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Research Article

Concerning Cut Point Spaces of Order Three

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A point p of a topological space X is a cut point of X if $X-\{p\}$ is disconnected. Further, if $X-\{p\}$ has precisely m components for some natural number $m \geq 2$ we will say that p has cut point order m. If each point y of a connected space Y is a cut point of Y, we will say that Y is a cut point space. Herein we construct a space S so that S is a connected Hausdorff space and each point of S is a cut point of order three. We also note that there is no uncountable separable cut point space with each point a cut point of order three and therefore no such space may be embedded in a Euclidean space.

1. Introduction

The study of cut points in topological spaces has long been of interest. Whyburn (e.g., [1–3]) studied heavily the role of cut points of metric continua. In particular, he showed that all cut points of a separable metric continuum are of order two except for a countable number.

Shimrat [4] proved that the following are equivalent for a nonempty connected separable metric space X: (1) X is locally connected and every point of X is a cut point; (2) X is locally arcwise connected, contains no simple closed curves, and has no end-points; (3) X is an open ramification. The reader is also referred to Stone [5].

Ward [6] showed that every metric space that is separable, connected and locally connected, and in which each point is a strong cut point (having cut point order two), is homeomorphic to the real line $\mathcal{R}$. Franklin and Krishnarao [7] have shown that the same characterization does not hold for Hausdorff spaces. Klieber [8] has provided a characterization similar to that of Ward's, namely that a separable Hausdorff space X is homeomorphic to $\mathcal{R}$ if every $x \in X$ is a strong cut point and the set of components of complements of points forms a subbase for the space X.

A comprehensive study of cut point spaces in the most general setting has been done by Honari and Bahrtampour [9]; the work is done without the assumption of any separation axioms. It is shown that each cut point is either open or closed and that every cut point space has infinitely many closed points and is noncompact. It is also shown that there is just one irreducible cut point space, to within a homeomorphism, namely the "Khalimsky line". This is a topology on the set $\mathbf{Z}$ of all integers, in which each odd integer is isolated and each even integer n has the smallest neighborhood $\{n-1, n, n+1\}$.

A natural question is whether a connected space may have each point be a cut point of fixed order greater than or equal to three. Herein, we complement the studies mentioned above by constructing a space S so that S is a connected Hausdorff space and each point of S is a cut point of order three. We also demonstrate in Section 4 that no cut point space with each point a cut point of order three may be embedded in a Euclidean space, and indeed that no such space that is uncountable can be separable, connected, and Hausdorff space.

2. Preliminaries

We will say that a point p of a topological space X is a cut point of X if $X - \{p\}$ is disconnected. Further, if $X - \{p\}$ has precisely m components for some natural number $m \geq 2$, we will say that p has cut point order m. If each point y of a connected space Y is a cut point of Y, we will say that Y is a cut point space. If N is a natural number greater than or equal to two and each point y of a cut point space Y has cut point order N, we will say that Y is a cut point space of order N.

For a space X and $A \subseteq X$, $\mathrm{Cl}(A)$ will denote the closure of A in X. For subsets A and B of space X, we will say that A and B are mutually separated if and only if $\mathrm{Cl}(A) \cap B = \varnothing$ and $A \cap \mathrm{Cl}(B) = \varnothing$. For points x and y in the Euclidean space $\mathcal{R}^2$, let $d(x, y)$ denote the Euclidean distance between x and y and, for $\epsilon > 0$, let $N(x, \epsilon)$ denote the open neighborhood $\{y : d(x, y) < \epsilon\}$.

3. Construction of cut point space S

We first construct a connected set in the plane, each point of which is a cut point of order two or three. The closure of this set is a well-known dendrite.

