{h}_{nm}}\right)^{-1}, \qquad v_m = \left(\prod_n \mu_{nm}q^{\mathbf{h}_{nm}}\right)^{-1}. \tag{5.16}$$

It follows from (5.14),

$$\tau_0(u) = \prod_n\left(1-\frac{u}{u_n}\right), \qquad \tau_M(u) = \prod_m v_m\prod_n\left(1-q^{-2\mathcal{J}_n}\frac{u}{u_n}\right). \tag{5.17}$$

For the normalization $Q(0) = \overline{Q}(0) = 1$ both equations in (5.12) are equivalent to the first line of (5.15):

$$\sum_{m=0}^{M}(-v)^m\tau_m(0) = \prod_m\left(1-\frac{v}{v_m}\right) = 0. \tag{5.18}$$

Suppose that $Q(u) \sim u^K$ and $\overline{Q}(u) \sim u^{\overline{K}}$ when $u \to \infty$. Then (5.12) and the second line of (5.15) provide the following conditions for $K$ and $\overline{K}$:

$$\prod_m \left(1 - q^{2K-2\mathcal{K}_m} \frac{v}{v_m}\right) = 0,$$
$$\prod_m \left(1 - q^{-2\overline{K}-2N-2\mathcal{K}_m} \frac{v}{v_m}\right) = 0. \tag{5.19}$$

The charges $\mathcal{J}_n$ and $\mathcal{K}_m$ are given by (3.23). Equation (5.18) has the solutions $v = v_m$, $m = 1, 2, 3, \ldots, M$. In what follows, we will assume the generic set of $\mu_{nm}$, so that all $v_m$ are different. Let $v = v_m$ correspond to $Q(u) = Q_m(u)$ and $\overline{Q}(u) = \overline{Q}_m(u)$. Then (5.19) defines the leading $u \to \infty$ asymptotic

$$Q_m(u) \sim u^{\mathcal{K}_m}, \qquad \overline{Q}_m(u) \sim u^{\overline{\mathcal{K}}_m}, \tag{5.20}$$

where

$$\overline{\mathcal{K}}_m = -\sum_n (1 + \mathbf{h}_{n,m}). \tag{5.21}$$

Thus, for the Fock space representation all $Q_m(u)$ are polynomials of the power $\mathcal{K}_m$. Functions $\overline{Q}_m(u)$ are rational functions, below we construct them in terms of $Q_m(u)$.

Instead of the Fock space representation $\mathrm{Spec}(\mathbf{h}) = 0, 1, 2, \ldots, \infty$, one may consider the anti-Fock space, $\mathrm{Spec}(\mathbf{h}) = -1, -2, -3, \ldots, -\infty$. Then equations in (4.46) are to be solved in a different way, and as a result $\overline{Q}_m(u)$ become polynomials and $Q_m(u)$ become rational functions.

The dual "$T$-$Q$" equations for $\mathrm{sl}_N$ correspond to the evident exchange $N \leftrightarrow M$, $u \leftrightarrow v$, and $\mathcal{J} \leftrightarrow \mathcal{K}$.

Turn now to the form of $\overline{Q}_m$ for the Fock space representation. The detailed "arithmetical" consideration of (5.12) as a set of linear equations allows one to conclude for instance

$$\overline{Q}_M(u) = \frac{W_M(u)}{V(u)}, \tag{5.22}$$

where

$$W_M(u) = \det||Q_j(q^{2i}u)v_j^i||_{i,j=1,\ldots,M-1}, \tag{5.23}$$

and $V(u)/V(q^2u) = v_1 v_2 \cdots v_M(\tau_M(q^M u)/\tau_0(q^2 u))$. All the other $\overline{Q}_m$ correspond to (5.22) and (5.23) with permuted set of indices of $Q_j$. Inhomogeneity of $v_m$ is important in this consideration since if $Q_m(0) = 1$, then $W_M(0) = v_1 \cdots v_{M-1} \prod_{1 \le i < j < M}(v_i - v_j)$, and therefore $W_M(u)$ is not zero. As well, for the generic $v_m$ all $Q_m(u)$ are functionally independent, their Wronskian

$$W(u) = \det||Q_j(q^{2(i-1)}u)v_j^{i-1}||_{i,j=1,\ldots,M} \tag{5.24}$$

obeys

$$\frac{W(u)}{W(q^2u)} = v_1v_2\cdots v_M\frac{\tau_M(q^Mu)}{\tau_0(u)} \tag{5.25}$$

with the initial condition $W(0) = \prod_{1\le i<j\le M}(v_i - v_j)$, therefore $W(u) \ne 0$.

