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Research Article

Regular Generalized ω-Closed Sets

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In 1982 and 1970, Hdeib and Levine introduced the notions of ω-closed set and generalized closed set, respectively. The aim of this paper is to provide a relatively new notion of generalized closed set, namely, regular generalized ω-closed, regular generalized ω-continuous, a-ω-continuous, and regular generalized ω-irresolute maps and to study its fundamental properties.

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1. Introduction

All through this paper (X,τ) and (Y,σ) stand for topological spaces with no separation axioms assumed, unless otherwise stated. Let $A \subseteq X$, the closure of A and the interior of A will be denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. A is regular open if $A = \mathrm{Int}(\mathrm{Cl}(A))$ and A is regular closed if its complement is regular open; equivalently A is regular closed if $A = \mathrm{Cl}(\mathrm{Int}(A))$, see [1]. Let (X,τ) be a space and let A be a subset of X. A point $x \in X$ is called a condensation point of A if for each $U \in \tau$ with $x \in U$, the set $U \cap A$ is uncountable. A is called ω-closed [2] if it contains all its condensation points. The complement of an ω-closed set is called ω-open. It is well known that a subset W of a space (X,τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open subsets of a space (X,τ), denoted by τ_ω or $\omega O(X)$, forms a topology on X finer than τ. The ω-closure and ω-interior, that can be defined in a manner similar to $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively, will be denoted by $\mathrm{Cl}_\omega(A)$ and $\mathrm{Int}_\omega(A)$, respectively. Several characterizations of ω-closed subsets were provided in [3, 2, 4]. Levine [5] introduced the notion of generalized closed sets and a class of topological spaces called $T_{1/2}$-spaces. He defined a subset A of a space (X,τ) to be generalized closed set (briefly g-closed) if $\mathrm{Cl}(A) \subseteq U$ whenever $U \in \tau$ and $A \subseteq U$. Generalized semiclosed [6] (resp., α-generalized closed [7], θ-generalized closed [8], generalized semi-preclosed [9], δ-generalized closed [10], ω-generalized closed [3, 11]) sets are defined by replacing the closure operator in Levine's original definition by the semiclosure (resp., α-closure, θ-closure, semi-preclosure, δ-closure, ω-closure) operator.

2. Regular generalized ω-closed sets

A subset A of (X,τ) is called regular generalized closed (simply, rg-closed) (see [12]) if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and U is regular open. Analogously, we begin this section by introducing the class of regular generalized ω-closed sets.

Definition 2.1. A subset A of (X,τ) is called regular generalized ω-closed (simply, $rg\omega$-closed) if $\mathrm{Cl}_\omega(A) \subset U$ whenever $A \subset U$ and U is regular open. A subset B of (X,τ) is called regular generalized ω-open (simply, $rg\omega$-open) if the complement of B is $rg\omega$-closed sets.

We have the following relation for $rg\omega$-closed with the other known sets:

$$\begin{array}{ccccc} \omega\text{-}c\text{-closed} & & & & \\ \Uparrow & & & & \\ \text{closed} & \Longrightarrow & g\text{-closed} & \Longrightarrow & rg\text{-closed} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \omega\text{-closed} & \Longrightarrow & g\omega\text{-closed} & \Longrightarrow & rg\omega\text{-closed} \end{array} \tag{2.1}$$

Example 2.2. Let $\mathbb{R}$ be the set of all real numbers, let $\mathbb{Q}$ be the set of all rational numbers, with the topology $\tau = \{\mathbb{R}, \phi, \mathbb{R} - \mathbb{Q}\}$. Then $A = \mathbb{R} - \mathbb{Q}$ is not $g\omega$-closed, since A is open, thus ω-open and $A \subseteq A$, $\mathrm{Cl}_\omega(A) \nsubseteq A$ (because A is not ω-closed). Also the only regular open set containing A is X. Thus A is $rg\omega$-closed.

Example 2.3. Let $X = \{a,b,c,d\}$, with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$. Then the set $\{a\}$ is not rg-closed, see [13]. But $\{a\}$ is $rg\omega$-closed set, since X is finite and τ_ω is discrete topology.

It is clear that if (X,τ) is a countable space, then $rg\omega(X,\tau) = \mathcal{P}(X)$, where $rg\omega(X,\tau)$ is the set of all $rg\omega$-closed subsets of X and $\mathcal{P}(X)$ is the power set of X.

Since every closed set is ω-closed we have the following.

Lemma 2.4. *For every subset A of (X,τ), $\mathrm{Cl}_\omega(A) \subset \mathrm{Cl}(A)$.*

The proof of the following result follows from the fact that every regular open set is an open set together with Lemma 2.4.