Consider the open interval $G_0 = (0,1) \times \{0\}$ on the x-axis in $\mathcal{R}^2$. Although not itself an element of the space, the origin will play a special role when we define the topology for our space and will be denoted by $\mathbb{O}$. Let D be the set of all dyadic rational numbers in $(0,1)$. That is, let $x \in D$ if and only if there is a positive integer n and a positive integer k such that $k \leq 2^{(n-1)}$ and $x = (2k-1)/2^n$. For each $x = (2k-1)/2^n \in D$, let I_x denote the open vertical interval $\{x\} \times (0, 1/2^n)$. Let G_1 be the set of all these intervals I_x. Next, for each interval g in G_1, add a collection of open horizontal intervals as was done for G_0. The midpoint p of each $g \in G_1$ should have an interval added of length half the length of g with left endpoint at p. Call this collection of open intervals G_2. Next add a

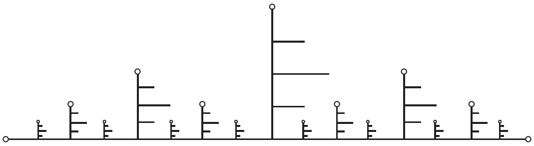

Figure 3.1

collection of open vertical intervals for each interval in G_2 in the same manner. Call this collection of open intervals G_3. Continue this process inductively. No two intervals in $\bigcup_{i>0} G_i$ should intersect. Let M_0 be the connected union of all these intervals; Figure 3.1 gives an indication of the first few steps in the construction of M_0.

Let T_0 be the set of cut points of M_0 of order three and let C_0 be the set of cut points of M_0 of order two. For each whole number n, let M_n denote the set of all sequences $(p_0, p_1, \dots, p_n)$ such that $p_n \in M_0$ and if $n > 0$, then $p_i \in C_0$ for each i such that $0 \le i < n$. If $p_0 \in M_0$, we may refer to (p_0) simply as p_0.

Let $S = \bigcup_{i=0}^{\infty} M_i$; S is the set of points (finite sequences) on which we will define a topology $\mathcal{T}$. If $p \in S$, then for each positive number ϵ we will define a subset $R(p,\epsilon)$ of S containing p. Let $B_p = \{R(p,\epsilon) : \epsilon > 0\}$. The members of B_p will be called regions and the union of all of the sets B_p for $p \in S$ will form a basis for $\mathcal{T}$.

Let $p \in S$. Then $p = (p_0, p_1, \dots, p_n)$ is in M_n, $p_n \in M_0$ and if $n > 0$, then for each i such that $0 \le i < n$, $p_i \in C_0$. Let $\epsilon > 0$.

We next define our regions $R(p,\epsilon)$.

(1) If $p_n \in T_0$, then
if $n = 0$, $R(p,\epsilon) = N(p_0,\epsilon) \cap T_0$, and
if $n > 0$, $R(p,\epsilon) = (p_0, p_1, \dots, p_{n-1}) \times (N(p_n,\epsilon) \cap T_0)$.

(2) If $p_n \in C_0$, then
if $n = 0$, $R(p,\epsilon) = \{p_0\} \cup (N(p_0,\epsilon) \cap T_0) \cup (p_0 \times (N(\mathbb{O},\epsilon) \cap T_0))$, and
if $n > 0$, $R(p,\epsilon) = (\{p\} \cup (p_0, p_1, \dots, p_{n-1}) \times (N(p_n,\epsilon) \cap T_0)) \cup ((p_0, p_1, \dots, p_n) \times (N(\mathbb{O},\epsilon) \cap T_0))$.

The next two lemmas are direct applications of our definitions.

Lemma 3.1. *If $p \in S$, $\epsilon > 0$, $\delta > 0$, and $\epsilon < \delta$, then $R(p,\epsilon) \subseteq R(p,\delta)$.*

Lemma 3.2. *If $p \in S$, $\epsilon > 0$, and $q \in R(p,\epsilon)$, then there is a positive number δ such that $R(q,\delta) \subseteq R(p,\epsilon)$.*

Theorem 3.3. *$B = \{B_p = R(p,\epsilon) : p \in S,\ \epsilon > 0\}$ is a basis for a topology $\mathcal{T}$ on S.*

Here, we must show that if a point p is in each of the regions U and V, there is a region containing p that is a subset of $U \cap V$. The proof is a direct application of Lemmas 3.1 and 3.2.