Using these "arithmetical" considerations, one may express the fundamental transfer matrices $\tau_m(u)$ of $\mathrm{sl}_M$ via determinants $\det\|Q_j(q^{2p_i}u)v_j^{p_i}\|_{i,j=1,\dots,M}$, where $p_i$ is a subset of $(0,1,\dots,M)$. Moreover, for more general sets of $p_i$, any transfer matrix of $\mathrm{sl}_M$ may be expressed as such a determinant [4].

The nested Bethe ansatz equations (see [17] and, e.g., [7]) may be derived from the generalized "$T$-$Q$" equations in an "arithmetical" way as well. Let

$$\begin{aligned}\mathcal{Q}_m(u) &= \det\|Q_j(q^{2i}u)v_j^i\|_{i,j=1,\dots,m},\\ \mathcal{P}_m(u) &= \det\|Q_j(q^{2i}u)v_j^i\|_{j=1,\dots,m;\,i=0,2,\dots,m},\end{aligned} \tag{5.26}$$

with $\mathcal{Q}_0 = 1$ and $\mathcal{P}_0 = 0$. Function $\mathcal{Q}_m(u)$ is a polynomial of the power $\mathcal{K}_1 + \mathcal{K}_2 + \cdots + \mathcal{K}_m$. Equations

$$\mathcal{P}_m(u)\mathcal{Q}_{m-1}(u) = \mathcal{P}_{m-1}(u)\mathcal{Q}_m(u) + \frac{1}{v_m}\mathcal{Q}_{m-1}(q^2u)\mathcal{Q}_m(q^{-2}u) \tag{5.27}$$

for $m = 1,2,\dots,M-1$ are just the determinant identities for (5.26). For $m = M$ the system (5.27) should be completed by an equation, following from (5.12):

$$\tau_1(u)\mathcal{Q}_{M-1}(u) = \tau_0(u)\mathcal{P}_{M-1}(u) + v_1\cdots v_{M-1}\tau_M(u)\mathcal{Q}_{M-1}(q^2u), \tag{5.28}$$

where $\tau_0$ and $\tau_M$ are given by (5.17). The nested Bethe ansatz equations are a closed system of algebraic equations for roots of $\mathcal{Q}_m(u)$ following from (5.27), (5.28).

## 6. Conclusion

This paper has a modest aim just to give a correct form of "$T$-$Q$" equations. We can say nothing about their solution. But we would like to note that from the point of view of three-dimensional models, the thermodynamical limit is the limit $N,M \to \infty$ with non-singular ratio $N : M$. The nested Bethe ansatz equations were never investigated in this limit since $\mathcal{Q}_m(u)$ (5.26) with finite $m$ has a finite number of roots. In addition, an excitation corresponds to a change of the structure of occupation numbers. From $\mathrm{sl}_M$ point of view of the previous section, it corresponds not only to a change of $\mathcal{K}_m$ related to the powers of $\mathcal{Q}$-operators, but as well it corresponds to a change of $\mathcal{J}_n$ which is a change of the $\mathrm{sl}_M$-structure of the nested Bethe ansatz.

Let us better conclude this paper by a brief comparison of two exactly integrable models in $2+1$ dimensional space-time. From the point of view of the algebra of observables, these models should be called "$q$-oscillator model" and "Weyl-algebra model." The last one is a quantum-mechanical reformulation of Zamolodchikov-Bazhanov-Baxter model of statistical mechanics, which has a long history [1–3, 18, 19]. Both models are based on two slightly different forms of local linear problem, the linear problem for Weyl-algebra model may be found in [16]. Solution of both classical models may be expressed in terms of algebraic geometry [8, 10]. Equations of motion may be understood as a canonical mapping conserving certain symplectic structure [5, 14]. Poisson structure allows an immediate quantization [5, 16]. Quantum-mechanical integrals of motion may be combined into a direct sum of transfer matrices for fundamental representations of either $\mathrm{sl}_N$ or $\mathrm{sl}_M$ [5, 13, 15], and finally, the solvability of the models is based on quantized auxiliary linear problem and remarkable features of their operator-valued matrices of coefficients, as can be seen in this paper and [12].

The continuous limit of both classical models may be illustrative. Equations of motion for six fields $q_j$, $q_j^*$, $j = 1,2,3$, follow from the action

$$A = \int d^3x\left[q_1^*\partial_1 q_1 + q_2^*\partial_2 q_2 + q_3^*\partial_3 q_3 + V(q^*,q)\right], \tag{6.1}$$

where for the classical continuous limit of $q$-oscillator model $V = q_1^* q_2^* q_3^* - q_1q_2q_3$, this is nothing but the model of three-wave resonant interaction [9]. For the classical continuous limit of Weyl-algebra model the potential is $V = (q_1^* - q_2)(q_2^* - q_3)(q_3^* - q_1)$ [11].

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