Theorem 2.5. *Every $g\omega$-closed set and rg-closed set are $rg\omega$-closed.*

Theorem 2.6. *Let A be an $rg\omega$-closed subset of (X,τ). Then $\mathrm{Cl}_\omega(A)-A$ does not contain any nonempty regular closed set.*

Proof. Let F be a regular closed subset of (X,τ) such that $F\subseteq \mathrm{Cl}_\omega(A)-A$. Then $F\subseteq X-A$ and hence $A\subseteq X-F$. Since A is $rg\omega$-closed set and $X-F$ is a regular open subset of (X,τ), $\mathrm{Cl}_\omega(A)\subseteq X-F$ and so $F\subseteq X-\mathrm{Cl}_\omega(A)$. Therefore $F\subseteq \mathrm{Cl}_\omega(A)\cap(X-\mathrm{Cl}_\omega(A))=\phi$. □

Theorem 2.7. *A subset A of (X,τ) is $rg\omega$-open if and only if $F\subseteq \mathrm{Int}_\omega(A)$ whenever F is a regular closed subset such that $F\subseteq A$.*

Proof. Let A be an $rg\omega$-open subset of X and let F be a regular closed subset of X such that $F\subseteq A$. Then $X-A$ is an $rg\omega$-closed set and $X-A\subseteq X-F$. Since $X-A$ is $rg\omega$-closed, $X-\mathrm{Int}_\omega(A)=\mathrm{Cl}_\omega(X-A)\subseteq X-F$. Thus $F\subseteq \mathrm{Int}_\omega(A)$. Conversely, if $F\subseteq \mathrm{Int}_\omega(A)$ where F is a regular closed subset of (X,τ) such that $F\subseteq A$, then for any regular open subset U such that $X-A\subseteq U$, we have $X-U\subseteq A$ and thus $X-U\subseteq \mathrm{Int}_\omega(A)$. That is, $X-\mathrm{Int}_\omega(A)=\mathrm{Cl}_\omega(X-A)\subseteq U$. Therefore X-A is $rg\omega$-closed. □

Lemma 2.8 [14]. *For every open U in a topological space X and every $A\subseteq X$,* $\mathrm{Cl}(U\cap A)=\mathrm{Cl}(U\cap \mathrm{Cl}(A))$.

Recall that two nonempty sets A and B of X are said to be separated if $\mathrm{Cl}(A)\cap B=\phi=A\cap \mathrm{Cl}(B)$.

Theorem 2.9. *If A and B are open, $rg\omega$-open, and separated sets, then $A\cup B$ is $rg\omega$-open.*

Proof. Let F be a regular closed subset of $A\cup B$. Then $F\cap \mathrm{Cl}(A)\subseteq A$, since A is open and by Lemma 2.8 we have $F\cap \mathrm{Cl}(A)$ is regular closed hence by Theorem 2.7 $F\cap \mathrm{Cl}(A)\subseteq \mathrm{Int}_\omega(A)$. Similarly, $F\cap \mathrm{Cl}(B)\subseteq \mathrm{Int}_\omega(B)$. Then we have $F\subseteq \mathrm{Int}_\omega(A\cup B)$ and hence $A\cup B$ is $rg\omega$-open. □

The following example shows that the union of $rg\omega$-open sets need not be $rg\omega$-open.

Example 2.10. Let X be an uncountable set and let A, B, C, D be subsets of X, such that each of them is uncountable set and the family $\{A,B,C,D\}$ is a partition of X. We defined the topology $\tau=\{\phi,X,\{A\},\{B\},\{A,B\},\{A,B,C\}\}$. Choose $x,y\notin A$ and $x\neq y$. Then $H=A\cup\{x\}$ and $G=A\cup\{y\}$ are $rg\omega$-closed, since only regular open set containing H,G is X. But $H\cap G=\{A\}$ and $\{A\}$ is regular open in X and $\mathrm{Cl}_\omega(A)\nsubseteq A$, since $\{A\}$ is not ω-closed. Thus $H\cap G$ is not $rg\omega$-closed. Therefore the union of $rg\omega$-open sets need not be $rg\omega$-open.

The proof of the following result is straightforward since τ_ω is a topology on X and thus omitted.

Theorem 2.11. *If A and B are $rg\omega$-closed sets, then $A\cup B$ is $rg\omega$-closed.*

Theorem 2.12. *Let A be a $rg\omega$-closed subset of (X,τ). If $B\subseteq X$ such that $A\subseteq B\subseteq \mathrm{Cl}_\omega(A)$, then B is also $rg\omega$-closed. Let B be a subset of (X,τ) and let A be an $rg\omega$-open subset such that* $\mathrm{Int}_\omega(A)\subseteq B\subseteq A$. *Then B is also $rg\omega$-open.*

The proof is obvious.

Theorem 2.13. *If A be an rgω-closed subset of (X,τ), then $\mathrm{Cl}_\omega(A) - A$ is rgω-open set.*

Proof. Let A be an $rg\omega$-closed subset of (X,τ) and let F be a regular closed subset such that $F \subseteq \mathrm{Cl}_\omega(A) - A$. By Theorem 2.6, $F = \phi$ and thus $F \subseteq \mathrm{Int}_\omega(\mathrm{Cl}_\omega(A) - A)$. By Theorem 2.7, $\mathrm{Cl}_\omega(A) - A$ is $rg\omega$-open set. □

We first recall the following lemmas to obtain further results for $rg\omega$-closed sets.