Theorem 3.4. *$(S,\mathcal{T})$ is Hausdorff.*

Proof. Suppose $p = (p_0, p_1, \dots, p_n)$ and $q = (q_0, q_1, \dots, q_m)$ are distinct elements of S. We consider two cases.

Case 1. Assume $m = n$. Select ϵ to be one third of the distance between p_n and q_n. Then $N(p_n,\epsilon) \cap N(q_n,\epsilon) = \varnothing$ and, therefore, $R(p,\epsilon) \cap R(q,\epsilon) = \varnothing$.

Case 2. Assume without loss of generality that $m > n$. Since for any point $x \in S$ and any $\epsilon > 0$, if $x \in M_n$, then $R(x,\epsilon) \subseteq M_n \cup M_{n+1}$, then $R(p,\epsilon) \cap R(q,\epsilon) = \varnothing$ unless $m = n+1$. In this case, we set ϵ to be less than one third of the distance from q_m to $\mathbb{O}$. Thus, we have that $N(q_m,\epsilon) \cap N(\mathbb{O},\epsilon) = \varnothing$. It follows that $R(p,\epsilon) \cap R(q,\epsilon) = \varnothing$. □

Theorem 3.5. *$(S,\mathcal{T})$ is connected.*

Proof. We begin by showing that M_0 with the subspace topology of S is connected. Assume S is not connected. Then there is a nonempty set $U \neq M_0$ open relative to M_0 such that no point is a boundary point of U. If $x \in U$, then there exists an $\epsilon_x > 0$ such that $N(x,\epsilon_x) \cap T_0 \subseteq U$. Moreover, if $p \in N(x,\epsilon_x) \cap C_0$, $p \in U$ since otherwise p is a boundary point of U. Thus, $U = [\bigcup_{x\in U}(N(x,\epsilon_x) \cap T_0)] \cup (C_0 \cap U) = \bigcup_{x\in U}[N(x,\epsilon_x)]$. Then U is a nonempty open set in M_0 with the subspace topology of R^2 such that no point is a boundary point of U, a contradiction.

We next show that $M_0 \cup M_1$ with the subspace topology of S is connected. If $p_0 \in C_0$, then p_0 is a limit point of $M_1(p_0) = p_0 \times M_0$ and $M_1(p_0)$ is connected since M_0 is connected. Now $M_1 = \bigcup_{x\in C_0} M_1(x)$ so $M_0 \cup M_1$ is the union of a collection of connected sets one of which, M_0, contains a limit point of each of the others so $M_0 \cup M_1$ is connected.

By a similar argument and by induction $\bigcup_{i=0}^{k} M_k$ is connected for each natural number k. It then follows that $S = \bigcup_{i=0}^{\infty} M_k$ is connected. □

Lemma 3.6. *With M_0 having the subspace topology of S, each point of T_0 is a cut point of order three in M_0 and each point of C_0 is a cut point of order two in M_0.*

Proof. Suppose $t \in T_0$. If M_0 were to have the subspace topology of the plane, it is clear that t would have cut point order three with $M_0 - \{t_0\} = K_1 \cup K_2 \cup K_3$ such that K_1, K_2, and K_3 are pairwise mutually separated and each is connected. We claim that K_1, K_2 and K_3 are also the pairwise mutually separated components of $M_0 - \{t_0\}$, where M_0 has the subspace topology of S.

We show that $\mathrm{Cl}(K_1) \cap K_2 = \varnothing$. Assume that $s \in \mathrm{Cl}(K_1) \cap K_2$. Then for each natural number j, $R(s,1/j) \cap K_1 \neq \varnothing$. Then $N(s,1/j) \cap K_1 \neq \varnothing$ and K_1 and K_2 are not mutually separated with M_0 having the subspace topology of the plane, a contradiction. In a similar way, $K_1 \cap \mathrm{Cl}(K_2) = \varnothing$ and K_1 and K_2 are mutually separated. By parallel arguments, the pairs K_1 and K_3 and K_2 and K_3, respectively, are mutually separated.

By a proof similar to that of Theorem 3.5, each of K_1, K_2, and K_3 is connected in S, and therefore $t \in T_0$ is a cut point of order three in $M_0 \subset S$.

Suppose $c \in C_0$. If M_0 were to have the subspace topology of the plane, it is clear that c would have cut point order two with $M_0 - \{c_0\} = K_1 \cup K_2$ such that K_1 and K_2 are mutually separated and each is connected. By an argument like that above, K_1 and K_2 are also the mutually separated components of $M_0 - \{c_0\}$, where M_0 has the subspace topology of S. Therefore, $c \in C_0$ is a cut point of order two in $M_0 \subset S$. □

Lemma 3.7. *If q_0 is a fixed element of C_0, then the collection of sequences $Q_0 = \{(q_0, p_1, \ldots, p_n)\}$ in S for all whole numbers n is connected. Furthermore, $Q_0 - \{q_0\}$ is connected.*

Proof. Note that $N_1' = \{q_0\} \times M_0$ is connected since M_0 is connected. Since q_0 is a limit point of N_1', $N_1 = N_1' \cup \{q_0\}$ is also connected. Similarly, $N_2'(x) = (q_0,x) \times M_0$ is connected for each $x \in C_0$. As before, (q_0,x) is a limit point of $N_2'(x)$ and a point of N_1'. Thus, $N_2' = \bigcup_{x\in C_0} N_2'(x)$ is the union of a collection of connected sets each having a limit point in N_1'. So we have that $N_2 = N_2' \cup N_1'$ is connected. Next define for each $(x_1,x_2) \in C_0 \times C_0$, $N_3'(x_1,x_2) = (q_0,x_1,x_2) \times M_0$. $N_3'(x_1,x_2)$ is connected and has a limit point $(q_0,x_1,x_2) \in N_2'$. Thus, $N_3' = \bigcup_{(x_1,x_2)\in C_0\times C_0} N_3'(x_1,x_2)$ is the union of a collection of connected sets each having a limit point in the connected set $N_2' \cup N_1'$ so $N_3' \cup N_2' \cup N_1'$ is connected. This process can be continued to define N_n' for each positive integer n to be the union of a collection of connected copies of M_0 each having a limit point in N_{n-1}' so that $N_1' \cup N_2' \cup \cdots \cup N_n'$ is connected and contains all points of Q_0 having $n+1$ or fewer coordinates. Thus, Q_0 and $Q_0 - \{q_0\} = \bigcup_{i>0} N_i'$ is connected. □

Theorem 3.8. *Each point of $(S,\mathcal{T})$ is a cut point of order three.*

Proof. If C is a component of $M_0 - \{p_0\}$ for some $p_0 \in M_0$, let C' denote $\{p = (x_0, p_1, p_2, \dots, p_n) \in S : n$ is a whole number, and $x_0 \in C\}$.

Let $p = (p_0, p_1, p_2, \dots, p_n)$ be a point of $(S,\mathcal{T})$. We now consider four cases.

Case 1. Suppose $n = 0$ and $p_0 \in T_0$. From Lemma 3.6, we have $M_0 - \{p_0\} = S_1 \cup S_2 \cup S_3$ so that S_i is a component of $M_0 - \{p_0\}$ for each $1 \le i \le 3$. Then $S - \{p_0\} = S_1' \cup S_2' \cup S_3'$. Note that each S_i', $1 \le i \le 3$ is connected follows from Lemma 3.7.

We show that $\mathrm{Cl}(S_1') \cap S_2' = \varnothing$ and $S_1' \cap \mathrm{Cl}(S_2') = \varnothing$. Assume that $t \in \mathrm{Cl}(S_1') \cap S_2'$. We now consider three cases.