Lemma 2.14 [3]. *If Y is an open subspace of a space X and A is a subset of Y, then $\mathrm{Cl}_{\omega|Y}(A) = \mathrm{Cl}_\omega(A) \cap (Y)$.*

Lemma 2.15. *If A is a regular open and rgω-closed subset of a space X, then A is ω-closed in X.*

The proof is obvious.

Theorem 2.16. *Let Y be an open subspace of a space X and $A \subseteq Y$. If A is rgω-closed in X, then A is rgω-closed in Y.*

Proof. Let U be a regular open set of Y such that $A \subseteq U$. Then $U = V \cap Y$ for some regular open set V of X. Since A is $rg\omega$-closed in X, we have $\mathrm{Cl}_\omega(A) \subseteq U$ and by Lemma 2.14, $\mathrm{Cl}_{\omega|Y}(A) = \mathrm{Cl}_\omega(A) \cap (Y) \subseteq V \cap Y = U$. Hence A is $rg\omega$-closed in X. □

Corollary 2.17. *If A is an rgω-closed regular open set and B is an ω-closed set of a space X, then $A \cap B$ is rgω-closed.*

Theorem 2.18. *Let A be an rgω-closed set. Then $A = \mathrm{Cl}_\omega(\mathrm{Int}_\omega(A))$ if and only if $\mathrm{Cl}_\omega(\mathrm{Int}_\omega(A)) - A$ is regular closed.*

Proof. If $A = \mathrm{Cl}_\omega(\mathrm{Int}_\omega(A))$, then $\mathrm{Cl}_\omega(\mathrm{Int}_\omega(A)) - A = \phi$ and hence $\mathrm{Cl}_\omega(\mathrm{Int}_\omega(A)) - A$ is regular closed. Conversely, let $\mathrm{Cl}_\omega(\mathrm{Int}_\omega(A)) - A$ be regular closed, since $\mathrm{Cl}_\omega(A) - A$ contains the regular closed set $\mathrm{Cl}_\omega(\mathrm{Int}_\omega(A)) - A$. By Theorem 2.6 $\mathrm{Cl}_\omega(\mathrm{Int}_\omega(A)) - A = \phi$ and hence $A = \mathrm{Cl}_\omega(\mathrm{Int}_\omega(A))$. □

Lemma 2.19 [3]. *Let (A,τ_A) be an antilocally countable subspace of a space (X,τ). Then $\mathrm{Cl}(A) = \mathrm{Cl}_\omega(A)$.*

We call (X,τ) an antilocally countable space if each nonempty open set is an uncountable set.

Corollary 2.20. *In an antilocally countable subspace of a space (X,τ), the concepts of rgω-closed set and rg-closed set coincide.*

Lemma 2.21 [3]. *Let (X,τ) and (Y,σ) be two topological spaces. Then $(\tau \times \sigma)_\omega \subseteq \tau_\omega \times \sigma_\omega$.*

Theorem 2.22. *If $A \times B$ is rgω-open subset of $(X \times Y, \tau \times \sigma)$, then A is rgω-open subset in (X,τ) and B is rgω-open subset in (Y,σ).*

Proof. Let F_A be a regular closed subset of (X,τ) and let F_B be a regular closed subset of (Y,σ) such that $F_A \subseteq A$ and $F_B \subseteq B$. Then $F_A \times F_B$ is regular closed in $(X \times Y, \tau \times \sigma)$ such that $F_A \times F_B \subseteq A \times B$. By assumption $A \times B$ is $rg\omega$-open in $(X \times Y, \tau \times \sigma)$ and so

$F_A \times F_B \subseteq \text{Int}_\omega(A \times B) \subseteq \text{Int}_\omega(A) \times \text{Int}_\omega(B)$ by Lemma 2.21. Therefore $F_A \subseteq \text{Int}_\omega$, $F_B \subseteq \text{Int}_\omega(B)$. Hence A, B are $rg\omega$-open. □

The converse of the above need not be true in general.

Example 2.23. Let $X = Y = \mathbb{R}$ with the usual topology τ. Let $A = \{\{\mathbb{R} - \mathbb{Q}\} \cup [\sqrt{2}, 5]\}$ and $B = (1,7)$. Then A and B are $rg\omega$-open (ω-open) subsets of $(\mathbb{R},\tau)$, while $A \times B$ is not $rg\omega$-open in $(\mathbb{R} \times \mathbb{R}, \tau \times \tau)$, since the set $F = [\sqrt{2},3] \times [3,5]$ is regular closed set contained in $A \times B$ and $F \nsubseteq \text{Int}_\omega(A \times B)$. The point $(\sqrt{2},4) \in F$ and $(\sqrt{2},4) \notin \text{Int}_\omega(A \times B)$, because if $(\sqrt{2},4) \in \text{Int}_\omega(A \times B)$, then there exist open set U containing $\sqrt{2}$ and open set V containing 4 such that $(U \times V) - (A \times B)$ is countable but $(U \times V) - (A \times B)$ is uncountable for any open set U containing $\sqrt{2}$ and open set V containing 4.