Case 1.1. Assume $t = (t_0) \in S_2'$ with $t_0 \in S_2 \cap T_0$. Let U be an open set in S with $t \in U$ that contains no point of S_1. Then $U \cap S_1' \ne \varnothing$ and $U \cap S_1' \subseteq T_0$. This implies that U contains a point $s = (s_0)$ with $s_0 \in S_1$ contrary to the definition of U.

Case 1.2. Assume $t = (t_0) \in S_2'$ with $t_0 \in C_0$. Let ϵ be a positive number such that $N(t_0,\epsilon)$ contains no point of S_1 in R^2. Let $U = R(t_0,\epsilon) = \{t_0\} \cup (N(t_0,\epsilon) \cap T_0) \cup (t_0 \times (N(\mathbb{O},\epsilon) \cap T_0))$. $U \cap S_1'$ must contain a point p in S. But if $p = (p_0)$, then $p \in N(t_0,\epsilon) \cap S_1$ contrary to the definition of ϵ. Also if $p = (p_0, p_1)$, then $p_0 = t_0 \notin S_1$ so $p \notin S_1'$.

Case 1.3. Assume $t = (t_0, t_1, \dots, t_n) \in S_2'$ with $n > 0$ and $t_0 \in S_2$. If $U = R(t,\epsilon)$, and $q \in U$, then $q = (q_0, q_1, \dots, q_m) \in U$ where $m = n$ or $m = n+1$. In either case $q_0 = t_0$ so $q \notin S_1'$, contrary to the assumption that $\mathrm{Cl}(S_1') \cap S_2' \ne \varnothing$.

Therefore, $\mathrm{Cl}(S_1') \cap S_2' = \varnothing$. By a parallel argument, $S_1' \cap \mathrm{Cl}(S_2') = \varnothing$. By similar arguments, $\mathrm{Cl}(S_1') \cap S_3' = \varnothing$ and $S_1' \cap \mathrm{Cl}(S_3') = \varnothing$, and $\mathrm{Cl}(S_2') \cap S_3' = \varnothing$ and $S_2' \cap \mathrm{Cl}(S_3') = \varnothing$. Therefore, S_1', S_2', and S_3' are pairwise mutually separated and p_0 is a cut point of order three.

Case 2. Suppose $n = 0$ and $p_0 \in C_0$. Suppose $M_0 - \{p_0\} = S_1 \cup S_2$ so that S_i is a component of $M_0 - \{p_0\}$ for each $1 \le i \le 2$. Then $S - \{p_0\} = S_1' \cup S_2' \cup T'$ where $T' = \{p = (p_0, p_1, \dots, p_n) : p \in S,\ n \ge 1\}$. S_1', S_2', and T' are pairwise mutually separated by arguments similar to those used in Case 1, and each of S_1', S_2', and T' is connected by Lemma 3.7. Thus, (p_0) is a cut point of order three.

Case 3. Suppose $n > 0$, $p = (p_0, p_1, \dots, p_n)$, and $p_n \in T_0$. Suppose $M_0 - \{p_n\} = S_1 \cup S_2 \cup S_3$ and without loss of generality assume that S_1 has $\mathbb{O}$ in its closure (if S_1 were to have the subspace topology of the plane). Let A_0 be the set of all points of S having a point of $M_0 - \{p_0\}$ as its first coordinate. For each positive integer $j < n$, let A_j be the set of all points of S whose first $j+1$ coordinates are $p_0, p_1, \dots, p_{j-1}, x$ where x is a point of $M_0 - \{p_j\}$. Let $A = \bigcup_{i=1}^{i=n-1} A_i$. If $i \in \{1,2,3\}$, let B_i be the set of all points of S whose first $n+1$ coordinates are $p_0, p_1, \dots, p_{n-1}, x$ where $x \in S_i$. A direct argument shows that $S - \{p\} = A \cup B_1 \cup B_2 \cup B_3$. We will show that $A \cup B_1$, B_2 and B_3 are mutually separated.

We show that $\mathrm{Cl}(A \cup B_1) \cap B_2 = \emptyset$. Assume that $t \in \mathrm{Cl}(A \cup B_1) \cap B_2$. We consider two cases.

Case 3.1. Assume $t = (t_0, t_1, \dots, t_n)$. Since $t \in B_2$, $t_n \in S_2$ and there is an $\epsilon > 0$ such that $N(t_n, \epsilon) \cap S_1 = \emptyset$. Let $U = R(t, \epsilon)$. If $x \in U$, $x = (x_0, x_1, \dots, x_k)$ for $k = n$ or $k = n+1$. In either case $x_n \in N(t_n, \epsilon)$ so $x_n \notin S_1$ and $x \notin B_1$. It remains to show that $A_1 \cap U = \emptyset$. If $x \in U$, $x_i = t_i = p_i$ for $0 \le i < n$. But if $x \in A$, there is an i, $0 \le i < n$ such that $x \in A_i$ and $x_i \ne p_i$.

Case 3.2. $t = (p_0, p_1, \dots, p_{n-1}, t_n, \dots, t_k)$ with $k > n$ and $t_n \in S_2 \cap C_0$. If U is a region containing t and x is in U, then x has the same first $k-1$ coordinates as t. But this means that $x_n = t_n \in S_2$, so x is not in S_1. As before, $x \notin A$, since $x_i = t_1 = p_i$ for $0 \le i < n$.

We now show that $(A_1 \cup B_1) \cap \mathrm{Cl}(B_2) = (A_1 \cap \mathrm{Cl}(B_2)) \cup (B_1 \cap \mathrm{Cl}(B_2)) = \emptyset$. Assume that $t \in (A_1 \cup B_1) \cap \mathrm{Cl}(B_2)$. We consider two cases.

Case 3.3. $t \in A_1 \cap \mathrm{Cl}(B_2)$, then $t = (t_0, t_1, \dots, t_j)$ for some whole number j, and since $t \in A$, there is an integer k such that $0 \le k < n$ such that $t_k \ne p_k$. If x is in the region $R(t, \epsilon)$, then $x_i = t_i$ for $0 \le i < n$. But this implies that $x_k = t_k \ne p_k$ and $x \notin B_2$, contrary to our assumption that $t \in Cl(B_2)$.

Case 3.4. $t \in B_1 \cap \mathrm{Cl}(B_2)$, then $t = (t_0, t_1, \dots, t_{n-1}, t_n, t_{n+1} \dots, t_k)$ with $t_n \in S_1$, $k \ge n$, and $t_n \ne p_n$. Since S_1 and S_2 are mutually separated, there is a positive number ϵ such that $N(t_n, \epsilon) \cap S_2 = \emptyset$. It follows that $R(t, \epsilon) \cap B_2 = \emptyset$, contrary to the assumption that $t \in Cl(B_2)$.

Therefore, $(A_1 \cup B_1)$ and B_2 are mutually separated. In a similar way, the pairs $(A_1 \cup B_1)$ and B_2 and B_2 and B_3, respectively, are mutually separated. Furthermore, it follows from Lemma 3.7 that each of $(A_1 \cup B_1)$, B_2, and B_3 is connected. Therefore, $p = (p_0, p_1, \dots, p_n)$ with $n > 0$ and $p_n \in T_0$ is a cut point of order three.