3. Regular generalized ω-$T_{1/2}$ space

Recall that a space (X,τ) is called $T_{1/2}$ [5] if every g-closed set is closed or equivalently if every singleton is open or closed, Dunham [15]. We introduce the following relatively new definition.

Definition 3.1. A space (X,τ) is a regular generalized ω-$T_{1/2}$ (simply, $rg\omega$-$T_{1/2}$) if every $rg\omega$-closed set in (X,τ) is ω-closed.

Theorem 3.2. *For a space (X,τ), the following are equivalent.*

(1) *X is a $rg\omega$-$T_{1/2}$.*

(2) *Every singleton is either regular closed or ω-open.*

Proof. (1)⇒(2) Suppose $\{x\}$ is not a regular closed subset for some $x \in X$. Then $X - \{x\}$ is not regular open and hence X is the only regular open set containing $X - \{x\}$. Therefore $X - \{x\}$ is $rg\omega$-closed. Since (X,τ) is $rg\omega$-$T_{1/2}$ space, $X - \{x\}$ is ω-closed and thus $\{x\}$ is ω-open.

(2)⇒(1) Let A be an $rg\omega$-closed subset of (X,τ) and $x \in \text{Cl}_\omega(A)$. We show that $x \in A$. If $\{x\}$ is regular closed and $x \notin A$, then $x \in (\text{Cl}_\omega(A) - A)$. Thus $\text{Cl}_\omega(A) - A$ contains a nonempty regular closed set $\{x\}$, a contradiction to Theorem 2.6. So $x \in A$. If $\{x\}$ is ω-open, since $x \in \text{Cl}_\omega(A)$, then for every ω-open set U containing x, we have $U \cap A \neq \phi$. But $\{x\}$ is ω-open then $\{x\} \cap A \neq \phi$. Hence $x \in A$. So in both cases we have $x \in A$. Therefore A is ω-closed. □

Theorem 3.3. *Let (X,τ) be an antilocally countable space. Then (X,τ) is a T_1-space if every $rg\omega$-closed set is ω-closed.*

Proof. Let $x \in X$, and suppose that $\{x\}$ is not closed. Then $A = X - \{x\}$ is not open, and thus A is $rg\omega$-closed (the only regular open set containing A is X). Therefore, by assumption, A is ω-closed, and thus $\{x\}$ is ω-open. So there exists $U \in \tau$ such that $x \in U$ and $U - \{x\}$ is countable. It follows that U is a nonempty countable open subset of $x \in X$, a contradiction. □

Definition 3.4. A map $f : X \to Y$ is said to be

(i) approximately closed [16] (a-closed) provided that $f(F) \subseteq \text{Int}(A)$ whenever F is a closed subset of X, A is a g-open subset of Y, and $f(F) \subseteq A$;

(ii) approximately continuous [16] (a-continuous) provided that $\mathrm{Cl}(A) \subseteq f^{-1}(V)$ whenever V is an open subset of Y, A is a g-closed subset of X, and $A \subseteq f^{-1}(V)$.

Definition 3.5. A map $f : X \to Y$ is said to be approximately ω-closed (simply, a-ω-closed) provided that $f(F) \subseteq \mathrm{Int}_\omega(A)$ whenever F is a regular closed subset of X, A is an $rg\omega$-open of Y, and $f(F) \subseteq A$.

Definition 3.6. A map $f : X \to Y$ is said to be approximately ω-continuous (simply, a-ω-continuous) provided that $\mathrm{Cl}_\omega(A) \subseteq f^{-1}(V)$ whenever V is a regular open subset of Y, A is an $rg\omega$-closed subset of X, and $A \subseteq f^{-1}(V)$.

The notions of a-closed (resp.; a-continuous) and a-ω-closed (resp.; a-ω-continuous) are independent.

Example 3.7. Let $X = \{a,b,c,d\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$. Let $f : (X,\tau) \to (X,\tau)$ be a function defined by $f(a) = a$, $f(b) = d$, $f(c) = b$, $f(d) = c$. Then f is a-ω-closed, since X is finite and thus τ_ω is a discrete topology, and f is not a-closed function. Because the set $A = \{b,c\}$ is g-open and $F = \{c,d\}$ is closed, $f(F) \subseteq A$, but $f(F) \nsubseteq \mathrm{Int}(A)$.

Example 3.8. Let $X = \mathbb{R}$ with the topology $\tau = \{\phi, X, \mathbb{R} - \mathbb{Q}\}$. Let $f : (X,\tau) \to (X,\tau)$ be a function defined by $f(x) = 0$, for all $x \in X$. Then f is a-closed, since for any closed set F of X, the only g-open set containing $f(F)$ is X. And f is not a-ω-closed function. Because the set $A = \mathbb{Q}$ is $rg\omega$-open and $F = \mathbb{R}$ is regular closed, $f(F) \subseteq A$, but $f(F) \nsubseteq \mathrm{Int}_\omega(A) = \phi$.

Theorem 3.9. *A space X is $rg\omega$-$T_{1/2}$-space if and only if every space Y and every function $f : X \to Y$ are a-ω-continuous.*

Proof. Let V be a regular open subset of Y and A is an $rg\omega$-closed subset of X such that $A \subseteq f^{-1}(V)$, since X is $rg\omega$-$T_{1/2}$-space then A is ω-closed thus $A = \mathrm{Cl}_\omega(A)$, hence $\mathrm{Cl}_\omega(A) \subseteq f^{-1}(V)$ and f is a-ω-continuous. Let A be a nonempty $rg\omega$-closed subset of X and let Y be the set X with the topology $\{Y, A, \phi\}$. Let $f : X \to Y$ be the identity mapping. By assumption f is a-ω-continuous. Since A is $rg\omega$-closed subset in X and open in Y such that $A \subseteq f^{-1}(A)$, it follows that $\mathrm{Cl}_\omega(A) \subseteq f^{-1}(A) = A$. Hence A is ω-closed in X and therefore X is $rg\omega$-$T_{1/2}$-space. □

Lemma 3.10. *If the regular open and regular closed sets of X coincide, then all subsets of X are $rg\omega$-closed (and hence all are $rg\omega$-open).*

Proof. Let A be any subset of X such that $A \subseteq U$ and U is regular open, then $\mathrm{Cl}_\omega(A) \subseteq \mathrm{Cl}_\omega(U) \subseteq \mathrm{Cl}(U) = U$. Therefore A is $rg\omega$-closed. □

Theorem 3.11. *If the regular open and regular closed sets of Y coincide, then a function $f : X \to Y$ is a-ω-closed if and only if $f(F)$ is ω-open for every regular closed subset F of X.*

Proof. Assume f is a-ω-closed by Lemma 3.10 all subsets of Y are $rg\omega$-closed. So for any regular closed subset F of X, $f(F)$ is $rg\omega$-closed in Y. Since f is a-ω-closed, $f(F) \subseteq \mathrm{Int}_\omega(f(F))$, therefore $f(F) = \mathrm{Int}_\omega(f(F))$ thus $f(F)$ is ω-open. Conversely if $f(F) \subseteq A$ where F is regular closed and A is $rg\omega$-open, then $f(F) = \mathrm{Int}_\omega(f(F)) \subseteq \mathrm{Int}_\omega(A)$ hence f is a-ω-closed. □

The proof of the following result for a-ω-continuous function is analogous and is omitted.

Theorem 3.12. *If the regular open and regular closed sets of X coincide, then a function $f : X \to Y$ is a-ω-continuous if and only if $f^{-1}(V)$ is ω-closed for every regular open subset V of Y.*

4. $rg\omega$-continuity

In this section, we will introduce some new classes of maps and study some of their characterizations. In [11, 3] a map $f : X \to Y$ is called ω-irresolute (resp., R-map [17]) if the inverse image of every ω-closed (resp., regular closed) subset of Y is ω-closed (resp., regular closed) in X. In [3], a map $f : X \to Y$ is called $g\omega$-closed if the image of every closed subset of X is $g\omega$-closed in Y. Relatively new definitions are given next.

Definition 4.1. A map $f : X \to Y$ is called $rg\omega$-closed (resp., ro-preserving, pre-ω-closed) if $f(V)$ is $rg\omega$-closed (resp., regular open, ω-closed) in Y for every closed (resp., regular open, ω-closed) subset V of X.

Example 4.2. Let $X = \{a,b,c,d\}$ with the topology $\tau = \{\phi, X, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$. Let $f : (X,\tau) \to (X,\tau)$ be a function defined by $f(a) = a$, $f(b) = b$, $f(c) = d$, $f(d) = c$. Then f is ro-preserving, since the family of all regular open sets of X is $\{\phi, X, \{a\}, \{b\}\}$. But if we defined $g : (X,\tau) \to (X,\tau)$ as $g(a) = c$, $g(b) = d$, $g(c) = a$, $g(d) = b$, then g is not ro-preserving function.

Definition 4.3. A map $f : X \to Y$ is called $rg\omega$-continuous (resp., $rg\omega$-irresolute) if the inverse image of every ω-closed (resp., $rg\omega$-closed) subset V of Y is $rg\omega$-closed subset of X.