Case 4. Suppose $n > 0$, $p = (p_0, p_1, \dots, p_n)$, and $p_n \in C_0$. Suppose $M_0 - \{p_n\} = S_1 \cup S_2$ and without loss of generality assume that S_1 has $\mathbb{O}$ in its closure (if S_1 were to have the subspace topology of the plane). Let A be defined exactly as was done in Case 3. For $j \in \{1,2\}$, let B_j be the set of all points of S whose first $n+1$ coordinates are $p_0, p_1, \dots, p_{n-1}, x$ where $x \in S_1$. Let B_3 be the set of all points of S whose first $n+1$ coordinates are $p_0, p_1, \dots, p_n$. Using arguments entirely similar to those already given it can be shown that each of $(A_1 \cup B_1)$, B_2, and B_3 is connected and that they are pairwise mutually separated. Therefore, $p = (p_0, p_1, \dots, p_n)$ with $n > 0$ and $p_n \in C_0$ is a cut point of order three. $\square$

4. Embedding cut point spaces

In Kuratowski [10, Theorem 1, page 160], it is shown that for a connected separable metric space Z, the set $Z - \{z\}$ is connected or is the union of two connected sets for every $z \in Z$ except for a countable set of points of Z. See also [2, Theorem 3.2]. The following is therefore immediate.

Theorem 4.1. *If X is a cut point space and each point p of X has cut point order m where $m \geq 3$, then X may not be separable and metric and thus may not be embedded in $\mathcal{R}^n$ for any $n \geq 2$.*

We now provide a similar theorem in the setting of separable, connected Hausdorff spaces. The referee correctly noted that in the following theorem we did not need X to be Hausdorff. We do need the points to be closed and some authors refer to such a space as a T_1-space. Thus, we have a slightly stronger theorem than is stated.

Theorem 4.2. *If X is a separable connected Hausdorff space, then X does not contain uncountably many points that separate X into three mutually separated connected sets.*

Proof. Assume that there is an uncountable set of points T of X that separate X into 3 mutually exclusive connected sets. Let $P = \{p_1, p_2, p_3, \ldots\}$ be a countable dense subset of X with $p_i \neq p_j$ if and only if $i \neq j$. For each two positive integers m and n, let $C_{m,n}$ be the set of all points of X that separate p_m from p_n. Note that if $x \in T$, then $X - \{x\}$ is the union of two mutually exclusive open sets, so x separates two points of P. Thus, each point of T is in $C_{m,n}$ for some choice of m and n. Thus, there exist integers i and j such that $M = T \cap C_{i,j}$ is uncountable. If $x \in M$, then $X - \{x\}$ is the union of three mutually separated sets, and x separates p_i from p_j, so these points belong to different components of $X - \{x\}$. For each $x \in M$, let A_x be the component containing p_i, B_x the component containing p_j, and C_x the other component. Note that C_x is open in X for each $x \in M$.

We now show that if x and y are two distinct points of M, then C_x does not intersect C_y. Assume to the contrary that there exist points x and y in M such that $C_x \cap C_y \neq \emptyset$. Now $X - \{x\} = A_x \cup B_x \cup C_x$. Note that $y \notin C_x$ since if it were, then $X - \{y\}$ would contain $A_x \cup B_x \cup \{x\}$ which is connected, so y would not separate p_i from p_j, contrary to the definition of M. So y is in A_x or B_x. First, assume $y \in B_x$. Then $X - \{y\}$ contains $\{x\}$, A_x, C_x, and C_y and the union of these sets is connected and thus a subset of A_y. Thus, we have that $C_y \subseteq A_y$, but these sets are mutually exclusive. Next assume that $y \in A_x$. In this case, we have $\{x\} \cup B_x \cup C_x \cup C_y$ is a connected subset of $X - \{y\}$ and thus of B_y. This is again a contradiction since C_y and B_y are mutually exclusive.

Therefore, the set of all C_x for all $x \in M$ is an uncountable collection of mutually exclusive open sets in X, contrary to the separability of X. □

Corollary 4.3. *If X is an uncountable, separable, and connected cut point space and each point p of X has cut point order 3, then X may not be Hausdorff and thus may not be embedded in $\mathcal{R}^n$ for any $n \geq 2$.*

The referee suggested that it might be possible to construct a more intuitive example of a cut point space of order three by modifying an example of Velicko [11]. Indeed, one that is metric and dendritic. We think this would be quite interesting but we were unable to construct such an example.

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