From the definition stated above we obtain the following diagram of implications:

$$\begin{array}{ccccc}
\text{continuous} & & & & \\
\Downarrow & & & & \\
\omega\text{-continuous} & \Longrightarrow & g\omega\text{-continuous} & \Longrightarrow & rg\omega\text{-continuous} \\
\Uparrow & & \Uparrow & & \Uparrow \\
\omega\text{-irresolute} & \Longrightarrow & g\omega\text{-irresolute} & \Longrightarrow & rg\omega\text{-irresolute}
\end{array} \tag{4.1}$$

Theorem 4.4. *Let $f : X \to Y$ be a surjective, $rg\omega$-irresolute, and pre-ω-closed map if X is $rg\omega$-$T_{1/2}$-space, then Y is also an $rg\omega$-$T_{1/2}$-space.*

Proof. Let A be $rg\omega$-closed subset of Y. Since f is an $rg\omega$-irresolute map, then $f^{-1}(A)$ is an $rg\omega$-closed subset of X. Since X is $rg\omega$-$T_{1/2}$-space, then $f^{-1}(A)$ is an ω-closed subset of X. Since f is a pre-ω-closed map, then $f(f^{-1}(A)) = A$ is an ω-closed subset of Y. Therefore Y is also $rg\omega$-$T_{1/2}$-space. □

Since every $g\omega$-closed set is $rg\omega$-closed, every $g\omega$-closed map is $rg\omega$-closed. Next we give new characterization of $g\omega$-closed maps.

Theorem 4.5. *A map $f : X \to Y$ is $g\omega$-closed if and only if for each $A \subseteq Y$ and each open set U containing $f^{-1}(A)$, there exists a $g\omega$-open subset V of Y such that $A \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. Let F be a $g\omega$-closed map, $A \subseteq Y$, and let U be an open set containing $f^{-1}(A)$. Then $V = Y - f(X - U)$ is $g\omega$-open subset of Y containing A and $f^{-1}(V) \subseteq U$. Conversely let F be closed subset of X and let H be an open subset of Y such that $f(F) \subseteq H$. Then $f^{-1}(Y - f(F)) \subseteq X - F$ and $X - F$ is open by hypothesis, there exists a $g\omega$-open subset V of Y such that $Y - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$. Therefore, $F \subseteq X - f^{-1}(V)$ and hence $f(F) \subseteq Y - V$. Since $Y - H \subseteq Y - f(F)$, $f^{-1}(Y - H) \subseteq f^{-1}(Y - f(F)) \subseteq f^{-1}(V) \subseteq X - F$, by taking complement, we get $F \subseteq X - f^{-1}(V) \subseteq X - f^{-1}(Y - f(F)) \subseteq X - f^{-1}(Y - H)$. Therefore $f(F) \subseteq Y - V \subseteq H$. Since $Y - V$ is $g\omega$-closed set and $\mathrm{Cl}_\omega(f(F)) \subseteq \mathrm{Cl}_\omega(Y - V) \subseteq H$, hence $f(F)$ is $g\omega$-closed. Thus f is a $g\omega$-closed map. □

Since every ω-closed set is $rg\omega$-closed, we have the following.

Theorem 4.6. *Every $rg\omega$-irresolute map is $rg\omega$-continuous map.*

Definition 4.7. A subset $A \subseteq X$ is said to be ω-c-closed provided that there is a proper subset B for which $A = \mathrm{Cl}_\omega(B)$. A map $f : X \to Y$ is said to be $g\omega$-c-closed if $f(A)$ is $g\omega$-closed in Y for every ω-c-closed subset $A \subseteq X$.

Since closed sets are obviously ω-c-closed, $g\omega$-closed maps are $g\omega$-c-closed. In a similar manner, we say a map $f : X \to Y$ is $rg\omega$-c-closed if $f(A)$ is $rg\omega$-closed in Y for every ω-c-closed subset $A \subseteq X$.

Theorem 4.8. *Let $f : X \to Y$ be an R-map and $rg\omega$-c-closed. Then $f(A)$ is $rg\omega$-closed in Y for every $rg\omega$-closed subset A of X.*

Proof. Let A be an $rg\omega$-closed subset of X and let U be a regular open subset of Y such that $f(A) \subseteq U$. Since f is an R-map, $f^{-1}(U)$ is a regular open subset of X and $A \subseteq f^{-1}(U)$. As A is an $rg\omega$-closed subset, $\mathrm{Cl}_\omega(A) \subseteq f^{-1}(U)$. Hence $f(\mathrm{Cl}_\omega(A)) \subseteq (U)$. Because $\mathrm{Cl}_\omega(A)$ is ω-c-closed and F is $rg\omega$-c-closed map, $f(\mathrm{Cl}_\omega(A))$ is $rg\omega$-closed. Therefore, $\mathrm{Cl}_\omega(f(A)) \subseteq \mathrm{Cl}_\omega(f(\mathrm{Cl}_\omega(A))) \subseteq f(\mathrm{Cl}_\omega(A)) \subseteq U$. Hence $f(A)$ is an $rg\omega$-closed subset of Y. □

Theorem 4.9. *Let $f : X \to Y$ be ro-preserving and ω-irresolute function, if B is $rg\omega$-closed in Y, then $f^{-1}(B)$ is $rg\omega$-closed in X.*

Proof. Let G be a regular open subset of X such that $f^{-1}(B) \subseteq G$. Then $B \subseteq f(G)$ and $f(G)$ is regular open. Since B is $rg\omega$-closed, then $\mathrm{Cl}_\omega(A) \subseteq f(G)$ and $f^{-1}(\mathrm{Cl}_\omega(B)) \subseteq G$. Since f is ω-irresolute then $f^{-1}(\mathrm{Cl}_\omega(B))$ is ω-closed and $\mathrm{Cl}_\omega(f^{-1}(\mathrm{Cl}_\omega(B))) = f^{-1}(\mathrm{Cl}_\omega(B))$, therefore $\mathrm{Cl}_\omega(f^{-1}((B))) \subseteq \mathrm{Cl}_\omega(f^{-1}(\mathrm{Cl}_\omega(B))) \subseteq G$ thus $f^{-1}(B)$ is $rg\omega$-closed in X. □

Theorem 4.10. *Let $f : X \to Y$ be a-ω-closed maps and ω-irresolute maps, if A is $rg\omega$-closed in Y, then $f^{-1}(A)$ is $rg\omega$-closed in X.*

Proof. Assume that A is an $rg\omega$-closed in Y and $f^{-1}(A) \subseteq U$, where U is a regular open subset of X. Taking complements we obtain $X - U \subseteq X - f^{-1}(A) \subseteq f^{-1}(Y - A)$ and $f(X - U) \subseteq Y - A$. Since f is a-ω-closed, $f(X - U) \subseteq \mathrm{Int}_\omega(Y - A) = Y - \mathrm{Cl}_\omega(A)$. It follows that $X - U \subseteq X - f^{-1}(\mathrm{Cl}_\omega(A))$ and $f^{-1}(\mathrm{Cl}_\omega(A)) \subseteq U$, since f is ω-irresolute, $f^{-1}(\mathrm{Cl}_\omega(A))$ is ω-closed thus we have $f^{-1}(A) \subseteq f^{-1}(\mathrm{Cl}_\omega(A)) \subseteq U$ and $\mathrm{Cl}_\omega(f^{-1}(A)) \subseteq \mathrm{Cl}_\omega(f^{-1}(\mathrm{Cl}_\omega(A))) = f^{-1}(\mathrm{Cl}_\omega(A)) \subseteq U$. Therefore $\mathrm{Cl}_\omega(f^{-1}(A)) \subseteq U$ and $f^{-1}(A)$ is $rg\omega$-closed in X. □

Theorem 4.11. *If $f : X \to Y$ is R-map and $rg\omega$-closed and A is g-closed subset of X, then $f(A)$ is $rg\omega$-closed.*

Proof. Let $f(A) \subseteq U$, where U is regular open subset of X then $f^{-1}(U)$ is regular open set containing A. Since A is g-closed, we have then $\mathrm{Cl}(A) \subseteq f^{-1}(U)$ and $f(\mathrm{Cl}(A)) \subseteq U$. Since f is $rg\omega$-closed, $f(\mathrm{Cl}(A))$ is $rg\omega$-closed. Therefore $\mathrm{Cl}_\omega(f(\mathrm{Cl}(A))) \subseteq U$ which implies that $\mathrm{Cl}_\omega(f(A)) \subseteq U$, hence $f(A)$ is $rg\omega$-closed. □

The proof of Theorem 4.8 can be easily modified to obtain the following result.

Theorem 4.12. *Let $f : X \to Y$ be a-ω-map and $rg\omega$-c-closed. Then $f(A)$ is $rg\omega$-closed subset of Y for every $rg\omega$-closed subset A of X.*

Theorem 4.13. *Let $f : X \to Y$ be R-map and pre-ω-closed. Then $f(A)$ is $rg\omega$-closed in Y for every $rg\omega$-closed subset A of X.*

Proof. Let A be any $rg\omega$-closed subset of X and let U be any regular open subset of Y such that $f(A) \subseteq U$. Since f is R-map, $f^{-1}(U)$ is regular open and $A \subseteq f^{-1}(U)$. As A is $rg\omega$-closed, $\mathrm{Cl}_\omega(A) \subseteq f^{-1}(U)$. Hence $f(\mathrm{Cl}_\omega(A)) \subseteq U$. Therefore $\mathrm{Cl}_\omega(f(A)) \subseteq \mathrm{Cl}_\omega(f(\mathrm{Cl}_\omega(A))) = f(\mathrm{Cl}_\omega(A)) \subseteq U$. Hence $f(A)$ is $rg\omega$-closed in Y. □

Definition 4.14. A map $f : X \to Y$ is said to be ω-contra-R-map if for every regular open subset V of Y, $f^{-1}(V)$ is ω-closed.

Example 4.15. Let $X = \mathbb{R}$ with the usual topology τ and let $Y = \{a, b, c, d\}$, with the topology $\sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then the function $f : (X, \tau) \to (Y, \sigma)$ defined by

$$f(x) = \begin{cases} a, & \text{if } x \in \mathbb{Q}, \\ c, & \text{if } x \notin \mathbb{Q}, \end{cases} \tag{4.2}$$

is ω-contra-R-map, since $\mathbb{Q}$ is ω-closed. But the function $f(x)$ defined by

$$f(x) = \begin{cases} a, & \text{if } x \in \mathbb{Q}, \\ b, & \text{if } x \notin \mathbb{Q}, \end{cases} \tag{4.3}$$

is not ω-contra-R-map, since the family of all regular open set in (Y, σ) is $\{\phi, Y, \{a\}, \{b\}\}$ and $f^{-1}(\{b\})$ is not ω-closed.

Theorem 4.16. *Let $f : X \to Y$ be ω-contra-R-map and $rg\omega$-c-closed. Then $f(A)$ is $rg\omega$-closed in Y for every subset A of X.*

Proof. Let A be any subset of X and let U be any regular open subset of Y such that $f(A) \subseteq U$. Then $A \subseteq f^{-1}(U)$. Since f is ω-contra-R-map, $f^{-1}(U)$ is ω-closed and so $\mathrm{Cl}_\omega(A) \subseteq \mathrm{Cl}\,\omega(f(U)) = f(U)$. Hence $f(\mathrm{Cl}_\omega(A)) \subseteq U$. As $\mathrm{Cl}_\omega(A)$ is ω-c-closed subset of X and f is $rg\omega$-c-closed map, $f(\mathrm{Cl}_\omega(A))$ is $rg\omega$-closed. Therefore $\mathrm{Cl}_\omega(f(A)) \subseteq \mathrm{Cl}_\omega(f(\mathrm{Cl}(A))) \subseteq f(\mathrm{Cl}_\omega(A)) \subseteq U$. Thus $f(A)$ is $rg\omega$-closed in Y. □

Theorem 4.17. *If map $f : X \to Y$ is $rg\omega$-continuous (resp., $rg\omega$-irresolute) and X is $rg\omega$-$T_{1/2}$, then f is ω-continuous (resp., $rg\omega$-irresolute).*

Proof. Let A be any closed (resp., ω-closed) subset of Y. Since f is an $rg\omega$-continuous (resp., $rg\omega$-irresolute) map, $f^{-1}(A)$ is an $rg\omega$-closed subset of X. As (X,τ) is $rg\omega$-$T_{1/2}$ space, $f^{-1}(A)$ is an ω-closed subset of X. Therefore, f is an ω-continuous (resp., $rg\omega$-irresolute). □

Theorem 4.18. *Let $f : X \to Y$ be a bijective, ro-preserving, and $rg\omega$-continuous map. Then f is $rg\omega$-irresolute map.*

Proof. Let V be any $rg\omega$-closed subset of X and let U be any regular open subset of Y such that $f^{-1}(V) \subseteq U$. Clearly $V \subseteq f(U)$. Since f is a ro-preserving map, $f(U)$ is regular open and, by assumption, V is $rg\omega$-closed set. Hence $\mathrm{Cl}_\omega(V) \subseteq f(U)$ and $f^{-1}(\mathrm{Cl}_\omega(V)) \subseteq U$. Since f is $rg\omega$-continuous and $\mathrm{Cl}_\omega(V)$ is ω-closed in Y, then $f^{-1}(\mathrm{Cl}_\omega(V))$ is a $rg\omega$-closed subset of U and so $\mathrm{Cl}_\omega(f^{-1}(\mathrm{Cl}_\omega(V))) \subseteq U$. Since $\mathrm{Cl}_\omega(f^{-1}(V)) \subseteq \mathrm{Cl}_\omega(f^{-1}(\mathrm{Cl}_\omega(V))) \subseteq U$, $\mathrm{Cl}_\omega(f^{-1}(V)) \subseteq U$. Therefore $f^{-1}(V)$ is an $rg\omega$-closed subset. Hence f is f $rg\omega$-irresolute map. □

Theorem 4.19. *A map $f : X \to Y$ is f $rg\omega$-closed if and only if for each subset B of Y and for each open set U containing $f^{-1}(B)$, there is an $rg\omega$-open set V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.*

Proof. Suppose f is $rg\omega$-closed, let B be a subset of Y, and U is an open set of X such that $f^{-1}(B) \subseteq U$. Then $f(X - U)$ is $rg\omega$-closed in Y. Let $V = Y - f(X - U)$, then V is $rg\omega$-open set and $f^{-1}(V) = f^{-1}(Y - f(X - U)) = X - (X - U) \subseteq U$ therefore V is an $rg\omega$-open set containing B such that $f^{-1}(V) \subseteq U$. Conversely suppose that F is a closed set of X then $f^{-1}(Y - f(F)) \subseteq X - F$, and $X - F$ is open. By hypothesis, there is an $rg\omega$-open set V of Y such that $Y - f(F) \subseteq V$ and $f^{-1}(V) \subseteq X - F$ therefore $F \subseteq X - f^{-1}(V)$. Hence $Y - V \subseteq f(F) \subseteq f(X - f^{-1}(V)) \subseteq Y - V$ implies that $f(F) = Y - V$, thus f is $rg\omega$-closed. □

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